A NOTE ON THE CARADUS CLASS $\mathfrak{F}$ OF BOUNDED LINEAR OPERATORS ON A COMPLEX BANACH SPACE

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1. In a recent paper (**1**) on meromorphic operators, Caradus introduced the class $\mathfrak{F}$ of bounded linear operators on a complex Banach space X. A bounded linear operator T is put in the class $\mathfrak{F}$ if and only if its spectrum consists of a finite number of poles of the resolvent of T. Equivalently, T is in $\mathfrak{F}$ if and only if it has a rational resolvent (**8**, p. 314).

Some ten years ago (in May, 1957), I discovered a property of the class $\mathfrak{F}$ which may be of interest in connection with Caradus' work, and is the subject of the present note.

2. THEOREM. *Let X be a complex Banach space. If T belongs to the class $\mathfrak{F}$, and the linear operator S commutes with every bounded linear operator which commutes with T, then there is a polynomial p such that $S = p(T)$.*

Suppose that T and S satisfy the hypothesis of the theorem. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the points of the spectrum of T, which by hypothesis are poles of the resolvent of T, and let $\nu_1, \nu_2, \ldots, \nu_n$ be the orders of those poles, respectively. Let M_r be the kernel (or "null manifold") of $(T - \lambda_r I)^{\nu_r}$ $(r = 1, 2, \ldots, n)$. Then $X = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ (**8**, p. 317, Theorem 5.9-E). For typographical convenience we write T_r for $T - \lambda_r I$ $(r = 1, 2, \ldots, n)$.

Now let x be any member of M_r (where r is any integer with $1 \leqq r \leqq n$). Choose a bounded linear functional f on X such that

$$(T_r^*)^{\nu_r} f = 0 \quad \text{but} \quad (T_r^*)^{\nu_r - 1} f \neq 0;$$

such an f exists since λ_r is also a pole of order ν_r of the resolvent of the adjoint T^* of T (**3**, p. 568, Theorem VII.3.7). We now consider the bounded linear operator

$$V = \sum_{s=1}^{\nu_r} T_r^{s-1} (x \otimes f) T_r^{\nu_r - s},$$

where $x \otimes f$ denotes the operator $y \to f(y)x$ on X into itself; cf. (**7**, p. 110). In view of our choice of x and f, we have:

$$T_r V = \sum_{s=1}^{\nu_r - 1} T_r^s (x \otimes f) T_r^{\nu_r - s}$$
$$= V T_r,$$

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so that V commutes with T_r, and thus with T. Hence (by hypothesis), V commutes with S.

Now

$$f,\ T_r^*f,\ (T_r^*)^2f, \ldots, (T_r^*)^{\nu_r-1}f$$

are clearly linearly independent (if $\sum_{s=1}^{\nu_r} \alpha_s\,(T_r^*)^{s-1}f = 0$, then

$$\sum_{s=1}^{\nu_r} \alpha_s (T_r^*)^{\nu_r+s-2}f = 0,$$

and hence $\alpha_1 = 0$, $\sum_{s=1}^{\nu_r} \alpha_s(T_r^*)^{\nu_r+s-3}f = 0$, and therefore $\alpha_2 = 0$, and so on), and thus a point y of X can be found such that

$$[(T_r^*)^{\nu_r-1}f](y) = 1, \qquad [(T_r^*)^{s-1}f](y) = 0 \quad (s = 1, 2, \ldots, \nu_r - 1),$$

that is,

$$f(T_r^{\nu_r-1}y) = 1, \qquad f(T_r^{\nu_r-s}y) = 0 \quad (s = 2, 3, \ldots, \nu_r)$$

(cf. **2**, p. 6, Theorem I.2.2, Corollary 2). Then $SVy = VSy$, and therefore

$$\sum_{s=1}^{\nu_r} ST_r^{s-1}(x \otimes f)T_r^{\nu_r-s}y = \sum_{s=1}^{\nu_r} T_r^{s-1}(x \otimes f)T_r^{\nu_r-s}Sy,$$

that is,

$$Sx = \sum_{s=1}^{\nu_r} f(T_r^{\nu_r-s}Sy)T_r^{s-1}x$$
$$= \sum_{s=1}^{\nu_r} f(T_r^{\nu_r-s}Sy)(T - \lambda_r I)^{s-1}x.$$

However, the choice of f and y was quite independent of the choice of $x \in M_r$. Hence,

$$Sx = p_r(T)x$$

for every $x \in M_r$, where p_r is the polynomial given by

$$p_r(\lambda) = \sum_{s=1}^{\nu_r} f(T_r^{\nu_r-s}Sy)(\lambda - \lambda_r)^{s-1}.$$

Having chosen a polynomial p_r as above for each $r = 1, 2, \ldots, n$, we now choose a polynomial p such that

$$p^{(s)}(\lambda_r) = p_r^{(s)}(\lambda_r) \qquad (s = 0, 1, 2, \ldots, \nu_r - 1;\ r = 1, 2, \ldots, n).$$

This can certainly be done; for example we can take

$$p = p_1 \cdot \phi_1 + p_2 \cdot \phi_2 + \ldots + p_n \cdot \phi_n,$$

where ϕ_r is given by

$$\phi_r(\lambda) = \left[\prod_{\substack{s=1;\\ s\neq r}}^{n} (\lambda - \lambda_s)^{\nu_s}\right]\Phi_r(\lambda),$$

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$\Phi_r(\lambda)$ being the sum of the first ν_r terms in the expansion of

$$\left[\prod_{\substack{s=1;\\ s\neq r}}^{n} (\lambda - \lambda_s)^{\nu_s}\right]^{-1}$$

as a power series in $\lambda - \lambda_r$ (this generalizes, in effect, the Lagrange interpolation formula, which corresponds to the case $\nu_1 = \nu_2 = \ldots = \nu_n = 1$; that such a generalization is possible is, of course, well known; cf. (**6**; **5**; **4**); the last two refer specifically to the Hermite interpolation formula, which corresponds to the case $\nu_1 = \nu_2 = \ldots = \nu_n = 2$). Then

$$p(T)x = p_r(T)x = Sx$$

for every $x \in M_r$ (**3**, p. 571, Theorem VII.3.16; **8**, p. 307, Theorem 5.8-B). Hence,

$$p(T)x = Sx$$

for every $x \in M_1 \oplus M_2 \oplus \ldots \oplus M_n = X$, and therefore $S = p(T)$, as required. Incidentally, $\phi_r(T)$ is the spectral projection of X onto M_r; cf. (**8**, § 5.9, p. 319, Problem 3).

Note. Since V is of finite rank, and thus a member of $\mathfrak{F}$, we have in fact proved the following, slightly stronger, result.

If $T \in \mathfrak{F}$, and the linear operator S commutes with every member $\mathfrak{F}$ which commutes with T, then there is a polynomial p such that $S = p(T)$.

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