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LIE POWERS OF FREE MODULES FOR CERTAIN GROUPS OF PRIME POWER ORDER

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To Laci Kovács on his 65th birthday

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Abstract

Let G be a finite group of order p^k, where p is a prime and $k \geqslant 1$, such that G is either cyclic, quaternion or generalised quaternion. Let V be a finite-dimensional free KG-module where K is a field of characteristic p. The Lie powers $L^n(V)$ are naturally KG-modules and the main result identifies these modules up to isomorphism. There are only two isomorphism types of indecomposables occurring as direct summands of these modules, namely the regular KG-module and the indecomposable of dimension $p^k - p^{k-1}$ induced from the indecomposable KH-module of dimension $p - 1$, where H is the unique subgroup of G of order p. Formulae are given for the multiplicities of these indecomposables in $L^n(V)$. This extends and utilises work of the first author and R. Stöhr concerned with the case where G has order p.

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1. Introduction

Let G be a group, K a field and V a KG-module. Let $L(V)$ denote the free Lie algebra on V, that is, the free Lie algebra which contains V as a subspace and which has every basis of V as a free generating set. For each positive integer n, let $L^n(V)$ be the homogeneous component of degree n in $L(V)$. The action of G on V extends naturally to $L(V)$, so that G acts on $L(V)$ by Lie algebra automorphisms. In this way $L(V)$ becomes a KG-module with each $L^n(V)$ as a submodule, called the nth Lie power of V.

Suppose now that G is finite and V is finite-dimensional. If the characteristic of K is zero then each $L^n(V)$ is semisimple and the character of this module is given in terms of the character of V by Brandt's character formula [3]. Thus $L^n(V)$ may be determined up to isomorphism, at least in principle, by the character orthogonality relations. In the case where K has prime characteristic p, and p divides the order of G, it is much more difficult to obtain information about the module structure of $L^n(V)$. Recent progress is described in [4] and [5]. The reader is also referred to [4] or [5] for further details of the background and underlying concepts.

The main result of the first author and Stöhr in [5] is a description of $L^n(V)$ in the case where K has prime characteristic p, G is cyclic of order p and V is a finite-dimensional free KG-module. The module $L^n(V)$ decomposes into a direct sum of modules isomorphic either to the regular KG-module or to the indecomposable of dimension $p - 1$. Here we extend this result to the case where G is an arbitrary cyclic p-group or (when $p = 2$) a quaternion or generalised quaternion group.

THEOREM 1. *Let G be a finite group of order p^k, where p is a prime and $k \geqslant 1$, such that G is either cyclic, quaternion or generalised quaternion. Let H be the unique subgroup of G of order p. Let K be a field of characteristic p and let V be a finite-dimensional free KG-module. Then, for each positive integer n, $L^n(V)$ is a direct sum of $r(n)$ copies of the regular KG-module and $s(n)$ copies of the indecomposable KG-module of dimension $p^k - p^{k-1}$ induced from the indecomposable KH-module of dimension $p - 1$, where*

$$s(n) = -\frac{1}{np^{k-1}} \sum_{\substack{d \\ p|d|n}} \mu(d)(\dim V)^{n/d}$$

and $r(n) = p^{-k} \dim L^n(V) - (1 - p^{-1})s(n)$.

In the equation for $s(n)$, μ is the Möbius function, dim V denotes the dimension of V as a K-space, and the summation is over all positive divisors d of n which are divisible by p. The second equation yields $r(n)$ because of Witt's formula for the dimension of $L^n(V)$:

$$\dim L^n(V) = \frac{1}{n} \sum_{d|n} \mu(d)(\dim V)^{n/d}.$$

Theorem 1 will be derived from the following more general result.

THEOREM 2. *Let G be a non-trivial finite p-group, where p is a prime, and let H be the subgroup generated by all elements of G of order p. Let K be a field of characteristic p and let V be a finite-dimensional free KG-module. Then, for each positive integer n, $L^n(V)$ is isomorphic to a module induced from some KH-module.*

Theorem 2 shows that there is a KH-module U such that $L^n(V) \cong U\!\uparrow^G$. Thus, to find $L^n(V)$ up to isomorphism it is sufficient to find U. However,

$$L^n(V\!\downarrow_H) \cong L^n(V)\!\downarrow_H \cong U\!\uparrow^G\!\downarrow_H.$$

If we assume that H is central in G then $U\!\uparrow^G\!\downarrow_H$ is isomorphic to the direct sum of $[G : H]$ copies of U. Thus we can find U up to isomorphism if we can find $L^n(V\!\downarrow_H)$. However, $V\!\downarrow_H$ is a free KH-module. Thus, when H is central in G, Theorem 2 reduces the problem of finding Lie powers of free KG-modules to the same problem for H.

Under the assumptions of Theorem 1, H is central in G and has order p. Thus Theorem 1 can be obtained from Theorem 2 by means of the main result of [5]. When G is any finite abelian p-group, Theorem 2 makes a reduction to the case where G is elementary abelian. At present, however, we are not able to deal with an elementary abelian p-group of order greater than p. The representation theory of such a group is complicated by the fact that it has infinitely many isomorphism types of indecomposable modules over a field of characteristic p.

The authors are grateful to Dr L. G. Kovács, Dr A. I. Papistas and Dr R. Stöhr for helpful discussions on the topics considered here. In particular, Dr Stöhr kindly pointed out a substantial simplification that could be made to our original proof of Lemma 3. The special case of Theorem 1 with $p = 2$ was obtained in the second author's Ph.D. thesis [7] and the special case where $p = 2$ and G is cyclic was also obtained independently by A. I. Papistas in unpublished work.

2. Preliminaries

Throughout the paper K denotes a field, and in this section K is an arbitrary field. All Lie algebras are Lie algebras over K, and all tensor products are taken with respect to K. By a *KG-module*, where G is a group, we mean a right module for the group algebra KG. As mentioned in Section 1, if V is a KG-module then the free Lie algebra $L(V)$ acquires the structure of a KG-module.

For any Lie algebra L, $[u, v]$ denotes the product of elements u and v of L, and expressions of the form $[u_1, u_2, \dots, u_n]$ denote left-normed products: thus, for $n \geqslant 3$,

$$[u_1, u_2, \dots, u_n] = [[u_1, \dots, u_{n-1}], u_n].$$

For subspaces $U_1, U_2, \dots, U_n$ of L, $[U_1, U_2, \dots, U_n]$ denotes the subspace of L spanned by all elements $[u_1, u_2, \dots, u_n]$ with $u_i \in U_i$ for $i = 1, \dots, n$.

By a *graded vector space over* K we mean a K-space V with a distinguished K-space decomposition $V = V_1 \oplus V_2 \oplus \cdots$. A *graded subspace of* V is then a subspace W such that $W = (W \cap V_1) \oplus (W \cap V_2) \oplus \cdots$.

Let L be a free Lie algebra over K and, for each positive integer n, let L_n be the nth homogeneous component of L with respect to a given free generating set for L. Then L is a graded vector space, $L = L_1 \oplus L_2 \oplus \cdots$. A subalgebra of L is called graded if it is graded as a subspace. By the theorem of Shirshov and Witt (see [8, Theorem 2.5]), every subalgebra of L is free on some free generating set. If Q is a subalgebra of L and W is a subspace of L such that Q is freely generated by a basis of W we say (with slight abuse of language) that *Q is freely generated by W* and write $Q = L(W)$. In this case every basis of W is a free generating set for Q. The notation $L(W)$ is used for the subalgebra of L generated by a subspace W only in the case where $L(W)$ is freely generated by W.

LEMMA 1. *Let Q be a graded subalgebra of the free Lie algebra L and write $Q = Q_1 \oplus Q_2 \oplus \cdots$ where $Q_i = Q \cap L_i$ for all i. For $i \geqslant 1$ let R_i be the subalgebra of Q generated by $Q_1 \oplus \cdots \oplus Q_i$ and let $R_0 = 0$. Let W be a subspace of Q which has the form $W = W_1 \oplus W_2 \oplus \cdots$, where $W_i = W \cap Q_i$ for all $i \geqslant 1$. Then Q is freely generated by W if and only if*

$$Q_i = (R_{i-1} \cap Q_i) \oplus W_i \quad \text{for all } i \geqslant 1. \tag{2.1}$$

PROOF. For the purposes of this proof, if X is any subset of L we write KX for the subspace of L spanned by X and $\langle X \rangle_L$ for the Lie subalgebra of L generated by X. For $i \geqslant 1$, let $E_i = Q_1 \oplus \cdots \oplus Q_i$ and let $E_0 = 0$. Thus, for all $i \geqslant 1$, $R_{i-1} = \langle E_{i-1} \rangle_L$. Also, write $E_i' = E_i \cap R_{i-1}$. Then

$$E_i' = Q_1 \oplus \cdots \oplus Q_{i-1} \oplus (R_{i-1} \cap Q_i).$$

If (2.1) holds, then any basis X_i of W_i is a basis for E_i modulo E_i'. Hence, by the proof of Theorem 2.5 of [8], Q is freely generated by $X_1 \cup X_2 \cup \cdots$. Thus Q is freely generated by W.

Conversely, suppose that Q is freely generated by W. For each $i \geqslant 1$, let X_i be a basis of W_i and let $X = X_1 \cup X_2 \cup \cdots$. Since $Q = \langle X \rangle_L$, Q is spanned by the set of all Lie monomials formed from elements of X. For $i \geqslant 1$, Q_i is spanned by all such monomials which belong to L_i. Therefore,

$$Q_i = \langle X_1 \cup \cdots \cup X_{i-1} \rangle_L \cap L_i + KX_i.$$

However, X is a free generating set for Q, and so

$$Q_i = \langle X_1 \cup \cdots \cup X_{i-1} \rangle_L \cap L_i \oplus KX_i. \tag{2.2}$$

Hence $Q_j \subseteq \langle X_1 \cup \cdots \cup X_j \rangle_L$ for $j = 1, \ldots, i-1$, and so $R_{i-1} = \langle X_1 \cup \cdots \cup X_{i-1} \rangle_L$. Thus (2.1) follows from (2.2). $\square$

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For subspaces U and V of any Lie algebra, let $V \wr U$ denote the subspace defined by

$$V \wr U = V + [V, U] + [V, U, U] + \cdots .$$

The following lemma is a version of 'Lazard elimination' (see [2, Chapter 2, Section 2.9, Proposition 10]) and it appears in [4, Lemma 2.2].

LEMMA 2. *Let G be any group and let U and V be KG-modules. Consider the free Lie algebra $L(U \oplus V)$ as a KG-module. Then U and $V \wr U$ freely generate free subalgebras $L(U)$ and $L(V \wr U)$, and there is a KG-module decomposition*

$$L(U \oplus V) = L(U) \oplus L(V \wr U).$$

Furthermore,

$$V \wr U = V \oplus [V, U] \oplus [V, U, U] \oplus \cdots ,$$

and, for each $n \geqslant 0$, there is a KG-module isomorphism

$$[V, \underbrace{U, \ldots, U}_{n}] \cong V \otimes \underbrace{U \otimes \cdots \otimes U}_{n}$$

under which $[v, u_1, u_2, \ldots, u_n]$ corresponds to $v \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_n$ for all $v \in V$ and all $u_1, \ldots, u_n \in U$.

We apply Lemma 2 to obtain two further results.

LEMMA 3. *Let G be any group and let $V_1, \ldots, V_m$ be KG-modules, where m is a positive integer. Consider the free Lie algebra $L(V_1 \oplus \cdots \oplus V_m)$ as a KG-module. Let Q be the ideal of $L(V_1 \oplus \cdots \oplus V_m)$ generated by the subspaces $[V_i, V_j]$ with $i \neq j$. Then there is a KG-module decomposition*

$$L(V_1 \oplus \cdots \oplus V_m) = L(V_1) \oplus \cdots \oplus L(V_m) \oplus Q.$$

Furthermore, Q is a free Lie subalgebra of $L(V_1 \oplus \cdots \oplus V_m)$ of the form $Q = L(W)$, where $W = W_2 \oplus W_3 \oplus \cdots$ such that, for each $n \geqslant 2$, W_n is a KG-submodule of $L^n(V_1 \oplus \cdots \oplus V_m)$ equal to the direct sum of all subspaces $[V_{i_1}, V_{i_2}, \ldots, V_{i_n}]$ with $i_1 > i_2 \leqslant i_3 \leqslant \cdots \leqslant i_n$. Furthermore, $[V_{i_1}, V_{i_2}, \ldots, V_{i_n}]$ is isomorphic to $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$ as a KG-module.

PROOF. By Lemma 2,

$$L(V_1 \oplus \cdots \oplus V_m) = L(V_1) \oplus L(Z_1),$$

where Z_1 is the direct sum of the subspaces $V_2, \dots, V_m$ and the subspaces

$$[V_{i_1}, \underbrace{V_1, \dots, V_1}_{n_1}]$$

with $i_1 > 1$ and $n_1 > 0$. Also,

$$[V_{i_1}, \underbrace{V_1, \dots, V_1}_{n_1}] \cong V_{i_1} \otimes \underbrace{V_1 \otimes \cdots \otimes V_1}_{n_1}$$

as KG-modules. Using the summand V_2 of Z_1 we obtain $L(Z_1) = L(V_2) \oplus L(Z_2)$, where Z_2 is the direct sum of the subspaces $V_3, \dots, V_m$ and the subspaces

$$[V_{i_1}, \underbrace{V_1, \dots, V_1}_{n_1}, \underbrace{V_2, \dots, V_2}_{n_2}]$$

with either $i_1 > 1, n_1 > 0, n_2 \geqslant 0$ or $i_1 > 2, n_1 = 0, n_2 > 0$. Also

$$[V_{i_1}, \underbrace{V_1, \dots, V_1}_{n_1}, \underbrace{V_2, \dots, V_2}_{n_2}] \cong V_{i_1} \otimes \underbrace{V_1 \otimes \cdots \otimes V_1}_{n_1} \otimes \underbrace{V_2 \otimes \cdots \otimes V_2}_{n_2}$$

as KG-modules. Continuing in this way we obtain

$$L(V_1 \oplus \cdots \oplus V_m) = L(V_1) \oplus \cdots \oplus L(V_m) \oplus L(W), \tag{2.3}$$

where W is the direct sum of the subspaces $[V_{i_1}, V_{i_2}, \dots, V_{i_n}]$ with $n \geqslant 2$ and $i_1 > i_2 \leqslant i_3 \leqslant \cdots \leqslant i_n$. Also,

$$[V_{i_1}, V_{i_2}, \dots, V_{i_n}] \cong V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$$

as KG-modules. Clearly, $[V_{i_1}, V_{i_2}, \dots, V_{i_n}]$ is a submodule of $L^n(V_1 \oplus \cdots \oplus V_m)$. Thus W can be written in the required form.

It remains to show that $L(W) = Q$, where Q is defined as in the statement of the lemma. Clearly $L(W) \subseteq Q$. Let D be the Lie algebra direct sum,

$$D = L(V_1) \oplus \cdots \oplus L(V_m),$$

formed from the Lie algebras $L(V_1), \dots, L(V_m)$ with componentwise multiplication. Let

$$\pi : L(V_1 \oplus \cdots \oplus V_m) \longrightarrow D$$

be the Lie algebra homomorphism given by $u\pi = u$ for all $u \in V_1 \cup \cdots \cup V_m$: this exists because $L(V_1 \oplus \cdots \oplus V_m)$ is free on $V_1 \oplus \cdots \oplus V_m$. As is well known, Q is the kernel of π. The restriction of π to the subspace $L(V_1) \oplus \cdots \oplus L(V_m)$ of $L(V_1 \oplus \cdots \oplus V_m)$ is clearly one-one and onto. Hence

$$L(V_1 \oplus \cdots \oplus V_m) = L(V_1) \oplus \cdots \oplus L(V_m) \oplus Q.$$

Since $L(W) \subseteq Q$ we obtain $L(W) = Q$ by (2.3). $\square$

LEMMA 4. *Let G be any group and let V be a KG-module. Let Q be a subalgebra of $L(V)$ which has the form $Q = L(U_r \oplus U_{r+1} \oplus \cdots)$ for some positive integer r, where, for each $i \geqslant r$, U_i is a free KG-submodule of $L^i(V)$. Let n be a positive integer such that $n \geqslant r$. Then, for each $i \geqslant r$, there exists a free KG-submodule X_i of $L^i(V)$ such that $Q = L(X_r) \oplus L(X_{r+1}) \oplus \cdots \oplus L(X_n) \oplus L(X_{n+1} \oplus X_{n+2} \oplus \cdots)$.*

PROOF. By Lemma 2,

$$L(U_r \oplus U_{r+1} \oplus \cdots) = L(U_r) \oplus L((U_{r+1} \oplus U_{r+2} \oplus \cdots) \wr U_r).$$

Also, $(U_{r+1} \oplus U_{r+2} \oplus \cdots) \wr U_r$ is the direct sum of the modules $U_{r+i} \wr U_r$ for $i \geqslant 1$, and $U_{r+i} \wr U_r$ is the direct sum of the modules

$$[U_{r+i}, \underbrace{U_r, \ldots, U_r}_{m}]$$

for $m \geqslant 0$. Furthermore,

$$[U_{r+i}, \underbrace{U_r, \ldots, U_r}_{m}] \cong U_{r+i} \otimes \underbrace{U_r \otimes \cdots \otimes U_r}_{m},$$

so this module is a free KG-submodule of $L^{r+i+mr}(V)$. Thus we may write

$$L(U_r \oplus U_{r+1} \oplus \cdots) = L(U_r) \oplus L(V_{r+1} \oplus V_{r+2} \oplus \cdots),$$

where, for $i \geqslant 1$, V_{r+i} is a free KG-submodule of $L^{r+i}(V)$. The result follows by induction on $n - r$. □

3. Main results

Let G be a non-trivial finite p-group, where p is a prime, and let H be the subgroup generated by all elements of G of order p. Let K be a field of characteristic p. For any KG-module V we write $V{\downarrow_H}$ for the KH-module obtained by restriction, and for any KH-module U we write $U{\uparrow^G}$ for the KG-module obtained by induction. We say that a KG-module is *H-induced* if it is isomorphic to a module induced from a KH-module. Clearly any direct sum of H-induced modules is H-induced.

PROOF OF THEOREM 2. Let V be a finite-dimensional free KG-module. Write $L = L(V)$ and $L_n = L^n(V)$ for all $n \geqslant 1$. We must prove that L_n is H-induced for all n. We use induction on n. Since $L_1 = V$, the module L_1 is free. Thus it is induced from a free KH-module. Hence we may assume that $n \geqslant 2$ and that the result is true for all smaller values of n.

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Let m be the index of H in G and let $\{c_1, \dots, c_m\}$ be a (right) transversal for H in G, where $c_1 = 1$. Since V is a free KG-module, there is a subspace Y of V such that $V = Y(KG) \cong Y \otimes KG$. Write $V_i = Y(KH)c_i$ for $i = 1, \dots, m$. Then, since H is normal in G, each V_i is a free KH-module isomorphic to V_1. Also, $V = V_1 \oplus \cdots \oplus V_m$ and $V \cong V_1\uparrow^G$.

By Lemma 3 applied to the group H, we have a KH-module decomposition $L = M \oplus Q$, where $M = L(V_1) \oplus \cdots \oplus L(V_m)$ and Q is the ideal of L generated by the subspaces $[V_i, V_j]$ with $i \neq j$. The subspaces $[V_i, V_j]$ are permuted under the action of G. Therefore Q is a KG-submodule of L. Also, $L(V_i) = L(V_1)c_i$ for $i = 1, \dots, m$, so that M is a KG-submodule of L. By Lemma 3, $Q = L(W)$ where W is a graded subspace of L. Thus Q is a graded subalgebra of L. Clearly M is a graded subspace of L. It follows that $L_n = (L_n \cap M) \oplus (L_n \cap Q)$. Thus it suffices to show that $L_n \cap M$ and $L_n \cap Q$ are H-induced.

It is easily verified that $L^n(V_i) = L^n(V_1)c_i$ for $i = 1, \dots, m$ and

$$L_n \cap M = L^n(V_1) \oplus \cdots \oplus L^n(V_m).$$

It follows that $L_n \cap M \cong L^n(V_1)\uparrow^G$. Therefore $L_n \cap M$ is H-induced.

By Lemma 3, $W = W_2 \oplus W_3 \oplus \cdots$ where, for $i \geqslant 2$, W_i is a free KH-submodule of L_i. We also write $W_1 = 0$. Since Q is a graded subalgebra of L, it has the form $Q = Q_1 \oplus Q_2 \oplus \cdots$ *where* $Q_i = Q \cap L_i$ *for all* i. *Clearly* $Q_1 = 0$. *For* $i \geqslant 1$ *let* R_i be the subalgebra of Q generated by $Q_1 \oplus \cdots \oplus Q_i$ and let $R_0 = 0$. Since Q is a KG-module, Q_i is a KG-module for each $i \geqslant 1$, and $R_{i-1} \cap Q_i$ is a submodule. Thus $Q_i/(R_{i-1} \cap Q_i)$ is a KG-module.

By Lemma 1, $Q_i = (R_{i-1} \cap Q_i) \oplus W_i$ for all $i \geqslant 1$. Since W_i is a free KH-module, $(Q_i/(R_{i-1} \cap Q_i))\downarrow_H$ is a free KH-module. Therefore, by the definition of H, $(Q_i/(R_{i-1} \cap Q_i))\downarrow_E$ is a free KE-module for every elementary abelian subgroup E of G. Therefore, by a theorem of Chouinard, [1, Theorem 5.2.4], $Q_i/(R_{i-1} \cap Q_i)$ is a projective KG-module. Since G is a p-group, this module is free (see, for example, [6, Theorem VII.7.15]). Therefore, for each $i \geqslant 1$, there is a KG-submodule U_i of Q_i such that U_i is free and $Q_i = (R_{i-1} \cap Q_i) \oplus U_i$. Note that $U_1 = 0$ and write $U = U_2 \oplus U_3 \oplus \cdots$. By Lemma 1, Q is freely generated by U, that is $Q = L(U)$.

By Lemma 4, we may write

$$Q = L(X_2) \oplus L(X_3) \oplus \cdots \oplus L(X_n) \oplus L(X_{n+1} \oplus X_{n+2} \oplus \cdots),$$

where each X_i is a free KG-submodule of L_i. It follows that $L_n \cap Q$ is the direct sum of the modules $L^{n/d}(X_d)$ where d ranges over the divisors of n in the set $\{2, \dots, n\}$. By the inductive hypothesis each summand is H-induced. Therefore $L_n \cap Q$ is H-induced, as required. □

PROOF OF THEOREM 1. Now let G have order p^k, with $k \geqslant 1$, and suppose that G is either cyclic, quaternion or generalised quaternion. Thus the subgroup H is central and is cyclic of order p. We write J_p to denote a regular KH-module and J_{p-1} to denote an indecomposable KH-module of dimension $p-1$ (this is isomorphic to the augmentation ideal of KH).

Let V be a finite-dimensional free KG-module and let n be a positive integer. By Theorem 2, there exists a KH-module U such that $L^n(V) \cong U{\uparrow^G}$. Therefore

$$L^n(V{\downarrow_H}) \cong L^n(V){\downarrow_H} \cong U{\uparrow^G}{\downarrow_H}.$$

Since H is central in G, $U{\uparrow^G}{\downarrow_H}$ is isomorphic to the direct sum of p^{k-1} copies of U. However $V{\downarrow_H}$ is a free KH-module. Therefore, by [5, Theorem 1], $L^n(V{\downarrow_H})$ is isomorphic to the direct sum of r_n copies of J_p and s_n copies of J_{p-1} where

$$s_n = -\frac{1}{n}\sum_{\substack{d \\ p|d|n}} \mu(d)(\dim V)^{n/d}$$

and $pr_n + (p-1)s_n = \dim L^n(V)$. It follows that U is isomorphic to the direct sum of $p^{-(k-1)}r_n$ copies of J_p and $p^{-(k-1)}s_n$ copies of J_{p-1}. Therefore the induced module $L^n(V)$ is isomorphic to the direct sum of $p^{-(k-1)}r_n$ copies of $J_p{\uparrow^G}$ and $p^{-(k-1)}s_n$ copies of $J_{p-1}{\uparrow^G}$.

Clearly $J_p{\uparrow^G}$ is a regular KG-module and is indecomposable (see [6, VII.5.2]). Since J_{p-1} remains indecomposable under any field extension it follows from Green's indecomposability theorem (see [6, VII.16.6]) that $J_{p-1}{\uparrow^G}$ is an indecomposable KG-module. Clearly it has dimension $p^k - p^{k-1}$. In the notation of Theorem 1, $r(n) = p^{-(k-1)}r_n$ and $s(n) = p^{-(k-1)}s_n$. This gives Theorem 1. □

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