Bull. Aust. Math. Soc. **89** (2014), 33–40
doi:10.1017/S0004972713000683

A HYPERSTABILITY RESULT FOR THE CAUCHY EQUATION

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(Received 12 October 2012; accepted 20 May 2013; first published online 22 August 2013)

Abstract

We prove a hyperstability result for the Cauchy functional equation $f(x+y) = f(x) + f(y)$, which complements some earlier stability outcomes of J. M. Rassias. As a consequence, we obtain the slightly surprising corollary that for every function f, mapping a normed space E_1 into a normed space E_2, and for all real numbers r, s with $r + s > 0$ one of the following two conditions must be valid:

$$\sup_{x,y\in E_1} \|f(x+y) - f(x) - f(y)\|\ \|x\|^r\ \|y\|^s = \infty,$$
$$\sup_{x,y\in E_1} \|f(x+y) - f(x) - f(y)\|\ \|x\|^r\ \|y\|^s = 0.$$

In particular, we present a new method for proving stability for functional equations, based on a fixed point theorem.

2010 *Mathematics subject classification*: primary 39B62; secondary 47H10, 39B82, 47H14, 47J20.

Keywords and phrases: hyperstability, Cauchy equation, additive function, restricted domain.

1. Introduction

The main motivation for the investigation of the stability of functional equations was given by Ulam in 1940 in his talk at the University of Wisconsin (see [17, 36]), where he presented the following unsolved problem, among others.

Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with

$$d(f(x), T(x)) < \varepsilon$$

for all $x, y \in G_1$?

For more information on this area of research and further references, see [18, 21]. Let us only mention that the following theorem seems to be the most classical result concerning stability of the Cauchy equation

$$T(x+y) = T(x) + T(y). \tag{1.1}$$

THEOREM 1.1. *Let E_1 and E_2 be two normed spaces, with E_2 complete. Take $c \geq 0$ and let $p \neq 1$ be a fixed real number. Let $f : E_1 \to E_2$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}.$$

Then there exists a unique solution $T : E_1 \to E_2$ of (1.1) with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|1 - 2^{p-1}|}, \qquad x \in E_1 \setminus \{0\}.$$

Theorem 1.1 is due to Aoki [1] for $0 < p < 1$ (see also [31]); Gajda [16] for $p > 1$; Hyers [17] for $p = 0$; and Th. M. Rassias [32] for $p < 0$ (see [33, page 326] and [4]). Quite often the result contained in the theorem is described as the Hyers–Ulam–Rassias stability of the Cauchy equation (1.1). It has motivated J. M. Rassias [29, 30] (see also [21, pages 50–51]) to prove the following theorem.

THEOREM 1.2. *Let E_1 and E_2 be two normed spaces, with E_2 complete. Take $c \geq 0$ and let p, q be real numbers with $p + q \in [0, 1)$. Let $f : E_1 \to E_2$ be an operator such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c\|x\|^p\|y\|^q, \qquad x, y \in E_1 \setminus \{0\}.$$

Then there exists a unique solution $T : E_1 \to E_2$ of (1.1) with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^{p+q}}{2 - 2^{p+q}}, \qquad x \in E_1 \setminus \{0\}.$$

We provide a complement for this result in the case $p + q < 0$; moreover, we do so on a restricted domain. Namely, we prove the following theorem (in which $\mathbb{N}$ denotes the set of positive integers).

THEOREM 1.3. *Let E_1 and E_2 be normed spaces, and $X \subset E_1 \setminus \{0\}$ be nonempty. Take $c \geq 0$ and let p, q be real numbers with $p + q < 0$. Assume that there exists a positive integer m_0 with*

$$nx \in X, \qquad x \in X,\ n \in \mathbb{N},\ n \geq m_0. \tag{1.2}$$

Then every operator $g : E_1 \to E_2$, satisfying the inequality

$$\|g(x+y) - g(x) - g(y)\| \leq c\|x\|^p\|y\|^q, \qquad x, y \in X,\ x + y \in X, \tag{1.3}$$

is additive on X, that is, fulfils the condition

$$g(x+y) = g(x) + g(y), \qquad x, y \in X,\ x + y \in X.$$

https://doi.org/10.1017/S0004972713000683 Published online by Cambridge University Press

Clearly the statement of Theorem 1.3 is much stronger than that of Theorem 1.1. Using the terminology proposed in [26], we name the property of equation (1.1) described in Theorem 1.3 *ϕ-hyperstability on X*, with $\phi(x, y) = c\|x\|^p\|y\|^q$ for $x, y \in X$.

Note that, as a consequence of Theorem 1.3, we obtain at once the slightly surprising corollary that every function f mapping a normed space E_1 into a normed space E_2 is either additive (that is, $f(x + y) = f(x) + f(y)$ for $x, y \in E_1$) or satisfies the condition

$$\sup_{x,y\in E_1} \|f(x + y) - f(x) - f(y)\| \, \|x\|^r \, \|y\|^s = \infty$$

for all real numbers r, s with $r + s > 0$.

2. Auxiliary results

The method of proof of Theorem 1.3 is based on a fixed point theorem in [5, Theorem 1] (see also [6, Theorem 2]). Our method can be considered to be an extension of the investigations in [2, 7, 22–24, 27, 28]. (For a survey on this subject, see [8].)

We need the following hypotheses. (Here, $\mathbb{R}_+$ stands for the set of nonnegative reals and A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$.)

(H1) $X \neq \emptyset$ is a set, E_2 is a Banach space, $f_1, \dots, f_k : X \to X$ and $L_1, \dots, L_k : X \to \mathbb{R}_+$.

(H2) $\mathcal{T} : {E_2}^X \to {E_2}^X$ satisfies

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in {E_2}^X, \ x \in X.$$

(H3) $\Lambda : {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ is given by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

We are now in a position to present the fixed point theorem mentioned above.

THEOREM 2.1. *Let (H1)–(H3) hold and let $\varepsilon : X \to \mathbb{R}_+$, $\varphi : X \to E_2$ satisfy the conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \qquad x \in X,$$
$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < \infty, \qquad x \in X.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x), \qquad x \in X.$$

Moreover, ψ is given by

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

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3. Proof of Theorem 1.3

Without loss of generality, we can assume that E_2 is complete, because otherwise we can simply replace E_2 by its completion. Note that, in view of the assumption that $p+q<0$, we must have $p<0$ or $q<0$. Therefore, it is sufficient to consider only the case where $q<0$.

Let f denote the restriction of g to the set X. Fix $m \in \mathbb{N}$ with $m \geq m_0$ and

$$m^{p+q} + (1+m)^{p+q} < 1.$$

Taking $y = mx$ in (1.3),

$$\|f((m+1)x) - f(x) - f(mx)\| \leq cm^q \|x\|^{p+q}, \quad x \in X. \tag{3.1}$$

Define operators $\mathcal{T} : {E_2}^X \to {E_2}^X$ and $\Lambda : {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ by

$$\begin{aligned} \mathcal{T}\xi(x) &:= \xi((m+1)x) - \xi(mx), \quad x \in X,\ \xi \in {E_2}^X, \\ \Lambda\delta(x) &:= \delta((m+1)x) + \delta(mx), \quad x \in X,\ \delta \in {\mathbb{R}_+}^X. \end{aligned}$$

Then Λ has the form described in (H3) with $k=2$, $f_1(x) = (m+1)x$, $f_2(x) = mx$, $L_1(x) = L_2(x) = 1$ for $x \in X$ and (3.1) can be written as

$$\|\mathcal{T} f(x) - f(x)\| \leq cm^q \|x\|^{p+q} =: \varepsilon(x), \quad x \in X.$$

Furthermore, (H2) is also valid.

Since

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \leq cm^q \|x\|^{p+q} \sum_{n=0}^{\infty} (m^{p+q} + (m+1)^{p+q})^n, \quad x \in X,$$

we have

$$\varepsilon^*(x) \leq \frac{cm^q \|x\|^{p+q}}{1 - m^{p+q} - (m+1)^{p+q}}, \quad x \in X.$$

Hence, according to Theorem 2.1, there is a solution $T_m : X \to E_2$ of the equation

$$T(x) = T((1+m)x) - T(mx) \tag{3.2}$$

such that

$$\|f(x) - T_m(x)\| \leq \frac{cm^q \|x\|^{p+q}}{1 - m^{p+q} - (m+1)^{p+q}}, \quad x \in X. \tag{3.3}$$

Moreover,

$$T_m(x) := \lim_{n\to\infty} \mathcal{T}^n f(x) \quad x \in X.$$

Next, it can be easily shown by induction that, for every $x, y \in X$ with $x+y \in X$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\|\mathcal{T}^n f(x+y) - \mathcal{T}^n f(x) - \mathcal{T}^n f(y)\| \leq c(m^{p+q} + (m+1)^{p+q})^n (\|x\|^p \|y\|^q). \tag{3.4}$$

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To this end, it is enough to observe that the case $n = 0$ is just (1.3) and, for every $k \in \mathbb{N}_0$ and $x, y \in X$ with $x + y \in X$,

$$\begin{aligned}\|\mathcal{T}^{k+1}f(x+y) - \mathcal{T}^{k+1}f(x) - \mathcal{T}^{k+1}f(y)\| \\ \le \|\mathcal{T}^{k}f((m+1)x+(m+1)y) - \mathcal{T}^{k}f((m+1)x) - \mathcal{T}^{k}f((m+1)y)\| \\ + \|\mathcal{T}^{k}f(mx+my) - \mathcal{T}^{k}f(mx) - \mathcal{T}^{k}f(my)\|.\end{aligned}$$

Letting $n \to \infty$ in (3.4), we obtain that

$$T_m(x+y) = T_m(x) + T_m(y), \qquad x, y \in X, \ x+y \in X. \tag{3.5}$$

Next, we prove that T_m is the unique function mapping X into E_2 that is additive on X and such that

$$\sup_{x \in X} \|f(x) - T_m(x)\| \, \|x\|^{-p-q} < \infty.$$

So, suppose that $T_0 : X \to Y$ is additive on X and satisfies

$$\sup_{x \in X} \|f(x) - T_0(x)\| \, \|x\|^{-p-q} < \infty.$$

Then there is a positive real constant M with

$$\|T_m(x) - T_0(x)\| \le M\|x\|^{p+q}, \qquad x \in X. \tag{3.6}$$

We can easily show by induction that, for each $j \in \mathbb{N}_0$,

$$\|T_m(x) - T_0(x)\| \le M\|x\|^{p+q} \sum_{n=j}^{\infty} (m^{p+q} + (m+1)^{p+q})^n, \qquad x \in X. \tag{3.7}$$

It is enough to note that the case $j = 0$ follows from (3.6) and, for each $l \in \mathbb{N}_0$,

$$\|T_m(x) - T_0(x)\| \le \|T_m((m+1)x) - T_0((m+1)x)\| + \|T_m(mx) - T_0(mx)\|, \qquad x \in X,$$

because T_m and T_0 are solutions to (3.2). Hence, letting $j \to \infty$ in (3.7), we get $T_m = T_0$.

Thus we have proved that, for each $m \in \mathbb{N}$, $m \ge m_0$, there exists a unique solution $T_m : X \to Y$ to (3.5) satisfying (3.3). The uniqueness of T_m means that

$$\|f(x) - T_k(x)\| \le \frac{cn^q\|x\|^{p+q}}{1 - n^{p+q} - (n+1)^{p+q}} \tag{3.8}$$

for every $x \in X$ and $k, n \in \mathbb{N}$, $n \ge m_0$ and $k \ge m_0$. In fact, if $k, n \in \mathbb{N}$, $n \ge k \ge m_0$, then

$$\|f(x) - T_n(x)\| \le \frac{cn^q\|x\|^{p+q}}{1 - n^{p+q} - (n+1)^{p+q}} \le \frac{ck^q\|x\|^{p+q}}{1 - k^{p+q} - (k+1)^{p+q}}, \qquad x \in X,$$

whence $T_n = T_k$, which yields (3.8).

Fixing k and letting $n \to \infty$ in (3.8), we get $f = T_k$. This implies that f is additive on the set X.

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4. Final remarks

We end the paper with some comments and corollaries.

REMARK 4.1. There arises a natural question: when, for $T_0 : E_1 \to E_2$ additive on $X \subset E_1$, is there an additive $T : E_1 \to E_2$ with $T(x) = T_0(x)$ for $x \in X$? This is the case when X is a subsemigroup of the group $(E_1, +)$ (see [25, Theorem 1.1, Ch. XVIII]). Some further information on this issue can be found in [34, Ch. 4]; an example of the extension procedure yielding such a result is provided in [35, pages 143–144].

Theorem 1.3 yields the following two simple corollaries, which correspond to the results in [3, 9–13, 15, 19] on the inhomogeneous Cauchy equation (4.2) and the cocycle equation (4.3).

COROLLARY 4.2. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $G : X^2 \to E_2$, and $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X$ with $x_0 + y_0 \in X$. Assume that (1.2) holds with some $m_0 \in \mathbb{N}$ and there are real p, q and $c > 0$ such that $p + q < 0$ and*

$$\|G(x, y)\| \leq c\|x\|^p\|y\|^q, \qquad x, y \in X,\ x + y \in X. \tag{4.1}$$

Then the functional equation

$$g_0(x + y) = g_0(x) + g_0(y) + G(x, y), \qquad x, y \in X,\ x + y \in X, \tag{4.2}$$

has no solutions in the class of functions $g_0 : X \to E_2$.

PROOF. Let $g_0 : X \to E_2$ be a solution to (4.2). Define $f : E_1 \to E_2$ by $f(x) = g_0(x)$ for $x \in X$ and $f(x) = 0$ for $x \in E_1 \setminus X$. Then (1.3) holds and consequently, by Theorem 1.3, f is additive on X, which means that $G(x_0, y_0) \neq 0$. This is a contradiction. □

COROLLARY 4.3. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $G : {E_1}^2 \to E_2$ satisfy the cocycle functional equation*

$$G(x, y) + G(x + y, z) = G(x, y + z) + G(y, z), \qquad x, y, z \in E_1, \tag{4.3}$$

and $G(x, y) = G(y, x)$ for $x, y \in E_2$. Assume that (1.2) holds with some $m_0 \in \mathbb{N}$ and there are real p, q and $c > 0$ such that $p + q < 0$ and (4.1) holds. Then $G(x, y) = 0$ for every $x, y \in X$ with $x + y \in X$.

PROOF. It is well known (see [14] or [20]) that G is coboundary, which means that there exists $g : E_1 \to E_2$ such that $G(x, y) = g(x + y) - g(x) - g(y)$ for $x, y \in E_1$. This means that $g_0 : X \to E_2$, given by $g_0(x) := g(x)$ for $x \in X$, is a solution to (4.2). So Corollary 4.2 yields the result. □

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