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AN INVARIANCE RELATIONSHIP FOR THE $G/G/1$ QUEUE

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Abstract

In this note, we show that for a stationary FCFS $G/G/1$ queue, the virtual waiting time and the time spent in the system by the customer in service have the same distribution. We assume that the latter is zero if the queue is empty.

VIRTUAL WAITING TIME; TIME IN SYSTEM BY CUSTOMER IN SERVICE; FCFS

We consider a stationary version of the FCFS $G/G/1$ queue in a framework identical to that of Miyazawa (1979), (1983). The input is denoted by $\{t_i, S_i\}_{i=-\infty}^{\infty}$ where t_i and S_i are the arrival time and service time of the ith customer ($i = 0, \pm 1, \cdots$). The input process is assumed to be a stationary random marked process defined on $(\Omega, \mathcal{F}, P)$. Let $L(t)$ be the number in the system at time t. Let $\tau_i(t)$ for $i = 1, \cdots, L(t)$ be the arrival times of the customers in the system at time t where $\tau_1(t) \leqq \tau_2(t) \leqq \cdots \leqq \tau_{L(t)}(t)$. Let $r_i(t)$ for $i = 1, \cdots, L(t)$ denote their residual service time at time t. Let us define

$$X(t) = (\tau_1(t), r_1(t), \tau_2(t), r_2(t), \cdots)$$

where $\tau_i(t) = r_i(t) = 0$ for $i > L(t)$. Let U_0 and U_1 denote the point processes associated with arrivals and departures respectively. Let $P_i (i = 0, 1)$ denote the Palm measure obtained by conditioning on an atom of U_i at time zero. Let E, E_1 and E_2 denote expectations with respect to the measures P, P_1 and P_2 respectively. Let $\lambda = EU_i(0, 1]$ for $i = 0, 1$ and $\rho = \lambda ES$.

Let $V(t)$ be the virtual waiting time at t, i.e., $V(t) = \sum_{i=1}^{\infty} r_i(t)$. Let $Y(t)$ be the time spent in the system by the customer in service at time t, i.e.,

$$Y(t) = \begin{cases} t - \tau_1(t) & \text{if } L(t) > 0 \\ 0 & \text{if } L(t) = 0. \end{cases}$$

Let $\tilde{V}(s, t)$ and $\tilde{Y}(s, t)$ denote $E(\exp(-sV(t)))$ and $E(\exp(-sY(t)))$ respectively.

Theorem. If $\rho < 1$ *and* U_0 *and* U_1 *are simple, then* $\tilde{V}(s, t) = \tilde{Y}(s, t)$.

Proof. Note that for $\rho < 1$, Miyazawa (1979) has shown the existence of U_1 and $X(t)$ satisfying (2.1) and (2.2) of Miyazawa (1983). We now use Lemma 3.1 of Miyazawa (1983) and define $f(t) = \exp(-sV(t))$. Then $Ef'(0) = sE(\exp(-sV(0)) \mid L(0) > 0)\rho$. Further, $E_0 f(0-) = E_0(\exp(-sV(0-))) = \tilde{W}_q(s)$ is the Laplace–Stieltjes transform (LST) of the waiting time distribution. Also, $E_0 f(0+) = E_0(\exp(-sV(0+))) = \tilde{W}(s)$ is the LST of the distribution of time spent in the system. So, from Lemma 3.1 of Miyazawa (1983),

$$sE(\exp(-sV(0)) \mid L(0) > 0)\rho = \lambda(\tilde{W}_q(s) - \tilde{W}(s)). \tag{1}$$

We now use the same lemma again with $f(t) = \exp(-sY(t))$. This time, $Ef'(0) = -sE(\exp(-sY(0)) \mid L(0) > 0)\rho$. A little thought should convince the reader that $E_1 f(0-) = \tilde{W}(s)$ and $E_1 f(0+) = \tilde{W}_q(s)$. So we have

$$-sE(\exp(-sY(0)) \mid L(0) > 0)\rho = \lambda(\tilde{W}(s) - \tilde{W}_q(s)). \tag{2}$$

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The result now follows from (1) and (2).

Remark. While the virtual waiting time has been studied extensively in the literature, very little work has been done on the time spent in the system by the customer in service. On page 101 of Prabhu (1965), there is a characterization of this stochastic process for the $GI/M/1$ queue. Recently, Sengupta (1989) has studied a more general Markov process which behaves much like $Y(t)$ and has applied the results to the $GI/PH/1$ queue. Using the notation of Sengupta (1989), the conditional distribution of $Y(t)$ given that the server is busy (for the $GI/PH/1$ queue) is given by $1 - \alpha \exp(Tx)e$. This result (together with the theorem above) considerably simplifies the characterization and the computation of the virtual waiting time distribution (see Remark 8, Section 3 of Sengupta (1989)). We note that the exponential form of the virtual waiting time distribution for the $GI/PH/1$ queue has also been proved by Asmussen (1988) and Ramaswami (1989). However, our results show that the same is true for the semi-Markovian queue also (Sengupta (1988)).

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