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ON LITTLEWOOD–PALEY FUNCTIONS ASSOCIATED WITH THE DUNKL OPERATOR

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Abstract

A Littlewood–Paley operator associated with the reflection part of the Dunkl operator is introduced and proved to be of type (p, p) for $1 < p < \infty$, based on boundedness of a generalised vector-valued singular integral. This fills a gap for $2 < p < \infty$ concerning the boundedness of a g-function in the Dunkl setting. The paper also supplies new proofs for $1 < p < \infty$ on the (p, p) boundedness of various g-functions associated with the Dunkl operator.

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1. Introduction

The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [3, 4]. The Littlewood–Paley g-functions in the Dunkl setting on the line were studied in an earlier paper [6], where their boundedness in norm was proved in several cases. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [2]). For the classical theory of the Littlewood–Paley g-functions, see [7–9, 11].

Assume that λ is a fixed nonnegative number. As in [3], for $1 < p < \infty$, we denote by $L^p_\lambda(\mathbb{R})$ the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|^p_{L^p_\lambda} := c_\lambda \int_{\mathbb{R}} |f(x)|^p |x|^{2\lambda}\, dx < \infty \quad \text{with } c_\lambda^{-1} = 2^{\lambda+1/2}\Gamma(\lambda + 1/2).$$

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The Dunkl operator on the line $\mathbb{R}$ involves a reflection and is defined by

$$(Df)(x) = f'(x) + \frac{\lambda}{x}(f(x) - f(-x)),$$

and the Dunkl transform of a function $f \in L^1_\lambda(\mathbb{R})$ is defined by

$$(\mathscr{F}_\lambda f)(\xi) := c_\lambda \int_{\mathbb{R}} f(x)E_\lambda(-ix\xi)|x|^{2\lambda}\,dx \quad \text{for } \xi \in \mathbb{R},$$

where E_λ is the Dunkl kernel

$$E_\lambda(z) = j_{\lambda-1/2}(iz) + \frac{z}{2\lambda+1} j_{\lambda+1/2}(iz) \quad \text{for } z \in \mathbb{C}$$

and $j_\alpha(z)$ is the normalised Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\alpha+1)}{n!\Gamma(n+\alpha+1)}\Big(\frac{z}{2}\Big)^{2n}.$$

The Dunkl transform $\mathscr{F}_\lambda$ was studied in [5] from the viewpoint of the signed hypergroup.

The operator $\Delta_\lambda = D_x^2 + \partial_y^2$ is called the λ-Laplacian. It can be written explicitly, for a given C^2 function u on the half-plane $\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y \in (0,\infty)\}$, as

$$(\Delta_\lambda u)(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\lambda}{x}\frac{\partial u}{\partial x} - \frac{\lambda}{x^2}(u(x,y) - u(-x,y)).$$

If $\Delta_\lambda u \equiv 0$, then u is said to be λ-harmonic on $\mathbb{R}^2_+$.

For $f \in L^p_\lambda(\mathbb{R})$, $1 \le p < \infty$, the associated Poisson integral, which is called the λ-Poisson integral in [3], is defined by

$$u_f(x,y) = (Pf)(x,y) = c_\lambda \int_{\mathbb{R}} f(t)P(x,-t,y)|t|^{2\lambda}\,dt \quad \text{for } (x,y) \in \mathbb{R}^2_+, \tag{1.1}$$

where $P(x,t,y)$ is the λ-Poisson kernel given by (cf. [3])

$$P(x,t,y) = \frac{\lambda}{\pi c_\lambda} \int_{-1}^{1} \frac{y(1+s)(1-s^2)^{\lambda-1}}{(y^2+x^2+t^2+2xts)^{\lambda+1}}\,ds. \tag{1.2}$$

The λ-Poisson integral $u_f(x,y)$ of $f \in L^p_\lambda(\mathbb{R})$ for $1 \le p < \infty$ is λ-harmonic on $\mathbb{R}^2_+$.

For $f \in L^p_\lambda(\mathbb{R})$, $1 \le p < \infty$, there are several possible g-functions of the Littlewood–Paley type, such as

$$g_1(f)(x) = \Big(\int_0^\infty \Big|\frac{\partial u_f}{\partial y}(x,y)\Big|^2 y\,dy\Big)^{1/2}, \quad g_x(f)(x) = \Big(\int_0^\infty \Big|\frac{\partial u_f}{\partial x}(x,y)\Big|^2 y\,dy\Big)^{1/2}$$

and

$$g_\nabla(f)(x) = \Big(\int_0^\infty |\nabla u_f(x,y)|^2 y\,dy\Big)^{1/2},$$

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where $\nabla = (\partial_x, \partial_y)$ is the gradient vector. In the Dunkl setting, the more apt substitutions for g_x and g_∇ are

$$g_D(f)(x) = \Big(\int_0^\infty |D_x u_f(x,y)|^2 y\,dy\Big)^{1/2} \quad \text{and} \quad g_{\nabla_\lambda}(f)(x) = \Big(\int_0^\infty |\nabla_\lambda u_f(x,y)|^2 y\,dy\Big)^{1/2},$$

where $\nabla_\lambda = (D_x, \partial_y)$ is the λ-gradient vector. Based upon a vector version of the multiplier theorem for the Dunkl transform, the boundedness of the operators g_1, g_D and g_{∇_λ} in $L^p_\lambda(\mathbb{R})$ was proved in [6] for $1 < p < \infty$, but that of the operator g_∇ in $L^p_\lambda(\mathbb{R})$ was only proved for $1 < p \le 2$. One of the contributions of the present paper is to fill in the gap for the operator g_∇, that is, to show that $\|g_\nabla(f)\|_{L^p_\lambda} \lesssim \|f\|_{L^p_\lambda}$ for $1 < p < \infty$.

For this purpose, we need to consider an operator associated to the reflection part of the Dunkl operator D, that is,

$$g_0(f)(x) = \Big(\int_0^\infty \Big|\frac{u(x,y) - u(-x,y)}{x}\Big|^2 y\,dy\Big)^{1/2}.$$

We shall also consider the operator

$$g_{\Delta_\lambda}(f)(x) = \Big(\int_0^\infty y\Delta_\lambda u^2(x,y)\,dy\Big)^{1/2}.$$

It is not difficult to see that there are close relationships between these g-functions $g_1(f)$, $g_x(f)$, $g_\nabla(f)$, $g_D(f)$, $g_{\nabla_\lambda(f)}$, $g_0(f)$ and $g_{\Delta_\lambda}(f)$; moreover $g_1(f)$, $g_D(f)$ and $g_{\nabla_\lambda}(f)$ are also closely related to the generalised Hilbert transform in the Dunkl setting. These relationships will be stated in Section 2. The boundedness of the operator g_0 in $L^p_\lambda(\mathbb{R})$ for $1 < p < \infty$ will be proved in Section 3, based on a lemma about vector-valued singular integrals. The boundedness of g_0 together with that of g_{∇_λ} implies the boundedness of g_∇ in $L^p_\lambda(\mathbb{R})$ for all $1 < p < \infty$. In Section 4, we give a new proof for all $1 < p < \infty$ of the (p, p) boundedness of various g-functions associated with the one-dimensional Dunkl operator, without using the vector version of the multiplier theorem for the Dunkl transform. Our proof seems to be more fundamental.

Throughout the paper, A denotes a positive number independent of variables and functions, which may be different on different occurrences. Also, $U \lesssim V$ means that $U \le cV$ for some positive constant c independent of variables and functions.

2. Several lemmas

Lemma 2.1. *Assume that $1 \le p < \infty$ and $f \in L^p_\lambda(\mathbb{R})$. Then:*

(i) $g_D(f) \le g_x(f) + \lambda g_0(f)$, $\quad g_x(f) \le g_D(f) + \lambda g_0(f)$;
(ii) $g_\nabla(f)^2 = g_1(f)^2 + g_x(f)^2$, $\quad g_{\nabla_\lambda}(f)^2 = g_1(f)^2 + g_D(f)^2$;
(iii) $g_{\nabla_\lambda}(f) \le g_\nabla(f) + \lambda g_0(f)$, $\quad g_\nabla(f) \le g_{\nabla_\lambda}(f) + \lambda g_0(f)$; *and*
(iv) $g_{\Delta_\lambda}(f)^2 = 2g_\nabla(f)^2 + \lambda g_0(f)^2$.

Proof. Parts (i) and (ii) follow from the definitions. Since

$$|\nabla_\lambda u(x,y)|^2 = |\nabla u(x,y)|^2 + 2\lambda \frac{\partial u(x,y)}{\partial x}\,\frac{u(x,y) - u(-x,y)}{x} + \lambda^2\Big(\frac{u(x,y) - u(-x,y)}{x}\Big)^2, \tag{2.1}$$

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integrating over $(0, \infty)$ with respect to $y\,dy$ yields the first inequality in part (iii) and the second one follows similarly. Finally, part (iv) is a consequence of the identity

$$\Delta_\lambda u^2(x, y) = 2|\nabla u(x, y)|^2 + \lambda\Big(\frac{u(x, y) - u(-x, y)}{x}\Big)^2. \tag{2.2}$$

□

Lemma 2.1 shows that $g_{\Delta_\lambda}(f)$ is essentially the largest of these g-functions. To state further relationships between them, we need the λ-Hilbert transform $\mathscr{H}_\lambda$, an analogue of the classical Hilbert transform. From [3], the λ-Hilbert transform $\mathscr{H}_\lambda f$ of a function f is defined as the limit of $v_f(x, y) = (Qf)(x, y)$ as $y \to 0+$, the conjugate λ-Poisson integral of f, which, together with the λ-Poisson integral $u_f(x, y) = (Pf)(x, y)$, satisfies the generalised Cauchy–Riemann equations

$$D_x u_f - \partial_y v_f = 0, \quad \partial_y u_f + D_x v_f = 0. \tag{2.3}$$

The conjugate λ-Poisson integral $v_f(x, y)$ is given explicitly by (cf. [3, (46) and (47)])

$$v_f(x, y) = (Qf)(x, y) = c_\lambda \int_{\mathbb{R}} f(t)Q(x, -t, y)|t|^{2\lambda}\,dt \quad \text{for } (x, y) \in \mathbb{R}^2_+,$$

where $Q(x, -t, y)$ is the conjugate λ-Poisson kernel

$$Q(x, -t, y) = \frac{\lambda\Gamma(\lambda + 1/2)}{2^{-\lambda-1/2}\pi} \int_{-1}^{1} \frac{(x - t)(1 + s)(1 - s^2)^{\lambda-1}}{(y^2 + x^2 + t^2 - 2xts)^{\lambda+1}}\,ds \quad \text{for } x, t \in \mathbb{R}, y \in (0, \infty).$$

The following proposition contains part of [3, Theorem 5.6 and Corollary 6.2].

Proposition 2.2. *For $f \in L^p_\lambda(\mathbb{R})$, $1 \le p < \infty$, the λ-Hilbert transform $\mathscr{H}_\lambda f$ exists almost everywhere, and the mapping $f \mapsto \mathscr{H}_\lambda f$ is (p, p) bounded for $1 < p < \infty$ and weakly-$(1, 1)$ bounded. Furthermore, if $1 < p < \infty$,*

$$(Qf)(x, y) = [P(\mathscr{H}_\lambda f)](x, y) \quad \text{for } f \in L^p_\lambda(\mathbb{R}). \tag{2.4}$$

For $f \in L^p_\lambda(\mathbb{R})$, $1 < p < \infty$, from (2.4), we have $v_f(x, y) = u_{\mathscr{H}_\lambda f}(x, y)$. By (2.3), $D_x u_{\mathscr{H}_\lambda f} - \partial_y v_{\mathscr{H}_\lambda f} = 0$ and $\partial_y u_{\mathscr{H}_\lambda f} + D_x v_{\mathscr{H}_\lambda f} = 0$, which, in conjunction with (2.3), implies that $D_x v_{\mathscr{H}_\lambda f} = -D_x u_f$ and $\partial_y v_{\mathscr{H}_\lambda f} = -\partial_y u_f$. In view of the continuity and integrability, the last two equations lead to $v_{\mathscr{H}_\lambda f}(x, y) = -u_f(x, y)$ and, again by (2.4), $u_{\mathscr{H}^2_\lambda f}(x, y) = -u_f(x, y)$. Therefore $\mathscr{H}^2_\lambda f = -f$, and then, for $1 < p < \infty$, by Proposition 2.2, there exists a constant $A_p > 0$ so that

$$A_p^{-1}\|f\|_{L^p_\lambda} \le \|\mathscr{H}_\lambda f\|_{L^p_\lambda} \le A_p\|f\|_{L^p_\lambda} \quad \text{for } f \in L^p_\lambda(\mathbb{R}).$$

Lemma 2.3. *Assume that $1 < p < \infty$ and $f \in L^p_\lambda(\mathbb{R})$. Then:*

(i) $g_D(f) = g_1(\mathscr{H}_\lambda f), g_1(f) = g_D(\mathscr{H}_\lambda f)$; *and*
(ii) $g_{\nabla_\lambda}(f)^2 = g_1(f)^2 + g_1(\mathscr{H}_\lambda f)^2 = g_D(f)^2 + g_D(\mathscr{H}_\lambda f)^2$.

Proof. From (2.4), $v_f(x, y) = u_{\mathscr{H}_\lambda f}(x, y)$. Consequently, from (2.3), $D_x u_f = \partial_y u_{\mathscr{H}_\lambda f}$ and $\partial_y u_f = -D_x u_{\mathscr{H}_\lambda f}$. The lemma follows. □

From Lemma 2.3, the operators g_{∇_λ}, g_D and g_1 have equivalent norm estimates.

Lemma 2.4. *If* $F(x,y) \in C^2(\mathbb{R}^2_+) \bigcap C(\overline{\mathbb{R}^2_+})$, $F(x,0) \in L^1_\lambda(\mathbb{R})$ *and*

$$F(x,y) = o((|x|+y)^{-2\lambda-1}), \quad |\nabla F(x,y)| = o((|x|+y)^{-2\lambda-2}) \quad \text{as } |x|+y \to \infty,$$

then

$$\iint_{\mathbb{R}^2_+} y\Delta_\lambda F(x,y)|x|^{2\lambda}\,dx\,dy = \int_{\mathbb{R}} F(x,0)|x|^{2\lambda}\,dx. \tag{2.5}$$

Proof. Let Ω be the half-disc $\{(x,y) : x^2+y^2 < N, y > 0\}$ for $N > 0$. If $u, v \in C^2(\overline{\Omega})$, then, from [3, (38)],

$$\iint_\Omega (v\Delta_\lambda u - u\Delta_\lambda v)|x|^{2\lambda}\,dx\,dy = \int_{\partial\Omega} |x|^{2\lambda}\Big(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\Big)\,ds,$$

where $\partial/\partial\mathbf{n}$ denotes the directional derivative of the outward normal. We take $u = F(x,y)$ and $v = y$, so that

$$\begin{aligned}\iint_\Omega y\Delta_\lambda F(x,y)|x|^{2\lambda}\,dx\,dy &= \int_{-N}^{N} F(x,0)|x|^{2\lambda}\,dx\\ &+ N^{2\lambda+1}\int_0^\pi \Big[N\frac{\partial F}{\partial r}(N\cos\theta, N\sin\theta) - F(N\cos\theta, N\sin\theta)\Big]|\cos\theta|^{2\lambda}\sin\theta\,d\theta.\end{aligned}$$

Letting $N \to \infty$ and using the assumptions, yields (2.5). □

Lemma 2.5. *If* $f \in \mathscr{D}(\mathbb{R})$, *the space of* C^∞ *functions on* $\mathbb{R}$ *with compact support, then*

$$u_f(x,y) = O((|x|+y)^{-2\lambda-1}), \quad |\nabla u_f(x,y)| = O((|x|+y)^{-2\lambda-2}) \quad \text{as } |x|+y \to \infty.$$

Proof. The two estimates are essentially contained in [6, Proposition 4]. Indeed, if we assume that supp $f \subset [-A, A]$ for some $A > 0$, then, for $|x| > 2A$, $|t| \le A$, from (1.2),

$$\begin{aligned}P(x,t,y) &= O((|x|+y)^{-2\lambda-1}),\\ \Big|\frac{\partial}{\partial x}P(x,t,y)\Big| + \Big|\frac{\partial}{\partial y}P(x,t,y)\Big| &= O((|x|+y)^{-2\lambda-2}).\end{aligned}$$

Thus the desired estimates follow. □

Lemma 2.6. *If* $\phi \in C^1(\mathbb{R})$ *and* $\phi(x) = o(|x|^{-2\lambda})$ *as* $x \to \infty$, *then* $\int_{\mathbb{R}} (D\phi)(x)|x|^{2\lambda}\,dx = 0$.

Proof. If $\lambda = 0$, the result is trivial. If $\lambda > 0$, we write $\phi = \phi_e + \phi_o$, where

$$\phi_e(x) = (\phi(x) + \phi(-x))/2, \quad \phi_o(x) = (\phi(x) - \phi(-x))/2.$$

It is obvious that $\int_{\mathbb{R}} (D\phi_e)(x)|x|^{2\lambda}\,dx = \int_{\mathbb{R}} \phi_e'(x)|x|^{2\lambda}\,dx = 0$. Moreover,

$$\int_{\mathbb{R}} (D\phi_o)(x)|x|^{2\lambda}\,dx = 2\int_0^\infty \Big(\phi_o'(x) + \frac{2\lambda}{x}\phi_o(x)\Big)x^{2\lambda}\,dx = 2\int_0^\infty (x^{2\lambda}\phi_o(x))'\,dx = 0.$$

The lemma is proved. □

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3. Boundedness of the operator g_0

THEOREM 3.1. *Assume that $\lambda > 0$. The operator g_0 is bounded on $L^p_\lambda(\mathbb{R})$ for $1 < p < \infty$.*

We need a lemma on boundedness of a variant of vector-valued singular integrals. The difference from the usual case is that the singularity of the kernel $K(x, t)$ occurs on both diagonals $x = \pm t$. We begin with the scalar-valued form, which follows from [1, Theorem 3.1].

LEMMA 3.2. *Suppose that T is a bounded linear operator on $L^2_\lambda(\mathbb{R})$ and there is a measurable function K on $\mathbb{R}^2$ so that, for $f \in L^2_\lambda(\mathbb{R})$ with compact support, $(Tf)(x)$ is given by the expression*

$$(Tf)(x) = c_\lambda \int_{\mathbb{R}} K(x, t) f(t)|t|^{2\lambda}\, dt \tag{3.1}$$

if both $x, -x \notin \operatorname{supp} f$ and both integrals $\int_{\mathbb{R}} K(x, t) f(t)|t|^{2\lambda}\, dt$ and $\int_{\mathbb{R}} K(t, x) f(t)|t|^{2\lambda}\, dt$ converge absolutely for almost all x in the range. If there are constants $c > 1$ and $A > 0$ so that the kernel K satisfies

$$\int_{||x|-|t||>c|t-t'|} (|K(x, t) - K(x, t')| + |K(t, x) - K(t', x)|)|x|^{2\lambda}\, dx \le A \tag{3.2}$$

for all $t, t' \in \mathbb{R}$ with $t \ne t'$, then T extends to a bounded operator from $L^p_\lambda(\mathbb{R})$ into itself for $1 < p < \infty$.

Although, from the perspective of domains of integration, the assumption in (3.2) is weaker then that in the usual case, the proof of the lemma proceeds by the same pattern as in [10, pages 20–22]. Indeed, by interpolation and duality, it suffices to show that the mapping $f \mapsto Tf$ is of weak-type $(1, 1)$. Taking the Calderón–Zygmund decomposition for $f \in L^1_\lambda(\mathbb{R})$ at a given height as $f = g + b$, where $b = \sum_k b_k$ with $\operatorname{supp} b_k \subset I_k$, the only modification of [10, pages 20–22] is, in evaluating $(Tb)(x)$, to integrate each $|(Tb_k)(x)|$ over $(cI_k)^c \cap (-cI_k)^c$ instead of $(cI_k)^c$; and then, invoking the cancellation of b_k, the condition (3.2) is applicable.

We shall apply the vector-valued analogue of the operator T described in Lemma 3.2, for which, under similar assumptions and with the symbols $|\cdot|$ denoting the norms in related Banach spaces, the proof goes through without difficulty. The details of such a transplantation for convolution-type operators are given in [9, pages 45–48].

LEMMA 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces. Suppose that T is a bounded linear operator from $L^2_\lambda(\mathbb{R}, \mathcal{H}_1)$ into $L^2_\lambda(\mathbb{R}, \mathcal{H}_2)$ and there is a measurable function K from $\mathbb{R}^2$ to $B(\mathcal{H}_1, \mathcal{H}_2)$ (the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$) so that, for $f \in L^2_\lambda(\mathbb{R}, \mathcal{H}_1)$ with compact support, $(Tf)(x)$ has the expression (3.1) if both $x, -x \notin \operatorname{supp} f$ and both the integrals $\int_{\mathbb{R}} K(x, t) f(t)|t|^{2\lambda}\, dt$ and $\int_{\mathbb{R}} K(t, x) f(t)|t|^{2\lambda}\, dt$ converge in the norm of $\mathcal{H}_2$ for almost all x in the range. If there are constants $c > 1$ and $A > 0$ so that the kernel K satisfies the condition (3.2) for*

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all $t, t' \in \mathbb{R}$ *with* $t \neq t'$, *then* T *extends to a bounded operator from* $L^p_\lambda(\mathbb{R}, \mathcal{H}_1)$ *into* $L^p_\lambda(\mathbb{R}, \mathcal{H}_2)$ *for* $1 < p < \infty$.

Proof of Theorem 3.1. We first show that g_0 is of type $(2, 2)$. Indeed, for $f \in \mathscr{D}(\mathbb{R})$, by Lemma 2.5,

$$u_f^2(x, y) = O((|x| + y)^{-4\lambda-2}),$$
$$|\nabla u_f^2(x, y)| = 2|u_f(x, y)|\,|\nabla u_f(x, y)| = O((|x| + y)^{-4\lambda-3}),$$

as $|x| + y \to \infty$; and then, by Lemma 2.4, $\|g_{\Delta_\lambda}(f)\|_{L^2_\lambda} = \|f\|_{L^2_\lambda}$. But from Lemma 2.1(iv), $g_0(f)(x) \le \lambda^{-1/2} g_{\Delta_\lambda}(f)(x)$, so that $\|g_0(f)\|_{L^2_\lambda} \le \lambda^{-1/2}\|f\|_{L^2_\lambda}$.

We now define $\mathcal{H}_1$ to be the space of complex numbers and $\mathcal{H}_2$ to be the L^2-space

$$\mathcal{H}_2 = \left\{\phi : |\phi|_{\mathcal{H}_2} := \left(\int_0^\infty |\phi(y)|^2 y\, dy\right)^{1/2} < \infty\right\}.$$

Thus $B(\mathcal{H}_1, \mathcal{H}_2)$ is isomorphic to $\mathcal{H}_2$. The associated operator T is given by

$$(Tf)(x) = \frac{u(x, y) - u(-x, y)}{x}.$$

We shall rewrite Tf in terms of a kernel function K. In fact, from (1.1),

$$(Tf)(x) = c_\lambda \int_{\mathbb{R}} f(t) K_y(x, t)|t|^{2\lambda}\, dt,$$

where

$$K_y(x, t) = \frac{1}{x}[P(x, -t, y) - P(-x, -t, y)].$$

It follows from (1.2) that

$$K_y(x, t) = \frac{1}{x}\frac{2\lambda}{\pi c_\lambda} \int_{-1}^1 \frac{ys(1 - s^2)^{\lambda-1}}{(y^2 + x^2 + t^2 - 2xts)^{\lambda+1}}\, ds,$$

and then integrating by parts gives

$$K_y(x, t) = \frac{2(\lambda + 1)}{\pi c_\lambda} \int_{-1}^1 \frac{yt(1 - s^2)^{\lambda}}{(y^2 + x^2 + t^2 - 2xts)^{\lambda+2}}\, ds. \tag{3.3}$$

In order to apply Lemma 3.3, we need to verify that $K_y(x, t)$ satisfies the required estimates. These are contained in the following lemma.

Lemma 3.4. *There is a constant* $A > 0$ *such that, for all* $t, t' \in \mathbb{R}$ *with* $t \neq t'$,

$$\int_{||x|-|t||>5|t-t'|} |K_y(x, t) - K_y(x, t')|_{\mathcal{H}_2} |x|^{2\lambda}\, dx \le A, \tag{3.4}$$

$$\int_{||x|-|t||>5|t-t'|} |K_y(t, x) - K_y(t', x)|_{\mathcal{H}_2} |x|^{2\lambda}\, dx \le A. \tag{3.5}$$

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PROOF. We put $\Lambda_y(x,t,s) = y^2 + x^2 + t^2 - 2xts$. We claim that, for $||x| - |t|| > 5|t - t'|$, $s \in (-1, 1)$,

$$\tfrac{1}{2}\Lambda_y(x,t,s) \le \Lambda_y(x,t',s) \le 2\Lambda_y(x,t,s). \tag{3.6}$$

Indeed, since

$$|\Lambda_y(x,t,s) - \Lambda_y(x,t',s)| \le |t - t'|(|t - t'| + 2|t - xs|),$$

$|t - t'| < 5^{-1}\Lambda_y(x,t,s)^{1/2}$ and $|t - xs| \le \Lambda_y(x,t,s)^{1/2}$, it follows that

$$|\Lambda_y(x,t,s) - \Lambda_y(x,t',s)| \le \tfrac{1}{2}\Lambda_y(x,t,s),$$

so that (3.6) is concluded. Direct calculation shows that

$$\left|\frac{\partial}{\partial t}[t\Lambda_y(x,t,s)^{-\lambda-2}]\right| \lesssim \frac{y + |x| + |t|}{\Lambda_y(x,t,s)^{\lambda+5/2}}, \tag{3.7}$$

and then, from (3.3), applying the mean value theorem in t and using (3.6) and (3.7),

$$|K_y(x,t) - K_y(x,t')| \lesssim \int_0^1 \frac{y|t - t'|(y + |x| + |t|)}{[y^2 + (|x| - |t|)^2 + 2|xt|(1 - s)]^{\lambda+5/2}}(1 - s)^\lambda\, ds.$$

Now making the substitution $\rho = 2|xt|(1 - s)/[y^2 + (|x| - |t|)^2]$,

$$|K_y(x,t) - K_y(x,t')| \lesssim \frac{y|t - t'|(y + |x| + |t|)}{|xt|^{\lambda+1}[y^2 + (|x| - |t|)^2]^{3/2}} \int_0^M \frac{\rho^\lambda}{(1 + \rho)^{\lambda+5/2}}\, d\rho,$$

where $M = 2|xt|/[y^2 + (|x| - |t|)^2]$. Since $\int_0^M \rho^\lambda(1 + \rho)^{-\lambda-5/2}\, d\rho \asymp [M/(M + 1)]^{\lambda+1}$, it follows that

$$|K_y(x,t) - K_y(x,t')| \lesssim \frac{(|x| + |t|)^{-2\lambda}}{[y^2 + (|x| - |t|)^2]^{3/2}}|t - t'|.$$

Thus

$$\begin{aligned} |K_y(x,t) - K_y(x,t')|_{\mathcal{H}_2} &\lesssim \frac{|t - t'|}{(|x| + |t|)^{2\lambda}}\left(\int_0^\infty \frac{y\, dy}{[y^2 + (|x| - |t|)^2]^3}\right)^{1/2} \\ &\lesssim \frac{|t - t'|}{(|x| + |t|)^{2\lambda}}(|x| - |t|)^{-2}, \end{aligned}$$

and so (3.4) is concluded.

We prove (3.5) similarly, since $(\partial/\partial x)[t\Lambda_y(x,t,s)^{-\lambda-2}]$ satisfies the same estimate as in (3.7). □

We return to the proof of Theorem 3.1. By Lemma 3.4, the operator T satisfies the conditions in Lemma 3.3 so that T extends to a bounded operator from $L^p_\lambda(\mathbb{R}, \mathcal{H}_1)$ into $L^p_\lambda(\mathbb{R}, \mathcal{H}_2)$ for $1 < p < \infty$, which is equivalent to the boundedness of g_0 in $L^p_\lambda(\mathbb{R})$. The proof of Theorem 3.1 is complete. □

4. Boundedness of the g-functions

THEOREM 4.1. *For $1 < p < \infty$, the g-functions g_1, g_x, g_D, g_∇, g_{∇_λ} and g_{Δ_λ} are all bounded operators on $L^p_\lambda(\mathbb{R})$.*

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By Lemma 2.1 and Theorem 3.1, the proof of the theorem is reduced to proving the following two claims.

(i) g_∇ is bounded on $L^p_\lambda(\mathbb{R})$ for $1 < p \le 2$.
(ii) g_{∇_λ} is bounded on $L^p_\lambda(\mathbb{R})$ for $2 \le p < \infty$.

4.1. Boundedness of g_∇ on $L^p_\lambda(\mathbb{R})$ for $1 < p \le 2$. Suppose, first, that $f \in \mathscr{D}(\mathbb{R})$ and is nonnegative. The positivity of the λ-Poisson integral $u_f(x, y)$ follows immediately from (1.1) and (1.2) . A direct calculation shows that

$$\Delta_\lambda u_f^p(x, y) = p(p-1)u_f^{p-2}(x, y)|\nabla u_f(x, y)|^2 + U(x, y),$$

where

$$U(x, y) = \frac{\lambda p}{x^2} u_f^{p-1}(x, y)(u_f(x, y) - u_f(-x, y)) - \frac{\lambda}{x^2}(u_f^p(x, y) - u_f^p(-x, y)), \tag{4.1}$$

or, equivalently,

$$|\nabla u_f(x, y)|^2 = \frac{1}{p(p-1)} u_f^{2-p}(x, y)[\Delta_\lambda u_f^p(x, y) - U(x, y)].$$

Observe that

$$\begin{aligned} g_\nabla(f)(x)^2 &= \frac{1}{p(p-1)} \int_0^\infty u_f^{2-p}(x, y)[\Delta_\lambda u_f^p(x, y) - U(x, y)]y\,dy \\ &\le \frac{1}{p(p-1)}(P^* f)^{2-p}(x)I(x), \end{aligned} \tag{4.2}$$

where $(P^* f)(x) = \sup_{y>0} |u_f(x, y)|$ is the λ-Poisson maximal function and

$$I(x) = \int_0^\infty [\Delta_\lambda u_f^p(x, y) - U(x, y)]y\,dy.$$

The integrand in $I(x)$ is nonnegative, and the integrability of $U(x, y)$ over $\mathbb{R}^2_+$ with respect to $y|x|^{2\lambda}\,dx\,dy$ follows from Lemma 2.5. Hence

$$\int_{\mathbb{R}} I(x)|x|^{2\lambda}\,dx = \iint_{\mathbb{R}^2_+} y\Delta_\lambda u_f^p(x, y)|x|^{2\lambda}\,dx\,dy - \int_0^\infty y \int_{\mathbb{R}} U(x, y)|x|^{2\lambda}\,dx\,dy.$$

Since the second term in (4.1) is odd in $x \in (-\infty, \infty)$,

$$\begin{aligned} \int_{\mathbb{R}} U(x, y)|x|^{2\lambda}\,dx &= \int_{\mathbb{R}} \frac{\lambda p}{x^2} u_f^{p-1}(x, y)(u_f(x, y) - u_f(-x, y))|x|^{2\lambda}\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \frac{\lambda p}{x^2}(u_f^{p-1}(x, y) - u_f^{p-1}(-x, y))(u_f(x, y) - u_f(-x, y))|x|^{2\lambda}\,dx \\ &\ge 0. \end{aligned}$$

Thus $\int_{\mathbb{R}} I(x)|x|^{2\lambda}\,dx \le \iint_{\mathbb{R}^2_+} y\Delta_\lambda u_f^p(x, y)|x|^{2\lambda}\,dx\,dy$ and, by Lemma 2.5, $F(x, y) := u_f^p(x, y)$ satisfies the conditions in Lemma 2.4, so that

$$\int_{\mathbb{R}} I(x)|x|^{2\lambda}\,dx \le \int_{\mathbb{R}} |f(x)|^p|x|^{2\lambda}\,dx. \tag{4.3}$$

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If $p = 2$, then, from (4.2) and (4.3), $\|g_\nabla(f)\|_{L^2_\lambda} \le 2^{-1/2}\|f\|_{L^2_\lambda}$. If $1 < p < 2$, applying Hölder's inequality to (4.2) gives

$$\|g_\nabla(f)\|_{L^p_\lambda} \le \frac{1}{p(p-1)}\|P^* f\|_{L^p_\lambda}^{(2-p)/2}\|I\|_{L^1_\lambda}^{1/2},$$

and then, by (4.3) and [3, Theorem 3.8], $\|g_\nabla(f)\|_{L^p_\lambda} \lesssim \|f\|_{L^p_\lambda}^{(2-p)/2}\|f\|_{L^p_\lambda}^{p/2} = \|f\|_{L^p_\lambda}$.

For general $f \in L^p_\lambda(\mathbb{R})$, $1 < p \le 2$, one may decompose f as a sum of its positive and negative parts, and use a density argument.

4.2. Boundedness of g_{∇_λ} on $L^p_\lambda(\mathbb{R})$ for $2 \le p < \infty$. Again we first consider the case for nonnegative $f \in \mathscr{D}(\mathbb{R})$. Assume that $p \ge 4$ and let q be the number so that $1/q + 2/p = 1$; thus $1 < q \le 2$. Then

$$\|g_{\nabla_\lambda}(f)\|^2_{L^p_\lambda} = \sup_\phi c_\lambda \int_{\mathbb{R}} g_{\nabla_\lambda}(f)(x)^2\phi(x)|x|^{2\lambda}\,dx \tag{4.4}$$

taken over all nonnegative $\phi \in \mathscr{D}(\mathbb{R})$ satisfying $\|\phi\|_{L^p_\lambda} \le 1$. If we define

$$J(f,\phi) = c_\lambda \int_{\mathbb{R}} g_{\nabla_\lambda}(f)(x)^2\phi(x)|x|^{2\lambda}\,dx,$$

then

$$J(f,\phi) = 4c_\lambda \int_0^\infty y \int_{\mathbb{R}} |\nabla_\lambda u_f(x,2y)|^2|x|^{2\lambda}\,dx\,dy. \tag{4.5}$$

Since $D_x u_f$ and $\partial_y u_f$ are λ-harmonic, it follows from (2.2) that $\Delta_\lambda(|\nabla_\lambda u_f(x,y)|^2) \ge 0$, and so, by [3, Theorem 4.7], $|\nabla_\lambda u_f(x,y)|^2$ is λ-subharmonic on $\mathbb{R}^2_+$. Furthermore, from Lemma 2.4, $\sup_{y>0} c_\lambda \int_{\mathbb{R}} |\nabla_\lambda u_f(x,y)|^2|x|^{2\lambda}\,dx < \infty$, and then, by [3, (45)],

$$|\nabla_\lambda u_f(x,2y)|^2 \le c_\lambda \int_{\mathbb{R}} |\nabla_\lambda u_f(t,y)|^2 P(x,-t,y)|t|^{2\lambda}\,dt \quad \text{for } (x,y) \in \mathbb{R}^2_+.$$

Thus, from (4.5),

$$\begin{aligned} J(f,\phi) &\le 4c_\lambda^2 \int_0^\infty y \int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla_\lambda u_f(t,y)|^2\phi(x)P(x,-t,y)|t|^{2\lambda}|x|^{2\lambda}\,dx\,dt\,dy \\ &= 4c_\lambda \int_0^\infty y \int_{\mathbb{R}} |\nabla_\lambda u_f(t,y)|^2 u_\phi(t,y)|t|^{2\lambda}\,dt\,dy, \end{aligned}$$

and, applying (2.1),

$$J(f,\phi)^{1/2} \le J_1(f,\phi)^{1/2} + \lambda J_2(f,\phi)^{1/2}, \tag{4.6}$$

where

$$J_1(f,\phi) = 4c_\lambda \int_0^\infty y \int_{\mathbb{R}} |\nabla u_f(x,y)|^2 u_\phi(x,y)|x|^{2\lambda}\,dx\,dy,$$

$$J_2(f,\phi) = 4c_\lambda \int_0^\infty y \int_{\mathbb{R}} \left|\frac{u(x,y)-u(-x,y)}{x}\right|^2 u_\phi(x,y)|x|^{2\lambda}\,dx\,dy.$$

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For $J_2(f,\phi)$, we apply Hölder's inequality and then Theorem 3.1 and [3, Theorem 3.8] to obtain

$$J_2(f,\phi) \le 4c_\lambda \int_{\mathbb{R}} g_0(f)(x)^2 (P^*\phi)(x)|x|^{2\lambda}\,dx \le 4\|g_0(f)\|^2_{L^p_\lambda}\|P^*\phi\|_{L^q_\lambda} \lesssim \|f\|^2_{L^p_\lambda}. \quad (4.7)$$

The evaluation of $J_1(f,\phi)$ is more difficult. First, for $h_1, h_2 \in C^2(\mathbb{R}^2_+)$, direct calculations show that

$$\Delta_\lambda(h_1h_2) = h_1\Delta_\lambda h_2 + h_2\Delta_\lambda h_1 + 2\langle \nabla h_1, \nabla h_2\rangle + \frac{\lambda}{x^2}(h_1(x,y) - h_1(-x,y))(h_2(x,y) - h_2(-x,y)).$$

Now we take $h_1 = u_f(x,y)^2$, $h_2 = u_\phi(x,y)$ so that $\Delta_\lambda h_2 = 0$, and then, by (2.2),

$$u_\phi(x,y)|\nabla u_f(x,y)|^2 = \tfrac{1}{2}\Delta_\lambda(u_f(x,y)^2 u_\phi(x,y)) - V(x,y), \quad (4.8)$$

where

$$V(x,y) = \langle \nabla u_f(x,y)^2, \nabla u_\phi(x,y)\rangle + \frac{\lambda}{2x^2}(u_f(x,y) - u_f(-x,y))^2 u_\phi(x,y) + \frac{\lambda}{2x^2}(u_f(x,y)^2 - u_f(-x,y)^2)(u_\phi(x,y) - u_\phi(-x,y)).$$

We note that

$$|V(x,y)| \le 2(P^*f)(x)|\nabla u_f(x,y)|\,|\nabla u_\phi(x,y)| + \frac{\lambda}{2}\Big(\frac{u(x,y) - u(-x,y)}{x}\Big)^2 (P^*\phi)(x) + \frac{\lambda}{2}[(P^*f)(x) + (P^*f)(-x)]\Big|\frac{u_f(x,y) - u_f(-x,y)}{x}\Big|\Big|\frac{u_\phi(x,y) - u_\phi(-x,y)}{x}\Big|,$$

and hence

$$\begin{aligned} &4c_\lambda \int_0^\infty y \int_{\mathbb{R}} |V(x,y)|\,|x|^{2\lambda}\,dx\,dy \\ &\quad\le 8c_\lambda \int_{\mathbb{R}} (P^*f)(x) g_\nabla(f)(x) g_\nabla(\phi)(x)|x|^{2\lambda}\,dx + 2\lambda c_\lambda \int_{\mathbb{R}} g_0(f)(x)^2 (P^*\phi)(x)|x|^{2\lambda}\,dx \\ &\qquad + 2\lambda c_\lambda \int_{\mathbb{R}} [(P^*f)(x) + (P^*f)(-x)] g_0(f)(x) g_0(\phi)(x)|x|^{2\lambda}\,dx. \end{aligned}$$

Applying Hölder's inequality to each term above and then Theorem 3.1 and [3, Theorem 3.8],

$$\begin{aligned} 4c_\lambda \int_0^\infty y \int_{\mathbb{R}} |V(x,y)|\,|x|^{2\lambda}\,dx\,dy &\le 8\|P^*f\|_{L^p_\lambda}\|g_\nabla(f)\|_{L^p_\lambda}\|g_\nabla(\phi)\|_{L^q_\lambda} \\ &\quad + 2\lambda\|g_0(f)\|^2_{L^p_\lambda}\|P^*\phi\|_{L^q_\lambda} + 4\lambda\|P^*f\|_{L^p_\lambda}\|g_0(f)\|_{L^p_\lambda}\|g_0(\phi)\|_{L^q_\lambda} \\ &\lesssim \|g_\nabla(f)\|_{L^p_\lambda}\|f\|_{L^p_\lambda} + \|f\|^2_{L^p_\lambda}. \end{aligned} \quad (4.9)$$

Incorporating (4.8) and (4.9) into the expression of $J_1(f,\phi)$,

$$J_1(f,\phi) \lesssim |J_3(f,\phi)| + [\|g_\nabla(f)\|_{L^p_\lambda}\|f\|_{L^p_\lambda} + \|f\|^2_{L^p_\lambda}], \quad (4.10)$$

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where

$$J_3(f,\phi) = 2c_\lambda \int_0^\infty y \int_{\mathbb{R}} \Delta_\lambda(u_f(x,y)^2 u_\phi(x,y))|x|^{2\lambda}\,dx\,dy.$$

To estimate $J_3(f,\phi)$, note that, for $y > 0$, $D_x(u_f(x,y)^2 u_\phi(x,y)) = o(|x|^{-2\lambda})$ as $x \to \infty$ by Lemma 2.5, and then, by Lemma 2.6, $\int_{\mathbb{R}} D_x^2(u_f(x,y)^2 u_\phi(x,y))|x|^{2\lambda}\,dx = 0$. Thus

$$J_3(f,\phi) = \lim_{\epsilon,M} 2c_\lambda \int_\epsilon^M y \int_{\mathbb{R}} \partial_y^2(u_f(x,y)^2 u_\phi(x,y))|x|^{2\lambda}\,dx\,dy,$$

where the limit is taken as $\epsilon \to 0+$, $M \to +\infty$. Changing the order of the integration and integrating by parts with respect to y,

$$J_3(f,\phi) = \lim_{\epsilon,M} 2c_\lambda \int_{\mathbb{R}} \Big[y\frac{\partial}{\partial y}(u_f(x,y)^2 u_\phi(x,y)) - u_f(x,y)^2 u_\phi(x,y)\Big]\Big|_\epsilon^M |x|^{2\lambda}\,dx. \quad (4.11)$$

From (1.2), $y|\partial_y P(x,-t,y)|$ is dominated by a multiple of $P(x,-t,y)$, so $y|\partial_y u_f(x,y)|$ does not exceed $(P^* f)(x)$ up to a multiple, and this is also true for ϕ. Hence the integrand in (4.11) is dominated by a multiple of $(P^* f)(x)^2 (P^*\phi)(x)$. Proceeding as in (4.7) or (4.9), $|J_3(f,\phi)| \lesssim \|f\|^2_{L^p_\lambda}$. Substituting this into (4.10) and then substituting the result and (4.7) into (4.6) yields

$$J(f,\phi) \lesssim \|g_\nabla(f)\|_{L^p_\lambda}\|f\|_{L^p_\lambda} + \|f\|^2_{L^p_\lambda}. \quad (4.12)$$

By Lemma 2.1 and Theorem 3.1,

$$\|g_\nabla(f)\|_{L^p_\lambda} \le \|g_{\nabla_\lambda}(f)\|_{L^p_\lambda} + \lambda\|g_0(f)\|_{L^p_\lambda} \lesssim \|g_{\nabla_\lambda}(f)\|_{L^p_\lambda} + \|f\|_{L^p_\lambda}.$$

Incorporating this into (4.12) and invoking (4.4),

$$\|g_{\nabla_\lambda}(f)\|^2_{L^p_\lambda} \lesssim [\|g_{\nabla_\lambda}(f)\|_{L^p_\lambda}\|f\|_{L^p_\lambda} + \|f\|^2_{L^p_\lambda}],$$

and thus $\|g_{\nabla_\lambda}(f)\|_{L^p_\lambda} \lesssim \|f\|_{L^p_\lambda}$.

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