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HANKEL MEASURES ON HARDY SPACE

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We characterise the complex measures μ on the open unit disk $\mathbf{D}$ such that $|\int_{\mathbf{D}} f^2\,d\mu| \leqslant C\|f\|_{H^2}^2$ for all f in the Hardy space H^2. The characterisation involves Carleson measures, the duality between H^1 and $BMOA$, and Hankel operators.

Let $\mathbf{D}$ be the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbf{C}$ and denote by dm the two-dimensional Lebesgue measure on $\mathbf{D}$. The boundary $\{z : |z| = 1\}$ of $\mathbf{D}$ will be written as $\partial\mathbf{D}$.

For $p \in [1,\infty)$, define H^p to be the Hardy space of all holomorphic functions f on $\mathbf{D}$ for which

$$\|f\|_{H^p} = \sup_{r\in(0,1)} \left[\frac{1}{2\pi}\int_{\partial\mathbf{D}} \left|f(r\zeta)\right|^p |d\zeta|\right]^{1/p} < \infty.$$

If $f \in H^p$, the radial limit $\lim_{r\to 1} f(r\zeta)$, written $f(\zeta)$, exists for almost every $\zeta \in \partial\mathbf{D}$; we may thus identify f with its boundary value.

The Fefferman–Stein duality theorem [4] tells us that $[H^1]^* \cong BMOA$ under the pairing:

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{\partial\mathbf{D}} f(\zeta)\overline{g(\zeta)}\,|d\zeta|,$$

where $BMOA$ is the space of all $f \in H^1$ such that

$$\|f\|_{BMOA} = \left|f(0)\right| + \sup_{w\in\mathbf{D}} \left\|f\circ\phi_w - f(w)\right\|_{H^1} < \infty,$$

and $\phi_w : \mathbf{D} \to \mathbf{D}$ is the Möbius transformation $z \mapsto (w-z)/(1-\overline{w}z)$, which interchanges w and 0 (see [1]).

When the author visited Lund University, Sweden, J. Peetre posed the following problem:

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PEETRE'S PROBLEM. *Let μ be a complex-valued measure on $\mathbf{D}$. What geometric properties must μ have in order that*

$$\left|\int_{\mathbf{D}} f^2\, d\mu\right| \leqslant C\|f\|_{H^2}^2 \quad \forall f \in H^2 \quad ? \tag{1}$$

Here and throughout the article, the letter C stands for a (variable) positive constant.

Before providing an answer to the problem, we would like to make two observations. First, if I is a subarc of $\partial\mathbf{D}$ with arclength $|I|$, then $S(I)$ denotes the *Carleson box*

$$\left\{r\zeta \in \mathbf{D} : 1 - \frac{|I|}{2\pi} \leqslant r < 1,\ \zeta \in I\right\}.$$

A complex measure μ on $\mathbf{D}$ satisfying

$$\sup_{I \subset \partial\mathbf{D}} \frac{|\mu|(S(I))}{|I|} < \infty$$

is called a *Carleson measure* (see [**7**]). It is known that μ is a Carleson measure (see [**5**, p.63], [**10**, p.170]) if and only if

$$\int_{\mathbf{D}} |f|^2\, d|\mu| \leqslant C\|f\|_{H^2}^2 \quad \forall f \in H^2.$$

A complex measure μ on $\mathbf{D}$ that satisfies condition (1) will be called a *Hankel measure*, because Peetre's problem originates from the study of Hankel matrices (see also [**6**, **9**]). It is clear that a Carleson measure must be a Hankel measure, but not conversely.

Second, any Hankel measure is Möbius invariant in the following sense. Let $\phi : \mathbf{D} \to \mathbf{D}$ be the Möbius mapping $z \mapsto (az + b)/(\bar{b}z + \bar{a})$, where $a, b \in \mathbf{C}$ and $|a|^2 - |b|^2 = 1$. Then if $f \in H^2$, so too is the function $\tilde{f} : z \mapsto [f \circ \phi(z)]/(\bar{b}z + \bar{a})$, and the mapping $f \mapsto \tilde{f}$ is an isometry on H^2. So if $d\mu(z)$ changes to $(d\mu \circ \phi(z))(\bar{b}z + \bar{a})^2$, then the best constant in Peetre's problem does not change. This feature corresponds to the conformal invariance of Carleson measures. See for example [**5**, p.239].

Before we state our solution to Peetre's problem, we need two more definitions. Given a measure μ on $\mathbf{D}$, we define the function P_μ and the Hankel operator K_μ (see [**8**]) by the formulas

$$P_\mu(z) = \int_{\mathbf{D}} \frac{1}{1 - z\bar{w}}\, d\mu(w);$$

$$(K_\mu f)(z) = \int_{\mathbf{D}} \frac{f(w)}{1 - wz}\, d\mu(w).$$

THEOREM. *Let μ be a complex measure on $\mathbf{D}$. Then the following are equivalent:*

(a) μ *is a Hankel measure.*

(b) $$\left|\int_{\mathbf{D}} f\,d\mu\right| \leqslant C\|f\|_{H^1} \quad \forall f \in H^1.$$

(c) $P_{\overline{\mu}}$ *is in* $BMOA$.

(d) $$\sup_{I\subset\partial\mathbf{D}} \frac{1}{|I|}\int_{S(I)} \left|\int_{\mathbf{D}} \frac{\overline{w}\,d\overline{\mu}(w)}{(1-\overline{w}z)^2}\right|^2 (1-|z|^2)\,dm(z) < \infty.$$

(e) $$\left|\int_{\mathbf{D}} f_1 f_2\,d\mu\right| \leqslant C\|f_1\|_{H^2}\|f_2\|_{H^2} \quad \forall f_1, f_2 \in H^2.$$

(f) K_μ *is a bounded operator on* H^2.

(g) *There exists* $F \in L^\infty(\partial\mathbf{D})$ *such that*

$$\int_{\mathbf{D}} f\,d\mu = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} f(\zeta)F(\zeta)\,d\zeta \quad \forall f \in H^1.$$

(h) $\sup_A \left|\int_{\mathbf{D}} A\,d\mu\right| < \infty$, *where the supremum ranges over all holomorphic atoms.*

PROOF: STEP 1. We show that (a), (b), (c) and (d) are equivalent.

Trivially, (b) implies (a). On the other hand, suppose (a) holds. To prove (b), take $f \in H^1$ with $f \not\equiv 0$. By [**3**, Theorem 2.8, p.24], there exist an inner factor g and an outer factor h such that $f = gh$ and $\|h\|_{H^1} \leqslant \|f\|_{H^1}$. Set $f_1 = (g-1)h/2$ and $f_2 = (g+1)h/2$. Then $f = f_1 + f_2$, and $\|f_k\|_{H^1} \leqslant \|f\|_{H^1}$, $k = 1, 2$. Since both f_1 and f_2 are not equal to 0 anywhere on $\mathbf{D}$, there are $g_1, g_2 \in H^2$ such that $f_1 = g_1^2$ and $f_2 = g_2^2$. Consequently,

$$\left|\int_{\mathbf{D}} f\,d\mu\right| \leqslant \left|\int_{\mathbf{D}} g_1^2\,d\mu\right| + \left|\int_{\mathbf{D}} g_2^2\,d\mu\right| \leqslant C\big(\|g_1\|_{H^2}^2 + \|g_2\|_{H^2}^2\big) \leqslant C\|f\|_{H^1},$$

that is, (b) holds.

By definition of $P_{\overline{\mu}}$,

$$\begin{aligned}\int_{\mathbf{D}} f(z)\,d\mu(z) &= \int_{\mathbf{D}} \left[\frac{1}{2\pi}\int_{\partial\mathbf{D}} \frac{f(\zeta)}{1-z\overline{\zeta}}|d\zeta|\right] d\mu(z)\\ &= \frac{1}{2\pi}\int_{\partial\mathbf{D}} f(\zeta)\left[\int_{\mathbf{D}} \frac{1}{1-z\overline{\zeta}}\,d\mu(z)\right]|d\zeta|\\ &= \frac{1}{2\pi}\int_{\partial\mathbf{D}} f(\zeta)\overline{P_{\overline{\mu}}(\zeta)}\,|d\zeta|\\ &= \langle f, P_{\overline{\mu}}\rangle.\end{aligned}$$

Thus, (c) holds if and only if (b) does, by the isomorphism between $[H^1]^*$ and $BMOA$.

An H^2-function f belongs to $BMOA$ if and only if $(1-|z|^2)|f'(z)|^2\,dm(z)$ is a Carleson measure (see [**10**, p.178]). Accordingly, (c) is equivalent to (d).

STEP 2. We verify that (b), (e) and (f) are equivalent.

An application of Schwarz's inequality to f_1f_2 (where $f_1, f_2 \in H^2$) shows that (b) implies (e). On the other hand, let (e) hold. By [**5**, p.87, Exercise 1], we see that every $f \in H^1$ can be factored as $f = f_1f_2$ with $f_1, f_2 \in H^2$ and $\|f_1\|_{H^2}^2 = \|f_2\|_{H^2}^2 = \|f\|_{H^1}$, so that (b) follows from the estimate

$$\left|\int_{\mathbf{D}} f\,d\mu\right| \leqslant C\|f_1\|_{H^2}\|f_2\|_{H^2} = C\|f\|_{H^1}.$$

When $f, g \in H^2$, the definition of K_μ implies that

$$\begin{aligned}\langle K_\mu f, g\rangle &= \frac{1}{2\pi}\int_{\partial\mathbf{D}} (K_\mu f)(\zeta)\overline{g(\zeta)}\,|d\zeta| \\ &= \frac{1}{2\pi}\int_{\partial\mathbf{D}}\left[\int_{\mathbf{D}} \frac{f(w)}{1-\zeta w}\,d\mu(w)\right]\overline{g(\zeta)}\,|d\zeta| \\ &= \int_{\mathbf{D}} f(w)\left[\frac{1}{2\pi}\int_{\partial\mathbf{D}} \frac{\overline{g(\zeta)}}{1-\zeta w}|d\zeta|\right]d\mu(w) \\ &= \int_{\mathbf{D}} f(w)\overline{g(\overline{w})}\,d\mu(w).\end{aligned}$$

This formula shows that (e) and (f) are equivalent.

STEP 3. We prove the equivalence of (b), (g) and (h).

It is obvious that (g) implies (b), thanks to the inequality

$$\left|\int_{\mathbf{D}} f\,d\mu\right| = \left|\frac{1}{2\pi i}\int_{\partial\mathbf{D}} f(\zeta)F(\zeta)\,d\zeta\right| \leqslant \|F\|_\infty\|f\|_{H^1} \quad \forall f \in H^1.$$

Conversely, if (b) is valid, then the linear functional T, defined by $T(f) = \int_{\mathbf{D}} f\,d\mu$, belongs to $[H^1]^*$. The Hahn–Banach theorem allows us to extend T to $L^1(\partial\mathbf{D})$, and hence produce $G \in L^\infty(\partial\mathbf{D})$ such that

$$T(f) = \frac{1}{2\pi}\int_{\partial\mathbf{D}} f(\zeta)\overline{G(\zeta)}\,|d\zeta| \quad \forall f \in L^1(\partial\mathbf{D}).$$

Take F such that $F(z) = \overline{zG(z)}$; then $F \in L^\infty(\partial\mathbf{D})$ and

$$\int_{\mathbf{D}} f\,d\mu = T(f) = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} f(\zeta)F(\zeta)\,d\zeta \quad \forall f \in H^1,$$

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and (g) holds.

In [**2**], atoms are defined to be functions $a : \partial\mathbf{D} \to \mathbf{C}$ which are either identically 1 or are supported on an open subarc I of $\partial\mathbf{D}$, and satisfy

$$\|a\|_\infty \leqslant \frac{1}{|I|} \quad \text{and} \quad \int_{\partial\mathbf{D}} a(\zeta)\,|d\zeta| = 0.$$

Holomorphic atoms A are the holomorphic projection of atoms a, that is,

$$A(z) = \frac{1}{2\pi}\int_{\partial\mathbf{D}} \frac{a(\zeta)}{1-\bar{\zeta}z}|\,d\zeta|.$$

By [**2**, Theorem V], every $f \in H^1$ can be represented as $\sum_{j=1}^{\infty}\lambda_j A_j$, where A_j are holomorphic atoms and $\lambda_j \in \mathbf{C}$ with $\sum_{j=1}^{\infty}|\lambda_n| \leqslant C\|f\|_{H^1}$. This decomposition, together with [**2**, Proposition VI], demonstrates that (b) implies (h) and vice versa. The proof of the theorem is complete. ∎

REMARK. Since $f_w(z) = (1-|w|^2)/(1-\overline{w}z)^2$ lies in H^1 and $\sup_{w\in\mathbf{D}}\|f_w\|_{H^1} < \infty$, a necessary condition for μ to be a Hankel measure is that

$$\sup_{w\in\mathbf{D}}\left|\int_{\mathbf{D}} \frac{1-|w|^2}{(1-\overline{w}z)^2}\,d\mu(z)\right| < \infty.$$

We do not know whether this condition is sufficient, too. However, it is worth mentioning (see [**5**, p.139]) that μ is a Carleson measure if and only if

$$\sup_{w\in\mathbf{D}}\int_{\mathbf{D}} \frac{1-|w|^2}{|1-\overline{w}z|^2}\,d|\mu|(z) < \infty.$$

As in [**6**] or [**8**], it is not hard to check that K_μ is a classical Hankel operator on H^2, and hence $K_\mu : H^2 \to H^2$ is a compact or S_p (the Schatten–von Neumann p-class, for $p \in (1,\infty)$) operator if and only if $P_{\overline{\mu}}$ belongs to $VMOA$, the space of all $f \in H^1$ with

$$\lim_{w\to\partial\mathbf{D}}\|f\circ\phi_w - f(w)\|_{H^1} = 0,$$

or to B_p, the p-Besov space of all holomorphic functions f on $\mathbf{D}$ such that

$$\int_{\mathbf{D}}|f'(z)|^p(1-|z|^2)^{p-2}\,dm(z) < \infty,$$

respectively. ∎

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