Nagoya Math. J., (2025), 1–22
DOI 10.1017/nmj.2025.10077

THE p-RATIONALITY OF HEIGHT-ZERO CHARACTERS

NGUYEN HUNG AND MANDI SCHAEFFER FRY

Abstract. We propose and present evidence for a conjectural global-local phenomenon concerning the p-rationality of height-zero characters. Specifically, if χ is a height-zero character of a finite group G and D is a defect group of the p-block of G containing χ, then the p-rationality of χ can be captured inside the normalizer $\mathbf{N}_G(D)$.

§1. Introduction

In recent years, there has been a growing interest in the study of rationality properties of characters of finite groups and their relationship to global-local properties, which relate the character theory of a finite group to p-local subgroups for a prime p. In this article, we are concerned with the p-rationality of height-zero characters of finite groups.

Let B be a p-block of a finite group G and let $\mathrm{Irr}(B)$ denote the set of ordinary irreducible characters of B. The *(p-)height* of a character $\chi \in \mathrm{Irr}(B)$ is given by

$$\mathbf{ht}(\chi) := \nu(\chi(1)) - \min_{\psi\in\mathrm{Irr}(B)}\{\nu(\psi(1))\},$$

where $\nu := \nu_p$ is the usual p-adic valuation function. We say that χ is *height-zero* if $\mathbf{ht}(\chi) = 0$. In other words, the height-zero characters of B are those characters in B whose degrees have the minimal possible p-part.

To measure how p-rational (or p-irrational) a character χ is, one considers the p-part of the conductor of its values $\{\chi(g) : g \in G\}$. Recall that every character value is a certain sum of roots of unity. Such a sum is called a *cyclotomic integer*. The conductor $c(\mathcal{S})$ of a collection $\mathcal{S}$ of cyclotomic integers is the smallest positive integer n such that $\mathcal{S} \subseteq \mathbb{Q}(\exp(2\pi i/n))$. For $\chi \in \mathrm{Irr}(G)$, we write $c(\chi) := c(\{\chi(g) : g \in G\})$ and call this the *conductor of* χ. The so-called *p-rationality level* of χ is defined as

$$\mathbf{lev}(\chi) := \nu(c(\chi)).$$

We put forward the following, which proposes that the p-rationality level of a height-zero character can be captured inside a local subgroup, namely, the defect normalizer.

CONJECTURE A. *Let p be a prime, G a finite group, and χ be a height-zero character in a block B of G with $\mathbf{lev}(\chi) \geq 2$. Suppose that D is a defect group of B. Then,*

$$\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)}).$$

REMARK 1.1. For $g \in G$, let $\mathbf{lev}(\chi(g)) := \nu_p(c(\chi(g)))$ – the p-rationality level of $\chi(g)$. It is easy to see that

$$\mathbf{lev}(\chi) = \max_{g\in G}\{\mathbf{lev}(\chi(g))\},$$

Received February 18, 2025. Revised June 6, 2025. Accepted June 6, 2025.
2020 Mathematics subject classification: Primary 20C15, 20C20, 20C33.
Keywords: height-zero characters, p-rationality, defect normalizer, global-local principals, Alperin–McKay–Navarro conjecture.

so there exists $g \in G$ such that $\mathbf{lev}(\chi(g)) = \mathbf{lev}(\chi)$. That is, there exists an element in the group that captures the p-rationality of χ. Conjecture A simply claims that such an element can be found in $\mathbf{N}_G(D)$.

Height-zero characters are well known for their nice behavior with respect to the global-local principle. Among the first observations of this was in Brauer's height zero conjecture, recently resolved in [23], which states that all irreducible characters in a block B have height zero if and only if the defect groups of B are abelian. Another example is the celebrated Alperin–McKay conjecture. (See, e.g., [25, Conjecture 9.5]. See also [30], where the conjecture was recently proven for $p = 2$.) The Alperin–McKay conjecture asserts that if b is the block of $\mathbf{N}_G(D)$ corresponding to B in Brauer's first main correspondence, then there exists a bijection between the height-zero characters in B and those in b. Conjecture A offers another global-local phenomenon for height-zero characters. In fact, we observe in §7 a relationship between Conjecture A and the well-known Alperin–McKay–Navarro (AMN) conjecture, which refines the Alperin–McKay conjecture to further include the action of Galois automorphisms.

Conjecture A is inspired by Navarro–Tiep's conjecture [28, Conjecture C]. In what follows, $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G and $\mathrm{Irr}_{p'}(G)$ the subset of $\mathrm{Irr}(G)$ consisting of characters of degree not divisible by p. Furthermore, for any positive integer n, we use $\mathbb{Q}_n$ to denote the n-th cyclotomic field $\mathbb{Q}_n := \mathbb{Q}(\exp(2\pi i/n))$.

Conjecture B ([28, Conjecture C], Conjecture C). *Let p be a prime, G a finite group, $P \in \mathrm{Syl}_p(G)$, and $\chi \in \mathrm{Irr}_{p'}(G)$ with $\mathbf{lev}(\chi) \geq 1$. Then,*

$$\mathbb{Q}_{p^{\mathbf{lev}(\chi)}} = \mathbb{Q}_p(\chi_P).$$

Conjecture B implies that if a p'-degree character χ has p-rationality level at least 2, then its level remains unchanged when restricted to a Sylow p-subgroup: $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$. (However, note that for $\mathbf{lev}(\chi) = 1$, this is false - see, Example 7.3.) Recall that a p'-degree character is a height-zero character lying in a block of maximal defect, which means that the defect groups are the Sylow p-subgroups of G. In such a case, Conjecture A only asserts that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(P)})$. In particular, Conjecture B implies Conjecture A in the case of p'-degree characters. However, it is important to note that for height-zero characters in general, $\mathbf{lev}(\chi)$ does not always equal $\mathbf{lev}(\chi_D)$, see the examples in §7.3. We refer the reader to [20], [28] for further discussion on Conjecture B.

What evidence do we have for Conjecture A? Our first main result confirms the cyclic-defect case.

Theorem C. *Let p be a prime and G a finite group. Let $B \in \mathrm{Bl}(G)$ be a p-block of G with cyclic defect group D and $\chi \in \mathrm{Irr}(B)$. Then, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$.*

Remark 1.2. The assumption on $\mathbf{lev}(\chi)$ in Conjecture A is essential. There are many examples with $\mathbf{lev}(\chi) = 1$ but $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = 0$, see again §7.3. Theorem C, however, shows that this cannot occur when a defect group D is cyclic.

The proof of Theorem C is based on Dade's cyclic-defect theory [6], [7]. When the defect groups of B are cyclic, the set $\mathrm{Irr}(B)$ is naturally partitioned into two types: *exceptional* characters and *non-exceptional* characters, see §3.2. While the characters of the latter type are always p-rational, we show that the p-rationality level of an exceptional character aligns with that of its associated (linear) character of the defect group D. Another key step is to

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

show that if an element $g \in G$ captures the p-rationality of a character χ in a block with cyclic defect groups, then its p-part g_p generates a defect group of the block. This is done in §3.

Our next result solves Conjecture B for prime-degree characters, and therefore confirms Conjecture A for characters whose degree is a prime different from p.

THEOREM D. *Let p be a prime and G a finite group. Let $\chi \in \mathrm{Irr}(G)$ be of prime degree not equal to p with $\mathbf{lev}(\chi) \geq 2$. Let $P \in \mathrm{Syl}_p(G)$. Then, $\mathbb{Q}_{p^{\mathbf{lev}(\chi)}} = \mathbb{Q}_p(\chi_P)$. In particular, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(P)}) = \mathbf{lev}(\chi_P)$.*

The proof of Theorem D is divided into two fundamentally different cases, depending on whether the character in question is primitive or imprimitive. The imprimitive case builds on ideas from the recent solution of Conjecture B for p-solvable groups [20], as detailed in §4. In contrast, the primitive case is reduced to analyzing the values of prime-degree characters of quasisimple groups, which is addressed in §§5 and 6.

§5.2 provides an additional evidence supporting Conjecture A for certain almost quasisimple groups. Finally, §7 discusses some consequences and examples related to Conjecture A, along with its connection to the well-known AMN conjecture.

To conclude this introduction, we mention recent work of Navarro, Ruhstorfer, Tiep, and Vallejo [27] concerning the determination of fields of values of p-height zero characters of finite groups. They proposed that an abelian number field F with conductor $p^a m$, where p does not divide m, arises as the field of values of a p-height zero irreducible character of a finite group if and only if $[\mathbb{Q}_{p^a} : (\mathbb{Q}_{p^a} \cap \mathbb{Q}_m F)]$ is not divisible by p. They reduced this conjecture to a question about blocks of quasisimple groups and ultimately verified it in the case $p = 2$.

§2. Galois automorphisms and p-rationality level

Here, we discuss briefly the relationship between the p-rationality level of a character and the action of Galois automorphisms. The notation here will often be referred to throughout.

Let G be a finite group and suppose that $|G| = n = p^b m$ with $(p,m) = 1$. Let $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Then,

$$\mathcal{G} \cong \mathcal{I} \times \mathcal{K},$$

where

$$\mathcal{I} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_m) \text{ and } \mathcal{K} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_{p^b})$$

are the subgroups of $\mathcal{G}$ of those automorphisms fixing p'-roots and p-power roots, respectively, of unity. Let

$$\mathcal{H} := \mathcal{I} \times \langle \sigma \rangle,$$

where $\sigma \in \mathcal{K}$ is such that its restriction to $\mathbb{Q}_m$ is the Frobenius automorphism $\zeta \mapsto \zeta^p$. The group $\mathcal{H}$ is an important ingredient in the McKay–Navarro and AMN Conjectures [26], which we will discuss further in §7.

It is well known that the Galois group $\mathcal{G}$ permutes the p-blocks of G. Let B be a p-block of G and $\mathcal{H}_B$ be the subgroup of $\mathcal{H}$ fixing B. Since $\mathcal{I}$ point-wisely fixes $\mathbb{Q}_m$, it fixes every Brauer character and thus every block of G. In particular,

$$\mathcal{I} \leq \mathcal{H}_B.$$

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

We define $\mathcal{I}' := \mathcal{I}$ if $p = 2$ and $\mathcal{I}' := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_{pm})$ if $p > 2$. Note that $\mathcal{I}'$ is the Sylow p-subgroup of $\mathcal{I}$. Note also that if H_1 and H_2 are groups with orders dividing n, then characters χ_1 of H_1 and χ_2 of H_2 have the same p-rationality level whenever they have the same stabilizer under $\mathcal{I}$. Further, the same can be said, replacing $\mathcal{I}$ with $\mathcal{I}'$, with the added assumption that $\mathbf{lev}(\chi_i) \geq 1$ for $i = 1, 2$ if $p > 2$.

In fact, as pointed out in [28, Section 4], there is one particular element of $\mathcal{I}'$ that captures this behavior. Namely, for $e \in \mathbb{Z}_{\geq 1}$, let σ_e be the element of $\mathcal{I}'$ mapping any p-power root of unity ω to ω^{1+p^e}. The Galois automorphism σ_e has been seen to play a pivotal role in consequences of the McKay–Navarro conjecture predicting global-local properties of finite groups, and it turns out that the stability of a character under σ_e is closely tied to its p-rationality level (see [28, Lemma 4.1]).

§3. Blocks of cyclic defect

The goal of this section is to prove Theorem C.

3.1. Small-defect blocks

We begin with an elementary upper bound for the p-rationality level in terms of a defect group, which allows us to easily control the level of characters in blocks of small defect.

Recall that if B is a p-block of a finite group G then its *defect* $d(B)$ is the nonnegative integer

$$d(B) := \nu(|G|) - \min_{\psi \in \mathrm{Irr}(B)} \{\nu(\psi(1))\}.$$

Moreover, the order of any defect group D of B is $|D| = p^{d(B)}$. We will also denote by $\exp(D)$ the exponent of D.

Throughout, for an integer n, we will write n_p and $n_{p'}$ for its p- and p'-parts, respectively, so that $n = n_p n_{p'}$, n_p is a power of p, and $(p, n_{p'}) = 1$. Similarly, for an element g of a finite group G, we will write g_p and $g_{p'}$ for the (unique) elements such that $g = g_p g_{p'}$ with $|g_p| = |g|_p$ and $|g_{p'}| = |g|_{p'}$.

Lemma 3.1. *Let $\chi \in \mathrm{Irr}(G)$ and B the p-block of G containing χ. Let D be a defect group of B. Then, $\mathbf{lev}(\chi) \leq \nu(\exp(D))$. In particular, $\mathbf{lev}(\chi) \leq d(B)$.*

Proof. Let $g \in G$ with $\chi(g) \neq 0$. Then, g_p belongs to a conjugate of D by [24, Corollary 5.9]. Therefore, $|g|_p \leq \exp(D)$, and we have

$$\chi(g) \in \mathbb{Q}_{|g|} \subseteq \mathbb{Q}_{|g|_p|g|_{p'}} \subseteq \mathbb{Q}_{\exp(D)|G|_{p'}}.$$

We now have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{\exp(D)|G|_{p'}}$, which implies that $c(\chi)$ divides $\exp(D)|G|_{p'}$, and the lemma follows. ∎

Corollary 3.2. *Let $\chi \in \mathrm{Irr}(G)$ belong to a p-block of defect 0, or defect 1 if $p = 2$. Then, $\mathbf{lev}(\chi) = 0$.*

Proof. This follows from Lemma 3.1. Note that $\mathbb{Q}_n = \mathbb{Q}_{2n}$ for odd n, so there are no characters of 2-rationality level 1. ∎

Lemma 3.3. *Let p be an odd prime. Suppose $\chi \in \mathrm{Irr}(G)$ belongs to a p-block of defect one with a defect group D. Then, $\mathbf{lev}(\chi) \in \{0, 1\}$ and $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$.*

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Proof. The first conclusion again follows from Lemma 3.1, so it remains to prove the second part. We have $\chi(1)_p = |G|_p/p$ and $\mathbf{ht}(\chi) = 0$. p-Blocks of defect one have been fully described in early work of Brauer [1], [2] (see also [24, Chapter 11]). Let $K := \mathbf{O}_{p'}(\mathbf{N}_G(D))$. Let $b \in \mathrm{Bl}(\mathbf{C}_G(D))$ be a root of B and $\xi \in \mathrm{Irr}(K)$ be the restriction (to K) of the canonical character in $\mathrm{Irr}(b)$ of B. In fact, ξ is the unique Brauer character in b. Let E be the subgroup of $\mathbf{N}_G(D)$ fixing b and $\overline{E} := E/\mathbf{C}_G(D)$ the inertial quotient of B and $e := |\overline{E}|$. The block B then contains precisely $e + (p-1)/e$ ordinary irreducible characters, where e of these are p-rational and, therefore, trivially satisfy the stated equality.

Suppose that χ is one of the remaining $(p-1)/e$ other characters, a so-called exceptional character. In this case, there exists $\lambda \in \mathrm{Irr}(D) - \{1_D\}$ and $\epsilon \in \{\pm 1\}$ such that

$$\chi(hk) = \epsilon(\lambda \times \xi)^{\mathbf{N}_G(D)}(hk)$$

for every $h \in D - \{1\}$ and $k \in K$, by [24, Chapter 11]. Note that $\chi(g) = 0$ whenever g_p is not conjugate to an element in D. Also, for each $h \in D - \{1\}$, a p'-element in $\mathbf{C}_G(h)$ must be inside K. Therefore, these elements hk capture all the non-zero values of χ. It follows that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_{D\times K})$, and thus $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_{\mathbf{N}_G(D)})$, as desired. ∎

3.2. Generalities on blocks with cyclic defect groups

To prove Theorem C for blocks of larger cyclic defect, we need to recall some basics on cyclic-defect theory, and refer the reader to [6] and [8, Chapter VII] for more details. The theory, developed by E. Dade in the sixties, generalizes Brauer's work on defect-one blocks mentioned above.

Let B be a block of a finite group G with cyclic defect group D of order p^a $(a \geq 1)$. Let $B_0 \in \mathrm{Bl}(\mathbf{N}_G(D))$ be the Brauer correspondent of B and $b_0 \in \mathrm{Bl}(\mathbf{C}_G(D))$ be a root of B; that is, $b_0^{\mathbf{N}_G(D)} = B_0$. Let $C := \mathbf{C}_G(D)$ and let E be the subgroup of $\mathbf{N}_G(D)$ fixing b_0. The inertial quotient E/C is then a cyclic group of p'-order acting Frobeniusly on D (as well as $\mathrm{Irr}(D)$). Let Λ be a complete set of representatives of the action of E on $\mathrm{Irr}(D) - \{1_D\}$. Then, $|\Lambda| = (p^a - 1)/e$, where $e := |E/C|$.

If $|\Lambda| = 1$, then D must have order p and B has precisely $e+1$ irreducible ordinary characters. Suppose that $|\Lambda| > 1$. Then, $\mathrm{Irr}(B)$ is partitioned into two naturally defined subsets $\mathrm{Irr}_{nex}(B)$ and $\mathrm{Irr}_{ex}(B)$. The former consists of precisely *e non-exceptional characters* $\{X_1, \ldots, X_e\}$. The latter consists of precisely $|\Lambda|$ *exceptional characters*, which are naturally labeled by the members of Λ:

$$\mathrm{Irr}_{ex}(B) = \{X_\lambda \mid \lambda \in \Lambda\}.$$

As noted in [26, p. 1135], the group $\mathcal{H}_B$ permutes the exceptional/non-exceptional characters among themselves.

For $0 \leq i \leq a$, let D_i be the (unique) subgroup of D containing the elements of order at most p^{a-i}; that is,

$$D_i \text{ is the subgroup of } D \text{ with } |D : D_i| = p^i.$$

Let

$$C_i := \mathbf{C}_G(D_i) \text{ and } N_i := \mathbf{N}_G(D_i).$$

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Assume now that $0 \le i \le a-1$. By [6, Proposition 1.6], the block $b_i := (b_0)^{C_i} \in \mathrm{Bl}(C_i)$ contains a unique Brauer character, say φ_i. By [6, Corollary 1.9], for $x \in D_i - D_{i+1}$, and y a p-regular element of C_i, we have for each $\lambda \in \Lambda$,

$$X_\lambda(xy) = \frac{\delta}{|C_i|} \sum_{h \in N_i} \lambda^h(x)(\varphi_i)^h(y), \tag{3.1}$$

for some $\delta \in \{\pm 1\}$ depending only on i; and for each $1 \le j \le e$,

$$X_j(xy) = \frac{\pm 1}{e|C_i|} \sum_{h \in N_i} (\varphi_i)^h(y).$$

We remark that, if $\chi \in \mathrm{Irr}(G)$ and $g \in G$, then $\chi(g)$ is a sum of $|g|$-th roots of unity. The nonexceptional characters X_j are therefore always p-rational.

3.3. Proof of Theorem C

We are ready to prove Theorem C, which we now restate.

THEOREM 3.4. *Let p be a prime and G a finite group. Let $B \in \mathrm{Bl}(G)$ be a p-block of G with cyclic defect group D. Then,* $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$ *for every $\chi \in \mathrm{Irr}(B)$.*

Proof. The case of defect zero follows from Corollary 3.2. We may assume that $|D| > 1$. We shall follow the notation described above. If $|\Lambda| = 1$ then, as mentioned already, the block B must have defect one, and we are done by Corollary 3.2 and Lemma 3.3. So we assume from now on that $|\Lambda| > 1$. If $\chi \in \mathrm{Irr}_{nex}(B)$ is a non-exceptional character of B, then χ is p-rational, and thus there is nothing to prove. We, therefore, assume furthermore that χ is one of the exceptional characters X_λ for some $\lambda \in \Lambda$.

We claim that

$$\text{if } \mathbf{lev}(X_\lambda) \ge 1, \text{ then } \mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda).$$

Let $\widetilde{D}$ be the unique subgroup of D of order p and set $\widetilde{N} := \mathbf{N}_G(\widetilde{D})$. Let $\widetilde{B} = (b_0)^{\widetilde{N}} = (B_0)^{\widetilde{N}}$, which is a block of $\widetilde{N}$ that has the same defect group D and Brauer correspondent B_0 as B. Dade proved in [7, Lemma 4.9] that the exceptional characters of $\widetilde{B}$ can be labeled by $\mathrm{Irr}_{ex}(\widetilde{B}) = \{\widetilde{X}_\lambda \mid \lambda \in \Lambda\}$ so that the bijection $X_\lambda \mapsto \widetilde{X}_\lambda$ from $\mathrm{Irr}_{ex}(B)$ to $\mathrm{Irr}_{ex}(\widetilde{B})$ satisfies $(\widetilde{X}_{\lambda_1} - \widetilde{X}_{\lambda_2})^G = \delta(X_{\lambda_1} - X_{\lambda_2})$ for some fixed $\delta \in \{\pm 1\}$ and every $\lambda_1, \lambda_2 \in \Lambda$. As mentioned in the proof of [26, Theorem 3.4], Dade's bijection commutes with the action of $\mathcal{H}_B$, and hence preserves the p-rationality level. This allows us, for the purpose of proving the claim, to assume that $\widetilde{D} \lhd G$.

Let $\widetilde{C} := \mathbf{C}_G(\widetilde{D})$ and $\widetilde{b} := (b_0)^{\widetilde{C}}$. By [6, §4], the exceptional characters of B are induced from (nontrivial) ordinary irreducible characters of $\widetilde{b}$. In fact, $\mathrm{Irr}(\widetilde{b})$ consists of $|D|$ characters $\{\chi_\lambda \mid \lambda \in \mathrm{Irr}(D)\}$ and $(\chi_{\lambda_1})^G = (\chi_{\lambda_2})^G$ if and only if $\lambda_1 = \lambda_2^z$ for some $z \in E$, so that

$$\mathrm{Irr}_{ex}(B) = \{(\chi_\lambda)^G \mid \lambda \in \Lambda\} \text{ and } X_\lambda = (\chi_\lambda)^G.$$

It was shown in [26, p. 1136], using the character-valued formula of χ_λ in [6, Lemma 3.2], that a Galois automorphism $\tau \in \mathcal{H}_B$ moves the character χ_λ in $\mathrm{Irr}(\widetilde{b})$ to the character in $\mathrm{Irr}((\widetilde{b})^\tau)$ labeled by χ_{λ^τ}. Recall the groups $\mathcal{I} := \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{p'}}) \le \mathcal{H}_B$ and the p-subgroup

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

$\mathcal{I}' \leq \mathcal{I}$ from §2. Note that every relevant block is point-wisely fixed by $\mathcal{I}$. It follows that, for every $\tau \in \mathcal{I}$, $(\chi_\lambda)^\tau = \chi_{\lambda^\tau}$, which implies that

$$\mathbf{lev}(\chi_\lambda) = \mathbf{lev}(\lambda).$$

Further, recall that to show $\mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda)$, it suffices to show they are stable under the same elements of $\mathcal{I}'$, assuming that $\mathbf{lev}(X_\lambda) \geq 1$ if $p \neq 2$.

For each $\tau \in \mathcal{I}'$, we have

$$X_\lambda^\tau = \left((\chi_\lambda)^G\right)^\tau = \left((\chi_\lambda)^\tau\right)^G = (\chi_{\lambda^\tau})^G.$$

Therefore, $(\chi_\lambda)^G$ is τ-invariant if and only if $(\chi_\lambda)^G = (\chi_{\lambda^\tau})^G$, which is equivalent to $\lambda^\tau = \lambda^z$ for some $z \in E$. We may assume that z has p'-order, because E/C has p'-order and C fixes every irreducible character of D. Further, note that the actions of z and of τ commute. Let $t := |\tau|$ be the order of τ, which is a p-power. Then, $\lambda = \lambda^{\tau^t} = \lambda^{z^t}$. Thus, $z^t \in C$ as E/C acts Frobeniusly on $\mathrm{Irr}(D)$. But $|z|$ and t are coprime, so $z \in C$ and $\lambda^\tau = \lambda$. We indeed have shown that, for every $\lambda \in \Lambda$,

$$\text{if } p = 2, \text{ then } \mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda)$$

and

$$\text{if } p > 2 \text{ and } \mathbf{lev}(X_\lambda) \geq 1, \text{ then } \mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda).$$

The proof of the claim is completed.

Note that the desired equality $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = \mathbf{lev}(\chi)$ is obvious when χ is p-rational. By the above claim, it suffices to prove the equality for those characters X_λ with $\mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda) \geq 1$. For convenience, let $\ell := \mathbf{lev}(\lambda)$. Since $\mathbf{lev}(X_\lambda) = \max\{\mathbf{lev}(X_\lambda(g)) : g \in G\}$, there exists some $g \in G$ such that

$$\mathbf{lev}(X_\lambda(g)) = \mathbf{lev}(X_\lambda) = \mathbf{lev}(\lambda) = \ell.$$

Our job now is to show that such an element g can be chosen to be inside $\mathbf{N}_G(D)$. In fact, we will see that, up to conjugation, this must be the case.

Note that X_λ takes value 0 on every element whose p-part is not conjugate to an element of D. For our purpose of analyzing the value $X_\lambda(g)$, we may, therefore, assume that $g_p \in D$. We next claim that $g_p \in D_0 - D_1$, so that g_p generates D.

Assume, to the contrary, that $g_p \in D_1$; that is, $|g_p| \leq p^{a-1}$. Then, there exists $1 \leq i \leq a-1$ such that $|g_p| = p^{a-i}$ and $g_p \in D_i - D_{i+1}$. Now $(g_p)^h \in D_i - D_{i+1}$ for every $h \in N_i$. We have

$$\mathbf{lev}(\lambda^h(g_p)) = \mathbf{lev}(\lambda((g_p)^{h^{-1}})) = \begin{cases} \ell - i & \text{if } i \leq \ell, \\ 0 & \text{if } i > \ell \end{cases}$$

for every $h \in N_i$. In any case,

$$\mathbf{lev}(\lambda^h(g_p)) \leq \ell - 1.$$

By the character-valued formula (3.1),

$$X_\lambda(g) = \frac{\delta}{|C_i|} \sum_{h \in N_i} \lambda^h(g_p)(\varphi_i)^h(g_{p'})$$

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

for some $\delta \in \{\pm 1\}$. Note that each value $(\varphi_i)^h(g_{p'})$ is p-rational. It follows that $X_\lambda(g)$, being a sum of complex numbers of level at most $\ell - 1$, must have level at most $\ell - 1$, which is a contradiction.

We have shown that $g_p \in D_0 - D_1$. In other words, g_p is a generator for D. Thus,

$$g_{p'} \in \mathbf{C}_G(g_p) = \mathbf{C}_G(D) \subseteq \mathbf{N}_G(D),$$

and it follows that

$$g = g_p g_{p'} \in \mathbf{N}_G(D).$$

Let $\chi := X_\lambda$. We then have

$$\ell = \mathbf{lev}(\chi(g)) \leq \mathbf{lev}(\chi_{\mathbf{N}_G(D)}) \leq \mathbf{lev}(\chi) = \ell,$$

implying that $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = \mathbf{lev}(\chi)$. The proof is complete. ∎

REMARK 3.5. Our proof of Theorem C indeed shows that, in view of Remark 1.1, if a group element captures the p-rationality of the character, then that element *must* lie inside the defect normalizer, up to conjugation. At the moment, we do not know if this is true for arbitrary defect.

We conclude this section with the confirmation of Conjecture A for all characters in blocks of defect at most 2.

PROPOSITION 3.6. *Let p be a prime, G a finite group, and χ be a height-zero character in a block B of G with $\mathbf{lev}(\chi) \geq 2$ and $d(B) \leq 2$. Suppose that D is a defect group of B. Then, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$.*

Proof. If $|D| \leq p^2$ then either D is cyclic or $\exp(D) \leq p$. The former case is solved by Theorem 3.4. In the latter case, $\mathbf{lev}(\chi) \leq 1$ by Lemma 3.1 and the result is trivial. ∎

§4. Imprimitive characters of prime degree

This section proves Theorem D in the case, where the character in question is imprimitive. Recall that a character $\chi \in \mathrm{Irr}(G)$ is termed *imprimitive* if there exists a subgroup $H < G$ and $\psi \in \mathrm{Irr}(H)$ such that $\chi = \psi^G$.

We shall need a p-local invariant of characters that was introduced recently in Isaacs–Navarro's solution [20] of Conjecture B for p-solvable groups.

DEFINITION 4.1. For a character Ψ (not necessarily irreducible) of a finite group G and a nonnegative integer i, let

$$\Delta_i(\Psi) := \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \mathbf{lev}(\chi) = i}} [\chi, \Psi]\chi$$

and, if one of $\Delta_i(\Psi)(1)$ is not divisible by p,

$$\ell(\Psi) := \max\{i \in \mathbb{Z}_{\geq 0} : \Delta_i(\Psi)(1) \not\equiv 0 \bmod p\}.$$

LEMMA 4.2. *Let Ψ be a character of a finite group G with $\mathbf{lev}(\Psi) = a$. We have:*

(i) $\Delta_i(\Psi)(1) \equiv 0 \bmod p$ *for every* $i \geq \max\{2, a+1\}$.
(ii) *If* $a \geq 1$, *then* $\mathbf{lev}(\Psi) \geq \ell(\Psi)$.

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Proof. Note that Part (ii) follows from (i), so it is sufficient to prove (i).

Clearly, $|G|_p \geq p^a$. Note that if $p=2$ then $a=0$ or is at least 2. Let $\mathcal{J}$ denote the (cyclic) p-group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{p^a|G|_{p'}})$ if $a>0$ or simply the p-group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{p|G|_{p'}})$ if $a=0$. Then, Ψ is $\mathcal{J}$-invariant. Since $\mathbb{Q}(\psi^\tau)=\mathbb{Q}(\psi)$ for every character ψ of G and every $\tau\in\mathcal{J}$, each $\Delta_i(\Psi)$ is $\mathcal{J}$-invariant, and hence $\mathcal{J}$ permutes the irreducible constituents of $\Delta_i(\Psi)$. Let $i>a$ if $a>0$ or $i\geq 2$ if $a=0$. Then, each constituent of $\Delta_i(\Psi)$, of level i, is not $\mathcal{J}$-invariant, and thus belongs to a $\mathcal{J}$-orbit of nontrivial length, which is necessarily a nontrivial p-power. As the irreducible constituents of $\Delta_i(\Psi)$ is a disjoint union of these orbits, the statement follows. ∎

The next result makes use of some ideas in the proof of [20, Theorem 3.5].

LEMMA 4.3. *Let $P\leq K\leq G$, where $P\in\mathrm{Syl}_p(G)$, $\chi\in\mathrm{Irr}_{p'}(G)$, and $\psi\in\mathrm{Irr}(K)$ such that $\chi=\psi^G$. Let $i\in\mathbb{Z}_{\geq 2}$. Then,*

$$\Delta_i(\chi_P)(1)\not\equiv 0 \bmod p \text{ if and only if } \Delta_i(\psi_P)(1)\not\equiv 0 \bmod p.$$

In particular, if $\max\{\ell(\chi_P),\ell(\psi_P)\}\geq 2$ then $\ell(\chi_P)=\ell(\psi_P)$.

Proof. Let X be a set of representatives for the double $K-P$ cosets in G, so that

$$G=\bigcup_{x\in X} KxP$$

is a disjoint union. We decompose

$$X=X_1\cup X_2,$$

where X_1 consists of those $x\in X$ such that $P\subseteq K^x$ and X_2 is, of course, the complement of X_1 in X. Using Mackey's theorem (see [18, Problem 5.6]), we have

$$\begin{aligned}\chi_P=\sum_{x\in X}\left((\psi^x)_{K^x\cap P}\right)^P &= \sum_{x\in X_1}\left((\psi^x)_{K^x\cap P}\right)^P+\sum_{x\in X_2}\left((\psi^x)_{K^x\cap P}\right)^P\\ &= \sum_{x\in X_1}(\psi^x)_P+\sum_{x\in X_2}\left((\psi^x)_{K^x\cap P}\right)^P.\end{aligned}$$

Therefore, for every $i\in\mathbb{Z}_{\geq 0}$,

$$\Delta_i(\chi_P)=\sum_{x\in X_1}\Delta_i((\psi^x)_P)+\sum_{x\in X_2}\Delta_i(((\psi^x)_{K^x\cap P})^P).$$

For each $x\in X_2$, note that $K^x\cap P$ is a proper subgroup of P, and it follows from [20, Lemma 3.1] that

$$\Delta_i\left(((\psi^x)_{K^x\cap P})^P\right)(1)\equiv 0 \bmod p,$$

for every $i\geq 2$. We now obtain

$$\Delta_i(\chi_P)(1)\equiv\sum_{x\in X_1}\Delta_i((\psi^x)_P)(1) \bmod p.$$

Let $x\in X_1$. Note that α is an irreducible constituent of ψ_P if and only in α^x is an irreducible constituent of $(\psi^x)_{P^x}$, and $\mathbf{lev}(\alpha)=\mathbf{lev}(\alpha^x)$. Thus, there is a natural bijection

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

between irreducible constituents of ψ_P and $(\psi^x)_P$ preserving the p-rationality level. In particular,

$$\Delta_i(\psi_P)(1) = \Delta_i((\psi^x)_P)$$

for every $x \in X_1$. The last congruence in the previous paragraph then yields

$$\Delta_i(\chi_P)(1) \equiv |X_1| \cdot \Delta_i(\psi_P)(1) \bmod p.$$

Now, by [20, Lemma 3.4], which states that $|X_1|$ is not divisible by p, the lemma follows. ∎

We can now prove Theorem D in the imprimitivity case.

THEOREM 4.4. *Let p be a prime and G be a finite group. Let $\chi = \lambda^G \in \mathrm{Irr}(G)$ for some linear character λ of a subgroup K of G of prime index not equal to p. Suppose $\mathbf{lev}(\chi) \geq 1$. Then, $\mathbb{Q}_{p^{\mathbf{lev}(\chi)}} = \mathbb{Q}_p(\chi_P)$.*

Proof. Since the conclusion is obvious when $\mathbf{lev}(\chi) = 1$, we assume that $a := \mathbf{lev}(\chi) \geq 2$. By the character-induction formula, we have

$$\mathbf{lev}(\lambda) \geq \mathbf{lev}(\chi) = a.$$

On the other hand, since λ is linear, we have $\mathbf{lev}(\lambda_P) = \ell(\lambda_P)$ and

$$\mathbf{lev}(\lambda) = \ell(\lambda_P) = \nu(\mathrm{ord}(\lambda)).$$

Furthermore, using Lemmas 4.2 and 4.3, we obtain

$$\mathbf{lev}(\chi_P) \geq \ell(\chi_P)$$

and

$$\ell(\chi_P) = \ell(\lambda_P).$$

The displayed (in)equalities imply that $a = \mathbf{lev}(\chi) = \mathbf{lev}(\chi_P) = \ell(\chi_P)$.

Note that $\mathbb{Q}_p(\chi_P) \subseteq \mathbb{Q}_{p^a}$ (see [28, Lemma 7.1]). Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q}_p(\chi_P))$. Since $[\mathbb{Q}_{p^a} : \mathbb{Q}_p] = p^{a-1}$, we have that τ has p-power order. Also, τ fixes χ_P, and hence τ permutes the linear constituents of χ_P of level a. As $\ell(\chi_P) = a$, it follows that $\Delta_a(\chi_P)(1) \not\equiv 0 \bmod p$, which implies that the number of the linear constituents of χ_P of level a is not divisible by p. Therefore, one of them must be τ-invariant, and, therefore, τ fixes $\mathbb{Q}_{p^a}$ or, in other words, τ is trivial. We have shown that $\mathbb{Q}_{p^a} = \mathbb{Q}_p(\chi_P)$, as desired. ∎

§5. Quasisimple groups

5.1. Theorem D for quasisimple groups

The main result of this section is the following, which proves Theorem D for quasisimple groups, when combined with Lemma 6.1 below and [28, Theorem A3]. This result will be used in §6 to prove Theorem D for primitive characters.

THEOREM 5.1. *Let M be a quasisimple group and $\chi \in \mathrm{Irr}(M)$ be of prime degree with $\mathbf{lev}(\chi) \geq 2$. Let p be a prime not equal to $\chi(1)$. Then,*

$$\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P),$$

where $P \in \mathrm{Syl}_p(M)$.

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Our next result reduces us to the case that G is a group of Lie type defined in characteristic distinct from p.

THEOREM 5.2. *Let p be a prime and let G be a quasisimple group such that $S = G/\mathbf{Z}(G)$ is an alternating group, a sporadic simple group, a simple group of Lie type with exceptional Schur multiplier, or a simple group of Lie type defined in characteristic p. Let $\chi \in \mathrm{Irr}(G)$ have height zero, lie in a block with nontrivial defect group D, and be such that $\mathbf{lev}(\chi) \geq 2$. Then, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_D)$. In particular, Conjecture A and Theorem 5.1 hold in these cases.*

Proof. When S is either a sporadic simple group or A_n with $5 \leq n \leq 7$, we have $\mathbf{lev}(\chi) \leq 1$ for all primes p and all $\chi \in \mathrm{Irr}(G)$ of height zero, which can be readily checked in [11]. If S is a group of Lie type with exceptional Schur multiplier or the Tits group ${}^2\mathrm{F}_4(2)'$, we see using [11] that $\mathbf{lev}(\chi) \leq 1$ for all height-zero characters of G except when $p = 2$ and $S = \mathrm{PSL}_3(4)$ with $4 \mid |\mathbf{Z}(G)|$; $S = \mathrm{B}_3(3)$ with $2 \mid |\mathbf{Z}(G)|$; $S = \mathrm{PSU}_4(3)$ with $4 \mid |\mathbf{Z}(G)|$; or $S = {}^2\mathrm{F}_4(2)'$. In the latter cases, $\mathbf{lev}(\chi) \leq 2$, and we in fact see using [11] that for every $\chi \in \mathrm{Irr}(G)$ and for every prime $p \mid |G|$, we have $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_D)$.

If $p = 2$ and S is A_n, then every irreducible character of S is p-rational (see [16, §3] for instance), and we are done. Now consider the case $p = 2$ and S is a simple group of Lie type defined in characteristic 2 with nonexceptional Schur multiplier. Note that by [13], the blocks with positive defect are in fact of maximal defect. That is, the nontrivial defect groups are Sylow 2-subgroups in this case. Then, we have $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_D)$ by [28, Theorem A3].

Finally, suppose that p is odd and G is a cover of an alternating group A_n with $n \geq 8$ or a quasisimple group of Lie type that is a quotient of $\mathbf{G}^F$ for some simple, simply connected algebraic group $\mathbf{G}$ over a field of characteristic p and a Steinberg endomorphism $F: \mathbf{G} \to \mathbf{G}$. It was shown in [28, Theorem 6.1] that, in this situation,

$$\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|_{p'}}(\sqrt{p})$$

for every $\chi \in \mathrm{Irr}(G)$. As $\mathbb{Q}_{|G|_{p'}}$ contains a primitive 4th root of unity and the conductor of $\sqrt{(-1)^{(p-1)/2}p}$ is p, it follows that $c(\chi)$ divides $p|G|_{p'}$. Then, $\mathbf{lev}(\chi) \leq 1$ and the conjecture trivially holds in this case. ∎

We are now ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Theorem 5.2, we may assume that $S = M/\mathbf{Z}(M)$ is a simple group of Lie type defined in characteristic $q_0 \neq p$, and that S has non-exceptional Schur multiplier. Further, we assume that p is odd, since the statement follows from [28, Theorem A3] if $p = 2$.

Now, [17, Theorem 4.2] gives a list of the possible (S, r) in this case, where $r = \chi(1)$ is the prime for which M has an irreducible character of degree r. By our assumptions, we are not in the cases listed in (i) or (vi) of [17, Theorem 4.2]. Then, S is one of:

$$\mathrm{PSL}_2(q), \mathrm{PSU}_n(q) (n \geq 3), \mathrm{PSL}_n(q) (n \geq 3), \text{ or } \mathrm{PSp}_{2n}(q),$$

with specifications on n, q, r, and χ in each case. Here, we have $S = G/\mathbf{Z}(G)$ and $M = G/Z$ for some $Z \leq \mathbf{Z}(G)$ and $G := \mathbf{G}^F$, where $\mathbf{G}$ is a simple, simply connected algebraic group and $F: \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism defining $\mathbf{G}$ over $\mathbb{F}_q$, where q is some power of q_0. We discuss each case separately.

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

(I) First, suppose we are in case (ii) of [17, Theorem 4.2], so $S = \mathrm{PSL}_2(q)$. Then, either $r = q = q_0$ and χ is the Steinberg character, which is rational; or q is odd and $\chi(1) = r = \frac{q-\epsilon}{2}$ for some $\epsilon \in \{\pm 1\}$; or q is a power of 2 and $r = q+\epsilon$ for some $\epsilon \in \{\pm 1\}$ is a Mersenne prime or Fermat prime. In the case q is odd, we have $c(\chi) \in \{q_0, 1\}$ as in [17, §5.2.2], so $\mathbf{lev}(\chi) = 0$.

So, assume q is a power of 2. Here, we have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{q-\epsilon}$, so we may assume $p \mid (q-\epsilon)$. In this case, χ is the restriction of a semisimple character $\widetilde{\chi}$ of $\widetilde{G} = \mathrm{GL}_2(q)$ indexed by a semisimple element with eigenvalues $\{\zeta^i, \zeta^{-i}\}$, where ζ is a primitive $(q-\epsilon)$ root of unity and $1 \le i < q-\epsilon$. Letting α be a generator for $\mathrm{Irr}(C_{q-\epsilon})$, we have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\alpha^i)$, so that $\mathbf{lev}(\chi) \le \mathbf{lev}(\alpha^i)$. Note that by [28, Lemma 4.1], $\mathbf{lev}(\chi)$ is the smallest positive integer e such that $\chi^{\sigma_e} = \chi$, since p is odd. (Recall from §2 that $\sigma_e \in \mathcal{I}'$ is the element mapping any p-power root of unity ω to ω^{1+p^e}.)

Now, the value of χ on a semisimple element g_j of $S = G = \mathrm{SL}_2(q)$ with eigenvalues $\{\zeta^j, \zeta^{-j}\}$ is $\zeta^{ij} + \zeta^{-ij}$. In particular, taking $m := (q-\epsilon)_{p'}$ and $h := g_m$, we have

$$\chi(h) = \zeta^{im} + \zeta^{-im},$$

which is stable under σ_e if and only if ζ^{im} is, since p is odd. (This is worked out, for example, as in [29, Lemma 2.1].) But this happens if and only if α^{im}, and hence α^i, is stable under σ_e. It follows that

$$\mathbf{lev}(\alpha^i) = \mathbf{lev}(\chi(h)) \le \mathbf{lev}(\chi).$$

This establishes that

$$\mathbf{lev}(\alpha^i) = \mathbf{lev}(\chi) = \mathbf{lev}(\chi|_P),$$

where P is a Sylow p-subgroup of S containing h.

(II) Next, suppose we are in case (iii) of [17, Theorem 4.2], so that $S = \mathrm{PSL}_n(q)$ with n an odd prime, $q = q_0^f \ge 3$ with q_0 a prime and f odd, and $r = (q^n-1)/(q-1)$ with $(n, q-1) = 1$. Note that this means $S = M = G$. Here, as in [17, §5.2.1], we have χ is one of the $q-2$ irreducible Weil characters of degree $r = (q^n-1)/(q-1)$. Note that χ extends to an irreducible Weil character $\widetilde{\chi}$ of $\widetilde{G} := \mathrm{GL}_n(q)$. Further, χ is determined by the irreducible constituent of $\widetilde{\chi}|_{\mathbf{Z}(\widetilde{G})}$.

Let α be a generator for $\mathrm{Irr}(\mathbf{Z}(\widetilde{G})) \cong C_{q-1}$. Then, for $i = 1, \ldots, q-2$, let $\widetilde{\chi}_i$ be an irreducible Weil character of $\widetilde{G}$ such that $\widetilde{\chi}_i|_{\mathbf{Z}(\widetilde{G})} = \widetilde{\chi}_i(1)\alpha^i$. Then, any such choice of $\widetilde{\chi}_i$ has the same restriction to G, and we let $\chi_i := \widetilde{\chi}_i|_G$ be that restriction. Let $\widetilde{\chi} := \widetilde{\chi}_i$ and $\chi := \chi_i$. As discussed in [17, §5.2.1], we have $\mathbb{Q}(\widetilde{\chi}_i) \subseteq \mathbb{Q}(\alpha^i)$, so that

$$\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\widetilde{\chi}) \subseteq \mathbb{Q}(\alpha^i) = \mathbb{Q}(\zeta^i) \subseteq \mathbb{Q}(\zeta),$$

where ζ is a primitive $(q-1)$-root of unity in $\overline{\mathbb{Q}}^{\times}$. From this, we may assume $p \mid (q-1)$, as otherwise $\mathbf{lev}(\chi) = 0$.

Now, following [12, p. 125], we see $\widetilde{\chi} = \mathrm{R}_L^{\widetilde{G}}(\lambda)$, where L is a Levi subgroup of the form $\mathrm{GL}_1(q) \times \mathrm{GL}_{n-1}(q)$ of $\widetilde{G}$, $\lambda \in \mathrm{Irr}(L)$ is the character of a module of the form

$$S(s,(1)) \otimes S(t,(n-1))$$

in the notation of [12, p. 125] with $s \ne t \in \mathbb{F}_q^{\times}$ and $s/t = \alpha^i$, and $\mathrm{R}_L^{\widetilde{G}}$ denotes Harish–Chandra induction. Here, by an abuse of notation, we also denote by α a generator of $\mathbb{F}_q^{\times} \cong C_{q-1}$.

Since multiplying by a linear character of $\widetilde{G}$ does not affect χ_i, we may further assume that $t=1$, so

$$\lambda = S(\alpha^i,(1))\otimes S(1,(n-1)).$$

(Note that for $\alpha^j \in \mathbb{F}_q^\times$, the module $S(\alpha^j,(k))$ affords the inflation of the linear character $\mathrm{GL}_k(q)/\mathrm{SL}_k(q)\cong \mathbb{F}_q^\times \to \overline{\mathbb{Q}}^\times$ defined by $\alpha\mapsto \zeta^j$.)

Now, let Q be a parabolic subgroup of $\widetilde{G}$ such that $L\le Q$ is the Levi complement in Q, and let $\hat{\lambda}$ be the inflation of λ to Q. Then,

$$\chi = \mathrm{Res}^{\widetilde{G}}_G \mathrm{Ind}^{\widetilde{G}}_Q(\hat{\lambda}) = \mathrm{Ind}^G_{Q\cap G}\mathrm{Res}^Q_{Q\cap G}(\hat{\lambda})$$

since $\widetilde{G}=GQ$. Note that $r=\chi(1)=[G:Q\cap G]$. Then, by the first half of the proof of Theorem 4.4, we have $\mathbf{lev}(\chi)=\mathbf{lev}(\chi_P)$, as desired.

(III) Now consider case (iv) of [17, Theorem 4.2]. Here, $S=\mathrm{PSU}_n(q)$ with n an odd prime, $r=(q^n+1)/(q+1)$, and $(n,q+1)=1$. Again, this means $S=M=G$ and χ is one of the q irreducible Weil characters of degree $r=(q^n+1)/(q+1)$ (see [17, §5.3]). Again, χ extends to an irreducible Weil character $\widetilde{\chi}$ of $\widetilde{G}:=\mathrm{GU}_n(q)$ and χ is determined by its values on $\mathbf{Z}(\widetilde{G})$. Letting α be a generator for $\mathrm{Irr}(\mathbf{Z}(\widetilde{G}))\cong C_{q+1}$, we have Weil characters $\chi_i=\widetilde{\chi}_i|_G$, where $\widetilde{\chi}_i|_{\mathbf{Z}(\widetilde{G})}=\widetilde{\chi}_i(1)\alpha^i$ as in (II), now for $i=1,\dots,q$. Here, we similarly have $\mathbb{Q}(\chi_i)\subseteq\mathbb{Q}(\alpha^i)\subseteq\mathbb{Q}(\xi)$, where ξ is a primitive $(q+1)$-root of unity in $\overline{\mathbb{Q}}^\times$. So, we assume $p\mid(q+1)$. Further, this establishes that

$$\mathbf{lev}(\chi_i)\le \mathbf{lev}(\alpha^i).$$

In this case, we use the explicit formula from [33, Theorem 4.1] for the values of χ_i. Namely, for $g\in\widetilde{G}$, we have

$$\chi_i(g)=\frac{(-1)^n}{q+1}\sum_{k=0}^{q}\xi^{-ik}(-q)^{\dim\mathrm{Ker}(g-\hat{\xi}^{-k})},$$

where $\hat{\xi}$ denotes a generator of the subgroup $C_{q+1}\le\mathbb{F}_{q^2}^\times$.

Now, let $m:=(q+1)_{p'}$ and let $\beta:=\hat{\xi}^m$ be a generator of the Sylow p-subgroup of $C_{q+1}\le\mathbb{F}_{q^2}^\times$. Let h be a semisimple p-element of G with eigenvalues $\{\beta,\beta^{-1},1,\dots,1\}$. Then, since n and p are odd, we have:

$$\begin{aligned}\chi_i(h)&=\frac{-1}{q+1}\left((\xi^{im}+\xi^{-im})(-q)+(-q)^{n-2}-\xi^{im}-\xi^{-im}+\sum_{k=1}^{q}\xi^{-ik}\right)\\&=(\xi^{im}+\xi^{-im})+\frac{1}{q+1}\left(q^{n-2}-\sum_{k=1}^{q}\xi^{-ik}\right)=(\xi^{im}+\xi^{-im})+\frac{q^{n-2}+1}{q+1}.\end{aligned}$$

Recall again that by [28, Lemma 4.1], $\mathbf{lev}(\chi)$ is the smallest positive integer e such that $\chi^{\sigma_e}=\chi$, since p is odd. From above, we see $\chi_i(h)$ is fixed by σ_e if and only if $(\xi^{im}+\xi^{-im})$ is fixed by σ_e. From here, we conclude similar to (I). Namely, $\chi_i(h)$ is then stable under σ_e if and only if ξ^{im} is stable under σ_e, if and only if the character α^{im} is stable under σ_e, so also α^i is. So we have $\mathbf{lev}(\chi_i(h))=\mathbf{lev}(\alpha^i)$. Then,

$$\mathbf{lev}(\chi_i)\ge\mathbf{lev}(\alpha^i),$$

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

forcing that

$$\mathbf{lev}(\alpha^i) = \mathbf{lev}(\chi_i) = \mathbf{lev}(\chi_i|_P),$$

where P is a Sylow p-subgroup of S containing h.

(IV) Finally, assume we are in case (v) of [17, Theorem 4.2]. Then, $S = \mathrm{PSp}_{2n}(q)$ and either

(a) $r = (q^n+1)/2$ with $n = 2^a \geq 2$ and $q = q_0^{2^k}$ with q_0 odd and $k \geq 0$; or
(b) $r = (3^n-1)/2$, where n is an odd prime and $q = 3$.

In case (a), we have as in [17, §5.4.1] that either q is a square and hence χ is rational-valued, or $k = 0$ so $q = q_0$ and $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\zeta_q)$, where ζ_q is a primitive qth root of unity in $\overline{\mathbb{Q}}^\times$. Then, χ is almost p-rational (hence has $\mathbf{lev}(\chi) \leq 1$) if $p = q_0$ and is p-rational (so $\mathbf{lev}(\chi) = 0$) if $p \neq q_0$. In case (b), we similarly have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\zeta_3)$ with ζ_3 a primitive 3rd root of unity. Again, $\mathbf{lev}(\chi) \leq 1$ if $p = 3$ and $\mathbf{lev}(\chi) = 0$ if $p \neq 3$. □

5.2. Further examples satisfying Conjecture A

We provide further evidence for Conjecture A among certain (almost)-quasisimple groups.

When $G = \mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_{q_0}$ for a prime q_0 and $F\colon \mathbf{G} \to \mathbf{G}$ is a Steinberg morphism, the set $\mathrm{Irr}(G)$ is partitioned into so-called rational Lusztig series $\mathcal{E}(G,s)$. Here, s ranges over semisimple elements, up to G^*-conjugacy, of $G^* = (\mathbf{G}^*)^{F^*}$, where $(\mathbf{G}^*, F^*)$ is dual to $(\mathbf{G}, F)$. Let $p \neq q_0$ be a nondefining prime and suppose that $s \in G^*$ is a semisimple element of order relatively prime to p. Then, we define $\mathcal{E}_p(G,s)$ to be the union of all $\mathcal{E}(G,st)$ where $t \in \mathbf{C}_{G^*}(s)$ is a p-element. A result of Digne and Michel yields that the set $\mathcal{E}_p(G,s)$ is a union of p-blocks. (See [4, Theorem 9.12].)

Our next examples concerning Conjecture A are the Suzuki and Ree groups. That is, these are the cases that F is not a Frobenius morphism.

Theorem 5.3. *Conjecture A holds for* ${}^2\mathrm{B}_2(q^2)$ *for* $q^2 = 2^{2n+1} > 2$ *and* ${}^2\mathrm{G}_2(q^2)$ *for* $q^2 = 3^{2n+1} > 3$. *Further, if Conjecture B holds for* ${}^2\mathrm{F}_4(q^2)$, *where* $q^2 = 2^{2n+1} > 2$, *then Conjecture A holds for* ${}^2\mathrm{F}_4(q^2)$ *for* $p \neq 3$.

Proof. By Theorems 5.2 and 3.4, we may assume that p is not the defining characteristic for G and that the Sylow p-subgroups of G are non-cyclic. Then, we are left to consider the case that $G = {}^2\mathrm{F}_4(q^2)$ with $q^2 = 2^{2n+1}$ and p is an odd prime dividing (q^2-1), (q^2+1), or (q^4+1).

If $p \nmid (q^2-1)$ [22, Bemerkung 1] yields that each $\mathcal{E}_p(G,s)$ for $s \in G^*$ a semisimple p'-element contains a unique block of positive defect, which, therefore, has as defect groups a Sylow p-subgroup of $\mathbf{C}_{G^*}(s)$, using [21, Lemma 2.6]. If instead $p \mid (q^2-1)$, we have by [22, Bemerkung 1] that each $\mathcal{E}_p(G,s)$ has one or three blocks of positive defect, but only one of these is noncyclic. In either case, a block B with non-cyclic defect groups has maximal defect for $p \neq 3$, completing the proof by our assumption that Conjecture B holds. □

Our next several examples will come from linear groups, especially in the case $p = 2$. Let $\widetilde{G} := \mathrm{GL}_n(q)$, $G = \mathrm{SL}_n(q)$, and $S = \mathrm{PSL}_n(q)$, and assume that q is odd and $p = 2$.

In this situation, consider a block $\widetilde{B}$ of $\widetilde{G}$ covering a block B of G. These can be chosen so that $\widetilde{B} = \mathcal{E}_2(\widetilde{G}, \widetilde{s})$ for some odd-order semisimple element $\widetilde{s}$ of $\widetilde{G}^* \cong \widetilde{G}$, by [4, Theorems 9.12 and 21.14]. Further, a Sylow 2-subgroup of $\mathbf{C}_{\widetilde{G}^*}(s)$ gives a defect group for $\widetilde{B}$ by [9, Corollary (5E)]. (See also [3], which shows the results of [9] also hold for $p = 2$.) Now, $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$

is a product $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s}) = \prod \mathrm{GL}_{m_i}(q^{d_i})$, where m_i and d_i correspond to the multiplicities and degrees of the eigenvalues of $\widetilde{s}$ and $n = \sum m_i d_i$.

THEOREM 5.4. *Let M be a quasisimple group with $M/\mathbf{Z}(M) = \mathrm{PSL}_2(q)$, where $q \geq 5$ is a prime power. Then, Conjecture A holds for M.*

Proof. Let q be a power of a prime q_0. In this case, the Sylow p-subgroups of M are cyclic unless $p \in \{2, q_0\}$. So, we may assume by Theorems 3.4 and 5.2 that $p = 2$ and by [28, Theorem A3] we need only consider blocks of M with non-maximal defect. We have $M \in \{G, S\}$, where $G = \mathrm{SL}_2(q)$ and $S = \mathrm{PSL}_2(q)$. Let $\widetilde{G} = \mathrm{GL}_2(q)$ and let $\bar{B}$ be a block of M with positive, non-maximal defect dominated by a block B of G. Let $\widetilde{B}$ be a block of $\widetilde{G}$ covering B. Then, as discussed above, there is some odd-order, semisimple element $\widetilde{s}$ of $\widetilde{G}^* \cong \widetilde{G}$ such that $\widetilde{B} = \mathcal{E}_2(\widetilde{G}, \widetilde{s})$ and a Sylow 2-subgroup of $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$ gives a defect group for $\widetilde{B}$. Then, with our assumption that $\bar{B}$ is not of maximal defect, we see that any such defect group has cyclic intersection with G, so that B (hence $\bar{B}$) has cyclic defect groups. Then, we again apply Theorem 3.4. ∎

PROPOSITION 5.5. *Let $\widetilde{G} = \mathrm{GL}_n(q)$ with q odd and let $\widetilde{B} = \mathcal{E}_2(\widetilde{G}, \widetilde{s})$ be a 2-block of $\widetilde{G}$ as discussed above. Further, suppose that $q \equiv -1 \pmod 4$ and that each d_i is odd, in the notation above. Then, $\widetilde{B}$ satisfies Conjecture A.*

Proof. Let D be a defect group for $\widetilde{B}$, such that $D = \prod D_i$ with $D_i \in \mathrm{Syl}_2(\mathrm{GL}_{m_i}(q^{d_i}))$. Note that in this case, $q^{d_i} \equiv -1 \pmod 4$ for each factor $\mathrm{GL}_{m_i}(q^{d_i})$ of $C := \mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$.

We claim that in this situation, each D_i/D_i' has exponent at most 2, and hence so does D/D'. We can see this from the description of Sylow 2-subgroups of $\mathrm{GL}_m(q^d)$ in [5]. Indeed, by the description in [5], such a group is a direct product of Sylow 2 subgroups of $\mathrm{GL}_{2^j}(q)$ for various powers 2^j of 2. Hence, it suffices to prove the claim for m a power of 2. A Sylow 2-subgroup P_{2^j} of $\mathrm{GL}_{2^j}(q^d)$ is an iterated wreath product $P_2 \wr C_2 \wr C_2 \cdots \wr C_2$, where $P_2 \in \mathrm{Syl}_2(\mathrm{GL}_2(q^d))$. Since the latter is semidihedral, we know $\exp(P_2/P_2') \leq 2$. For $j \geq 2$, we have $P_{2^j} = P_{2^{j-1}} \wr C_2$. Then, $P_{2^j}/P_{2^j}' \cong P_{2^{j-1}}^2 / \langle (P_{2^{j-1}}')^2, [P_{2^{j-1}}^2, C_2] \rangle \times C_2$, and we can see inductively that $\exp(P_{2^j}/P_{2^j}') \leq 2$. This proves our claim. (See also [19, Proposition 4.3]).

Hence, by [19, Theorem D], each odd-degree character in the principal block $B_0(C)$ of C is 2-rational. But then the same is true for the height-zero characters of $\widetilde{B}$ by the main Theorem of [32], since these correspond to the height-zero characters in $B_0(C)$ by Jordan decomposition using [9, Theorem (7A)]. (See also [3], which shows the results of [9] continue to hold for $p = 2$.) ∎

REMARK 5.6. We remark that the same proof shows that when $\widetilde{G} = \mathrm{GU}_n(q)$ with $q \equiv 1 \pmod 4$ and $\widetilde{B} = \mathcal{E}_2(\widetilde{G}, \widetilde{s})$ is a 2-block of $\widetilde{G}$ with $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s}) = \prod \mathrm{GL}_{m_i}^{\eta}(q^{d_i})$ with $q^{d_i} \equiv -\eta \pmod 4$ for each i, then Conjecture A holds for $\widetilde{B}$.

Further, using the description of Sylow 2-subgroups of $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, $O_{2n+1}(q)$, $\mathrm{SO}_{2n}^{\pm}(q)$, and $O_{2n}^{\pm}(q)$ in [5] and arguing similarly to before, we see that each of these groups also satisfy $\exp(P/P') \leq 2$ for a Sylow 2 subgroup P. So, taking G to be a classical-type group $\mathrm{CSp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, or $\mathrm{CSO}_{2n}^{\pm}(q)$ and a block B such that $B = \mathcal{E}_2(G, s)$ (again applying [4, Theorems 9.12 and 21.14]), we may obtain analogous examples using [10] in place of [9]. This could be further extended to classical types whose center is disconnected in the cases that each $\chi \in \mathrm{Irr}(B)$ lies in a series $\mathcal{E}(G, st)$, where $\mathbf{C}_{\mathbf{G}^*}(st)$ is connected, using [31] in place of [32].

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

§6. Primitive characters of prime degree

This section handles the primitivity case of Theorem D. We begin with an easy observation.

LEMMA 6.1. *Let p be a prime, G a finite group, $P \in \mathrm{Syl}_p(G)$, and $\chi \in \mathrm{Irr}_{p'}(G)$ with $\mathbf{lev}(\chi) \geq 2$. To prove Conjecture B, it suffices to show that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$ and, additionally, $\mathbb{Q}_4 \subseteq \mathbb{Q}(\chi_P)$ if $p = 2$.*

Proof. Let $a := \mathbf{lev}(\chi) = \mathbf{lev}(\chi_P) \geq 2$. We aim to show that $\mathbb{Q}_p(\chi_P) = \mathbb{Q}_{p^a}$. First, we have $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}_{p^a}$ (see [28, Lemma 7.1]). If p is odd then all the subfields of $\mathbb{Q}_{p^a}$ containing $\mathbb{Q}_p$ are of the form $\mathbb{Q}_{p^b}$ for $1 \leq b \leq a$ (see the proof of [28, Theorem 2.3]), and, therefore, the fact that $\mathbf{lev}(\chi_P) = a$ forces $\mathbb{Q}_p(\chi_P)$ to be the entire $\mathbb{Q}_{p^a}$.

Let $p = 2$ and assume that $\mathbb{Q}_4 \subseteq \mathbb{Q}(\chi_P)$. Now all the subfields of $\mathbb{Q}_{2^a}$ containing $\mathbb{Q}_4$ are again of the form $\mathbb{Q}_{2^b}$ for $1 \leq b \leq a$, and we still have $\mathbb{Q}_{2^a} = \mathbb{Q}(\chi_P)$. ∎

LEMMA 6.2. *Let $K \trianglelefteq G$ and $\overline{\chi} \in \mathrm{Irr}(G/K)$. Let χ be the inflation of $\overline{\chi}$ up to G. Let $P \in \mathrm{Syl}_p(G)$ and $\overline{P} := PK/K \in \mathrm{Syl}_p(G/K)$. Then,*

$$\mathbf{lev}(\chi) = \mathbf{lev}(\overline{\chi}) \text{ and } \mathbf{lev}(\chi_P) = \mathbf{lev}(\overline{\chi}_{\overline{P}}).$$

Proof. This follows from the fact that the sets of values of χ and $\overline{\chi}$, as well as those of χ_P and $\overline{\chi}_{\overline{P}}$, are the same. ∎

LEMMA 6.3. *Let $K_1, \ldots, K_n$ be finite abelian extensions of $\mathbb{Q}$. Let $K := K_1 \cdots K_n$ denote the smallest subfield of $\mathbb{C}$ containing all K_i. Then, K is also a finite abelian extension of $\mathbb{Q}$ and*

$$\mathbf{lev}(K) = \max\{\mathbf{lev}(K_i) : 1 \leq i \leq n\}.$$

Proof. It is easy to see that $K \subseteq \mathbb{Q}_{\mathrm{lcm}(c(K_1),\ldots,c(K_n))}$, so K is a finite abelian extension of $\mathbb{Q}$.

Suppose $a := \max\{\mathbf{lev}(K_i) : 1 \leq i \leq n\}$. Then, $K_i \subseteq \mathbb{Q}_{p^a}\mathbb{Q}_{c(K_i)_{p'}}$ and thus $K \subseteq \mathbb{Q}_{p^a}\mathbb{Q}_{\prod_i c(K_i)_{p'}}$, implying that $\mathbf{lev}(K) \leq a$. The lemma follows as it is clear that $\mathbf{lev}(K_i) \leq \mathbf{lev}(K)$ for every i. ∎

For the remainder of this section, a finite group G is called *almost quasisimple* if there exists a nonabelian simple group S such that $S \trianglelefteq G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$.

We shall need the following rather technical result, extracted from [17].

LEMMA 6.4. *Let G be an almost quasisimple irreducible primitive subgroup of $\mathrm{GL}(r,\mathbb{C})$, where r is a prime, and $\chi \in \mathrm{Irr}(G)$ the corresponding character. Let S be the socle of $G/\mathbf{Z}(G)$ and M the last term in the derived series of G. Let π be the set of prime divisors of $r|\mathbf{Z}(M)|$ and Z_π denotes the Hall π-subgroup of (the abelian group) $Z := \mathbf{Z}(G)$. Let $N := Z_\pi M$. Then, the following hold.*

- (i) *M is quasisimple with $M/\mathbf{Z}(M) = S$ and χ_M is irreducible.*
- (ii) *$\varphi := \chi_N \in \mathrm{Irr}(N)$ and φ has the canonical extension $\widehat{\varphi}$ to G, in the sense of [25, Corollary 6.2]. Furthermore, $\chi = \widehat{\varphi}\lambda$ for some linear character λ of G/N.*
- (iii) *$\mathbb{Q}(\chi) = \mathbb{Q}(\chi_M)\mathbb{Q}(\mu)\mathbb{Q}(\lambda)$, where λ is as in (ii) and μ is the irreducible (linear) character of Z_π lying under χ.*

Proof. This follows from [17, Proposition 3.5 and Corollary 4.4] and their proofs. ∎

THEOREM 6.5. *Let p be a prime and G a finite group. Let χ be an irreducible primitive character of G of prime degree not equal to p. Suppose $\mathbf{lev}(\chi) \geq 2$. Then, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$ for $P \in \mathrm{Syl}_p(G)$. Furthermore, if $p = 2$ then $\mathbb{Q}_4 \subseteq \mathbb{Q}(\chi_P)$.*

Proof. We may assume that χ is faithful, by modding out by its kernel and using Lemma 6.2 if necessary. Let

$$r := \chi(1).$$

Then, G is an irreducible primitive subgroup of $\mathrm{GL}(r,\mathbb{C})$. By [17, Lemma 4.1], we are in one of the following situations.

(A) G is almost quasisimple.
(B) G contains a normal r-subgroup $R = \mathbf{Z}(R)E$, where E is an irreducible extraspecial r-group of order r^3, and either $R = E$ or $\mathbf{Z}(R) \cong C_4$.

A. Consider the case, when G is almost quasisimple. As in Lemma 6.4, we use Z for the center of G, S for the socle of G/Z, and M the last term in the derived series of G. Also, π is the set of prime divisors of $r|\mathbf{Z}(M)|$ and Z_π denotes the Hall π-subgroup of Z. Set $N := Z_\pi M$.

By Lemmas 6.3 and 6.4, we have

$$\mathbf{lev}(\chi) = \max\{\mathbf{lev}(\chi_M), \mathbf{lev}(\mu), \mathbf{lev}(\lambda)\}$$

for some linear characters μ of Z_π and λ of G/N.

(i) Suppose first that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_M)$. Recall that M is quasisimple and $\chi_M \in \mathrm{Irr}(M)$, by Lemma 6.4(i). Using Theorem 5.1, we obtain

$$\mathbf{lev}(\chi_M) = \mathbf{lev}(\chi_Q),$$

for every $Q \in \mathrm{Syl}_p(M)$. Choosing such a Q that is contained in P, we have

$$\mathbf{lev}(\chi) \geq \mathbf{lev}(\chi_P) \geq \mathbf{lev}(\chi_Q) = \mathbf{lev}(\chi_M) = \mathbf{lev}(\chi),$$

and the first statement of the theorem follows. When $p = 2$, we know from [28, Theorem A3] that $\mathbb{Q}(\chi_Q) = \mathbb{Q}_{2^{\mathbf{lev}(\chi_M)}} \supseteq \mathbb{Q}_4$, and thus $\mathbb{Q}(\chi_P) \supseteq \mathbb{Q}_4$, as wanted.

(ii) Next, suppose that $\mathbf{lev}(\chi) = \mathbf{lev}(\mu)$. Let $Q \in \mathrm{Syl}_p(Z_\pi)$. Recall that $\mu \in \mathrm{Irr}(Z_\pi)$ lies under χ and Z_π is central in G. Hence, χ_{Z_π} is a rational multiple of μ, and we deduce that χ_Q is a rational multiple of μ_Q as well. Since $\mathbf{lev}(\chi) \geq 2$, we have $\mathbb{Q}(\chi_Q) = \mathbb{Q}(\mu_Q) \supseteq \mathbb{Q}_4$, so it remains to show that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$.

Observe that

$$\mathbf{lev}(\chi_Q) = \mathbf{lev}(\mu_Q).$$

Clearly, $\mathbf{lev}(\mu) = \mathbf{lev}(\mu_Q)$, as μ is linear. Altogether, we have

$$\mathbf{lev}(\chi) \geq \mathbf{lev}(\chi_Q) = \mathbf{lev}(\mu_Q) = \mathbf{lev}(\mu) = \mathbf{lev}(\chi),$$

implying that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_Q)$, and we are done again.

(iii) Finally, suppose that $a = \mathbf{lev}(\chi) = \mathbf{lev}(\lambda) > \max\{\mathbf{lev}(\chi_M), \mathbf{lev}(\mu)\}$. Notice that N is a central product of Z_π and M and $\varphi \in \mathrm{Irr}(N)$ lies over $\varphi_M \in \mathrm{Irr}(M)$ and $\mu \in \mathrm{Irr}(Z_\pi)$. Therefore,

$$\mathbb{Q}(\varphi) = \mathbb{Q}(\chi_M)\mathbb{Q}(\mu).$$

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

It follows from Lemma 6.3 that

$$\mathbf{lev}(\varphi) = \max\{\mathbf{lev}(\chi_M), \mathbf{lev}(\mu)\} < a.$$

As $\mathbb{Q}(\widehat{\varphi}) = \mathbb{Q}(\varphi)$ (see [25, Corollary 6.4]), we then have $\mathbf{lev}(\widehat{\varphi}) < a$.

Again, by Lemma 6.4, $\chi = \widehat{\varphi}\lambda$. So $\chi_P = \widehat{\varphi}_P\lambda_P$. Assume to the contrary that $\mathbf{lev}(\chi_P) \leq a-1$. Then, for every $g \in P$, we have $\widehat{\varphi}(g) \neq 0$ and

$$\mathbf{lev}(\lambda(g)) = \mathbf{lev}(\chi(g)\widehat{\varphi}(g)^{-1}) \leq \max\{\mathbf{lev}(\chi(g)), \mathbf{lev}(\widehat{\varphi})\} \leq a-1,$$

which implies that $\mathbf{lev}(\lambda) = \mathbf{lev}(\lambda_P) \leq a-1$, a contradiction. We have proved that $\mathbf{lev}(\chi_P) \geq a$, which forces $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$, as desired.

Now suppose that $p = 2$. If either $\mathbf{lev}(\chi_M) \geq 2$ or $\mathbf{lev}(\mu) \geq 2$ then it was already known from parts (i) and (ii) that $\mathbb{Q}_4 \subseteq \mathbb{Q}(\chi_P)$. So let us assume that both χ_M and μ are 2-rational. Both φ and $\widehat{\varphi}$ are then 2-rational as well, as $\mathbb{Q}(\widehat{\varphi}) = \mathbb{Q}(\varphi) = \mathbb{Q}(\chi_M)\mathbb{Q}(\mu)$. It follows that $\widehat{\varphi}_P$ is rational-valued. Now we see that $\mathbb{Q}(\chi_P) \supseteq \mathbb{Q}_4$, using $\chi_P = \widehat{\varphi}_P\lambda_P$ together with the facts that λ is linear and $\mathbf{lev}(\lambda) \geq 2$.

B. Next we consider the situation in which G contains a normal r-subgroup $R = \mathbf{Z}(R)E$, where E is an irreducible extraspecial r-group of order r^3, and either $R = E$ or $\mathbf{Z}(R) \cong C_4$. By the main result of [20], we may assume that G is non-solvable. Using [17, Theorem 6.1], we obtain

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi_Z) = \mathbb{Q}(\exp(2i\pi/a)),$$

where $Z := \mathbf{Z}(G)$ and $a \in \mathbb{Z}^+$ is a certain divisor of $\exp(Z)$ divisible by r.

Let θ be the (unique) irreducible constituent of χ_Z, so that $\chi_Z = \alpha\theta$ for some $\alpha \in \mathbb{Z}^+$. Let $Q \in \mathrm{Syl}_p(Z)$. As θ is linear, we then have

$$\mathbf{lev}(\chi) \geq \mathbf{lev}(\chi_Q) = \mathbf{lev}(\theta_Q) = \mathbf{lev}(\theta) = \mathbf{lev}(\chi_Z) = \mathbf{lev}(\chi),$$

which implies that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_P)$. Also, $\mathbb{Q}(\chi_P) \supseteq \mathbb{Q}(\chi_Q) = \mathbb{Q}(\mu_Q) \supseteq \mathbb{Q}_4$. The proof is complete. ∎

Theorem D now readily follows from Theorems 4.4 and 6.5, and Lemma 6.1.

§7. Further discussion

We have seen the connection between Conjecture A and Navarro–Tiep's Conjecture B. We now discuss another connection, this time with the well-known AMN conjecture. In particular, we shall explain how Conjecture A implies that the conjectural AMN bijection should respect the p-rationality level of the defect-normalizer restrictions.

7.1. Connection with the AMN conjecture

Keep the notation from §2, so that $|G| = n = p^b m$ with $(p,m) = 1$ and $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \cong \mathcal{I} \times \mathcal{K}$. Recall that $\mathcal{H} = \mathcal{I} \times \langle\sigma\rangle$, where $\sigma \in \mathcal{K}$ is such that its restriction to $\mathbb{Q}_m$ is the Frobenius automorphism $\zeta \mapsto \zeta^p$. The McKay–Navarro conjecture predicts that, for $P \in \mathrm{Syl}_p(G)$, there should exist an $\mathcal{H}$-equivariant bijection from $\mathrm{Irr}_{p'}(G)$ to $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$. Such a bijection necessarily preserves the p-rationality level of characters (see [14, §2], for instance).

As noted in §3, the Galois group $\mathcal{G}$ permutes the p-blocks of G. Let B be a p-block of G and $\mathcal{H}_B$ be the subgroup of $\mathcal{H}$ fixing B. Recall that $\mathcal{I} \leq \mathcal{H}_B$.

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Now let D be a defect group of B and $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ be the Brauer correspondent of B. The group $\mathcal{H}_B$ then permutes (and preserves the height) the ordinary characters in B and b. The AMN conjecture [26, Conjecture B] asserts that there exists a bijection

$$^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$$

that commutes with the action of $\mathcal{H}_B$. That is,

$$(\chi^\tau)^* = (\chi^*)^\tau$$

for every $\chi \in \mathrm{Irr}_0(B)$ and every $\tau \in \mathcal{H}_B$.

Let $\mathbb{F}$ be the fixed field in $\mathbb{Q}_n$ of $\mathcal{H}_B$. The AMN conjecture then implies that

$$\mathbb{F}(\chi) = \mathbb{F}(\chi^*)$$

for all $\chi \in \mathrm{Irr}_0(B)$. As $\mathcal{I} \leq \mathcal{H}_B$, we have $\mathbb{F} \subseteq \mathbb{Q}_m$, and thus $\mathbf{lev}(\mathbb{F}) = 0$. It follows that the conjectural bijection * preserves the p-rationality level:

$$\mathbf{lev}(\chi) = \mathbf{lev}(\chi^*).$$

The following is the AMN conjecture with the defect-normalizer restriction incorporated.

Conjecture 7.1. *Let p be a prime and G a finite group. Let $B \in \mathrm{Bl}(G)$ be a p-block of G with defect group D and $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ be its Brauer correspondent. Let $\mathcal{H}_B$ be the subgroup of $\mathcal{H}$ fixing B. Then, there exists an $\mathcal{H}_B$-equivariant bijection $^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ such that $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = \mathbf{lev}(\chi^*)$ for every $\chi \in \mathrm{Irr}_0(B)$ of p-rationality level at least 2.*

Theorem 7.2. *Conjecture 7.1 follows from the AMN conjecture and Conjecture A. Conversely, Conjecture A follows from Conjecture 7.1.*

Proof. We keep the above notation. First, the AMN conjecture implies that there exists an $\mathcal{H}_B$-equivariant bijection $^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ such that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi^*)$. Conjecture A then implies that $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = \mathbf{lev}(\chi^*)$ for all $\chi \in \mathrm{Irr}_0(B)$ of level at least 2.

For the converse statement, let χ be a height-zero character in a block B and assume that there exists a bijection $^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ that commutes with the action of $\mathcal{H}_B$ such that $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) = \mathbf{lev}(\chi^*)$. As mentioned, we then have $\mathbf{lev}(\chi) = \mathbf{lev}(\chi^*)$, and it follows that $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$, as wanted. □

Conjecture 7.1 might be compared with the following conjecture of Navarro and Tiep, which combines Conjecture B and the McKay–Navarro conjecture.

Conjecture 7.3 ([28], Condition D). *Let p be a prime, G a finite group, and $P \in \mathrm{Syl}_p(G)$. Then, there exists an $\mathcal{H}$-equivariant bijection $^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ such that $\mathbb{Q}_p(\chi_P) = \mathbb{Q}_p(\chi^*_P)$ for all $\chi \in \mathrm{Irr}_{p'}(G)$.*

7.2. Consequences of Conjecture A

A character χ is termed *almost p-rational* if its conductor is not divisible by p^2, or, equivalently, $\mathbf{lev}(\chi) \leq 1$ (see [15]).

Consequence 7.4. *Let χ be a height-zero character of a finite group G and D a defect group of the p-block of G containing χ. Assume Conjecture A holds. Then, χ is almost p-rational if and only if $\chi_{\mathbf{N}_G(D)}$ is almost p-rational. In particular, when $p = 2$, χ is 2-rational if and only if $\chi_{\mathbf{N}_G(D)}$ is 2-rational.*

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press

Proof. It is clear that if χ is almost p-rational then so is $\chi_{\mathbf{N}_G(D)}$. Conversely, if χ is not almost p-rational then $\mathbf{lev}(\chi) \geq 2$, and it follows from Conjecture A that $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) \geq 2$. □

CONSEQUENCE 7.5. *Let G be a finite group with abelian Sylow p-subgroups. Assume that Conjecture A holds. Then, $\mathbf{lev}(\chi) = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$ for every $\chi \in \mathrm{Irr}(G)$ of p-rationality level at least 2 and D a defect group of the p-block of G containing χ.*

Proof. This follows from the solution of the "if" implication of Brauer's height zero conjecture [21]. □

CONSEQUENCE 7.6. *Let χ be an irreducible 2-height zero character of a finite group G. Suppose that $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$ is a quadratic number field, where $d \not\equiv 1 \pmod 4$ is a square-free integer. Assume that Conjecture A holds. Then, $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_{\mathbf{N}_G(D)})$, where D a defect group of the p-block of G containing χ.*

Proof. Note that $c(\sqrt{d}) = 4|d|$ when $d \not\equiv 1 \pmod 4$ is a square-free integer. Therefore, by the hypothesis, $\mathbf{lev}(\chi) \geq 2$. By Conjecture A, we then have $\mathbf{lev}(\chi_{\mathbf{N}_G(D)}) \geq 2$. In particular, $\chi_{\mathbf{N}_G(D)}$ is not rational, implying that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_{\mathbf{N}_G(D)})$. □

7.3. Examples

We end with examples to justify some of our claims from §1.

First, for height-zero characters χ in general, $\mathbf{lev}(\chi)$ does not always equal $\mathbf{lev}(\chi_D)$, where D is a defect group of the p-block containing χ. For example, when $p = 2$, the group `SmallGroup`(24,4) has characters of degree 2 with $\mathbf{lev}(\chi) = 2$ and $\mathbf{lev}(\chi_D) = 0$; the group `SmallGroup`(48,5) has characters of degree 2 with $\mathbf{lev}(\chi) = 3$ and $\mathbf{lev}(\chi_D) = 2$; and several similar examples can be found among the `SmallGroup` library in [11]. There are also examples for odd primes: when $p = 3$, the group `SmallGroup`(108,19) has characters of degree 3 with $\mathbf{lev}(\chi) = 2$ and $\mathbf{lev}(\chi_D) = 1$.

Next, the assumption $\mathbf{lev}(\chi) \geq 2$ in Conjecture A is necessary. For example, the group $2.\mathsf{A}_{10}.2$ with $p = 5$ has characters with degree 432 with $\mathbf{lev}(\chi) = 1$ but $\mathbf{lev}(\chi_P) = 0 = \mathbf{lev}(\chi_{\mathbf{N}_G(P)})$. Further, $2.\mathsf{A}_{11}$ with $p = 3$ has characters of degree 1584 in blocks having non-maximal defect with $\mathbf{lev}(\chi) = 1$ and $\mathbf{lev}(\chi_D) = 0 = \mathbf{lev}(\chi_{\mathbf{N}_G(D)})$.

Acknowledgement. The authors are grateful to Thomas Breuer, Gabriel Navarro, and Benjamin Sambale for several helpful conversations on topics related to this work. Part of this work was completed while the first author was visiting the Department of Mathematics at the University of Denver. He thanks the department and faculty members Alvaro Arias, Mandi Schaeffer Fry, Michael Kinyon, and Petr Vojtechovsky for their hospitality.

The first author gratefully acknowledges the support of the AMS-Simons Research Enhancement Grant (AWD-000167 AMS) and the UA Faculty Research Grant (FRG 1747). The second author gratefully acknowledges support from the National Science Foundation, Award No. DMS-2100912, and her former institution, Metropolitan State University of Denver, which held the award and allowed her to serve as PI. The authors are grateful to the anonymous referee for several helpful comments.

References

[1] R. Brauer, *On groups whose order contains a prime number to the first power I*, Amer. J. Math. **64** (1942), 401–420.

[2] R. Brauer, *On groups whose order contains a prime number to the first power II*, Amer. J. Math. **64** (1942), 421–440.

[3] M. Broué, *Les ℓ-blocs des groupes $GL(n,q)$ et $U(n,q^2)$ et leurs structures locales*, Astérisque, **133–134** (1986), 159–188.

[4] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, Cambridge University Press, Cambridge, 2004.

[5] R. Carter and P. Fong, *The Sylow 2-subgroups of the finite classical groups*, J. Algebra **1** (1964), 139–151.

[6] E. C. Dade, *Blocks with cyclic defect groups*, Ann. Math. **84** (1966), 20–48.

[7] E. C. Dade, *Counting characters in blocks with cyclic defect groups I*, J. Algebra **186** (1996), 934–969.

[8] W. Feit, *The Representation Theory of Finite Groups*, North-Holland Publ. Co., New York, 1982.

[9] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Invent. Math. **69** (1982), 109–153.

[10] P. Fong and B. Srinivasan, *The blocks of finite classical groups*, J. Reine Angew. Math. **396** (1989), 122–191.

[11] The GAP Group, *GAP groups, algorithms, and programming, version 4.11.0*, 2020. http://www.gap-system.org

[12] R. Guralnick and P. H. Tiep, *Low-dimensional representations of special linear groups in cross characteristics*, Proc. London Math. Soc. (3) **78** (1999), 116–138.

[13] J. E. Humphreys, *Defect groups for finite groups of Lie type*, Math. Z. **119** (1971), 149–152.

[14] N. N. Hung, *The continuity of p-rationality and a lower bound for p'-degree irreducible characters of finite groups*, Trans. Amer. Math. Soc. **377** (2024), 323–344.

[15] N. N. Hung, G. Malle, and A. Maróti, *On almost p-rational characters of p'-degree*, Forum Math. **34** (2022), 1475–1496.

[16] N. N. Hung and P. H. Tiep, *Degrees and fields of values of irreducible characters*, Eur. J. Math. **9** (2023), no. 2, Article No. 37. 17pp.

[17] N. N. Hung, P. H. Tiep, and A. E. Zalesski, *The field of values of prime-degree characters and Feit's conjecture*, Math. Z. **309** (2025), Article No. 32.

[18] I. M. Isaacs, *Character Theory of Finite Groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[19] I. M. Isaacs, M. W. Liebeck, G. Navarro, and P. H. Tiep, *Fields of values of odd-degree irreducible characters*, Adv. Math. **354** (2019), 106757. 26 pp.

[20] I. M. Isaacs and G. Navarro, *Primes and conductors of characters*, J. Algebra **653** (2024), 42–53.

[21] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. Math. **178** (2013), 321–384.

[22] G. Malle, *Die unipotenten Charaktere von ${}^2F_4(q^2)$*, Commun. Algebra **18** (1990), no. 7, 2361–2381.

[23] G. Malle, G. Navarro, A. A. Schaeffer Fry, and P. H. Tiep, *Brauer's height zero conjecture*, Ann. of Math. **200** (2024), no. 2, 557–608.

[24] G. Navarro, *Characters and Blocks of Finite Groups*, London Mathematical Society Lecture Note Series, 50, Cambridge University Press, Cambridge, 1998.

[25] G. Navarro, *Character Theory and the McKay Conjecture*, Cambridge Studies in Advanced Mathematics, 175, Cambridge University Press, Cambridge, 2018.

[26] G. Navarro, *The McKay conjecture and Galois automorphisms*, Ann. Math. **160** (2004), 1129–1140.

[27] G. Navarro, L. Ruhstorfer, P. H. Tiep, and C. Vallejo, *The fields of values of the height zero characters*, Adv. Math. **443** (2024), Article No. 109589. 25 pp.

[28] G. Navarro and P. H. Tiep, *The fields of values of characters of degree not divisible by p*, Forum Math. Pi **9** (2021), 1–28.

[29] A. Peña, F. Pryor, and A. A. S. Fry, *Galois action and cyclic defect groups for $Sp_6(2^a)$*, Involve **17** (2024), no. 2, 273–292.

[30] L. Ruhstorfer, *The Alperin–McKay and Brauer's height zero conjecture for the prime 2*, Ann. Math. **201** (2025), no. 2, 379–457.

[31] A. A. Schaeffer Fry, J. Taylor, and C. R. Vinroot, *Galois automorphisms and a unique Jordan decomposition in the case of connected centralizer*, J. Algebra **664B** (2025), 123–149.

[32] B. Srinivasan and C. R. Vinroot, *Galois group action and Jordan decomposition of characters of finite reductive groups with connected center*, J. Algebra **558** (2020), 708–727.
[33] P. H. Tiep and A. E. Zalesski, *Some characterizations of the Weil representations of the symplectic and unitary groups*, J. Algebra **192** (1997), 130–165.

Nguyen Hung (Corresponding Author)
Department of Mathematics
The University of Akron
Akron, OH 44325
United States
hungnguyen@uakron.edu

Mandi Schaeffer Fry
Department of Mathematics
University of Denver
Denver, CO 80210
United States
mandi.schaefferfry@du.edu

https://doi.org/10.1017/nmj.2025.10077 Published online by Cambridge University Press