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AN EXAMPLE OF A POSITIVE NON-SYMMETRIC KERNEL SATISFYING THE COMPLETE MAXIMUM PRINCIPLE

Dedicated to Professor Yukinari TÔKI on his sixtieth birthday

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§1. Introduction and preliminaries.

One of the main problems in potential theory is to determine the class of kernels satisfying the domination principle or the complete maximum principle. As to positive symmetric kernels this is settled to a certain extent, but as to non-symmetric kernels we have not yet obtain satisfactorily large explicit classes. In this note we shall give a class of positive non-symmetric convolution kernels on the real line satisfying the complete maximum principle.

Let K be a positive Radon measure on the real line $\boldsymbol{R}$. Given an essentially bounded Lebesgue measurable function f vanishing outside a compact set, the convolution $K*f$ is called a *potential* relative to a positive kernel K. This is a Lebesgue measurable function on $\boldsymbol{R}$. Throughout this note we shall identify two functions which differ on a set of Lebesgue measure zero.

We shall say that the kernel K satisfies the *complete maximum principle*, when the following statement is valid: if f and g are essentially bounded Lebesgue measurable functions vanishing outside a compact set and $K*f \leqq K*g + 1$ on the set $\{x \in \boldsymbol{R}\,;\, f(x) > 0\}$, then the same inequality holds almost everywhere in $\boldsymbol{R}$. A sufficient condition for the complete maximum principle is the existence of a so-called *submarkovian resolvent* $\{K_p\}$ $(p > 0)$, a family of positive kernels such that for each $p > 0$, $K - K_p = pK*K_p = pK_p*K$ and $pK_p*1 \leqq 1$, and $K = K_0 = \lim_{p \downarrow 0} K_p$. The outline of the proof: we have, for each $p > 0$,

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$$K + \frac{1}{p}\varepsilon = \frac{1}{p}\left(\sum_{n=0}^{\infty}(pK_p)^n\right),$$

where $(pK_p)^0 = \varepsilon$, the Dirac measure on the origin, and $(pK_p)^n = (pK_p)^{n-1}*(pK_p)$. This shows that $K + (1/p)\varepsilon$ satisfies the complete maximum principle, so that K itself satisfies the complete maximum principle.

§2. Negative-definite functions.

A complex-valued continuous function $\lambda(x)$ on $\boldsymbol{R}$ is said to be *negative-definite*, when it holds that

i) $\lambda(0) \geqq 0$, $\lambda(-x) = \overline{\lambda(x)}$,

ii) for any n points $x_1, x_2, \cdots, x_n$ on $\boldsymbol{R}$ and for any n complex numbers $\xi_1, \xi_2, \cdots, \xi_n$ with $\sum_{j=1}^{n}\xi_j = 0$,

$$\sum_{j,k}\lambda(x_j - x_k)\xi_j\bar{\xi}_k \leqq 0 .$$

As easily seen, $\lambda(x) = ax^2 + ibx + c$ is negative-definite, if a and c are non-negative and b is real. $\lambda(x) = c - (p - ix)^{-1}$ with $p > 0$ and $c \geqq p^{-1}$ is also negative-definite, since

$$-(p - ix)^{-1} = -\int_0^{\infty} e^{-py}e^{ixy}dy .$$

The sum of negative-definite functions $\lambda_1, \lambda_2, \cdots, \lambda_n$ is evidently negative-definite. In this section we shall show that for specific λ_j's, $\lambda = \left(\sum_{j=1}^{n}\lambda_j^{-1}\right)^{-1}$ is negative-definite.

LEMMA 1. *Let* $0 < p_1 < p_2 < \cdots < p_n$ *and* $\mu_1, \mu_2, \cdots, \mu_n, \nu_1, \nu_2, \cdots, \nu_n$ *be non-negative numbers, and let* $f(z)$ *be a complex-valued function* $\left(\sum_{j=1}^{n}\left(\frac{\mu_j}{p_j + z} + \frac{\nu_j}{p_j - z}\right)\right)^{-1}$. *If* $\sum_{j=1}^{n}\mu_j \neq \sum_{j=1}^{n}\nu_j$, $f(z)$ *is equal to*

$$az + b + \sum_{k=1}^{2n-1}\frac{c_k}{z - \alpha_k} \tag{1}$$

with $b > 0$, $\alpha_k c_k > 0$. *If* $\sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}\nu_j$, *it is equal to*

$$az^2 + bz + c + \sum_{k=1}^{2n-2}\frac{d_k}{z - \alpha_k} \tag{2}$$

with $a < 0$, $c > 0$, $\alpha_k d_k > 0$.

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Proof. First we consider the case that $\sum_{j=1}^{n} \mu_j \neq \sum_{j=1}^{n} \nu_j$. In this case $f(z)$ is equal to $\prod_{j=1}^{n} (p_j^2 - z^2)/\psi(z)$, where $\psi(z) = \sum_{k=1}^{n} ((\mu_k + \nu_k)p_k + (\nu_k - \mu_k)z) \cdot \prod_{j\neq k} (p_j^2 - z^2)$ is a polynomial of degree $2n-1$: the coefficient c of z^{2n-1} is $(-1)^{n-1} \sum_{k=1}^{n} (\nu_k - \mu_k)$. Hence $f(z)$ is written as

$$az + b + \frac{\varphi(z)}{\psi(z)}$$

with a polynomial $\varphi(z)$ of degree less than $2n-1$ and $a = \left(\sum_{k=1}^{n} (\mu_k - \nu_k)\right)^{-1}$. Remark that $\psi(z)$ has $2n-1$ simple zeros α_k's and β_k's: in the case that c is positive and n is odd, or c is negative and n is even,

$$\beta_n < -p_n < \beta_{n-1} < -p_{n-1} < \cdots < -p_2 < \beta_1 < -p_1$$
$$p_1 < \alpha_1 < p_2 < \cdots < p_{n-1} < \alpha_{n-1} < p_n ,$$

and in the case that c is negative and n is odd, or c is positive and n is even

$$-p_n < \beta_{n-1} < -p_{n-1} < \cdots < -p_2 < \beta_1 < -p_1$$
$$p_1 < \alpha_1 < p_2 < \cdots < p_{n-1} < \alpha_{n-1} < p_n < \alpha_n .$$

In any case we have

$$\frac{\varphi(z)}{\psi(z)} = \sum_k \frac{\varphi(\alpha_k)}{\psi'(\alpha_k)} \frac{1}{z - \alpha_k} + \sum_k \frac{\varphi(\beta_k)}{\psi'(\beta_k)} \frac{1}{z - \beta_k}$$
$$\varphi(z) = \prod_{j=1}^{n} (p_j^2 - z^2) - (az + b)\psi(z)$$
$$\frac{\varphi(\alpha_k)}{\psi'(\alpha_k)} = \prod_{j=1}^{n} \frac{(p_j^2 - \alpha_k^2)}{\psi'(\alpha_k)} > 0$$
$$\frac{\varphi(\beta_k)}{\psi'(\beta_k)} = \prod_{j=1}^{n} \frac{(p_j^2 - \beta_k^2)}{\psi'(\beta_k)} < 0 .$$

Therefore $\alpha_k\varphi(\alpha_k)/\psi'(\alpha_k)$ and $\beta_k\varphi(\beta_k)/\psi'(\beta_k)$ are positive. We put $z = 0$ in (1) and obtain

$$b = \left(\sum_{j=1}^{n} \frac{\mu_j + \nu_j}{p_j}\right)^{-1} + \sum_{k=1}^{2n-1} \frac{c_k}{\alpha_k} > 0 .$$

Now suppose that $\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \nu_j$ $(\neq 0)$. Then

$$f(z) = az^2 + bz + c + \frac{\varphi(z)}{\psi(z)}$$

with a polynomial $\varphi(z)$ of degree less than $2n-1$ and $a = -\left(\sum_{k=1}^{n} (\mu_k + \nu_k)p_k\right)^{-1} < 0$. $\psi(z)$ has $2(n-1)$ simple zeros α_k's and β_k's such that

$$p_k < \alpha_k < p_{k+1}\,, \qquad -p_{k+1} < \beta_k < -p_k \ (1 \leqq k \leqq n-1)\,.$$

Similarly in the first case it holds that

$$\frac{\varphi(z)}{\psi(z)} = \sum_{k=1}^{n-1} \frac{\varphi(\alpha_k)}{\psi'(\alpha_k)} \frac{1}{z-\alpha_k} + \sum_{k=1}^{n-1} \frac{\varphi(\beta_k)}{\psi'(\beta_k)} \frac{1}{z-\beta_k}$$

$$\varphi(z) = \prod_{j=1}^{n} (p_j^2 - z^2) - (az^2 + bz + c)\psi(z)$$

$$\frac{\varphi(\alpha_k)}{\psi'(\alpha_k)} = \prod_{j=1}^{n} \frac{(p_j^2 - \alpha_k^2)}{\psi'(\alpha_k)} > 0$$

$$\frac{\varphi(\beta_k)}{\psi'(\beta_k)} = \prod_{j=1}^{n} \frac{(p_j^2 - \beta_k^2)}{\psi'(\beta_k)} < 0\,.$$

Hence the product of α_k and d_k in (2) is positive for each k and

$$c = \left(\sum_{j=1}^{n} \frac{\mu_j + \nu_j}{p_j}\right)^{-1} + \sum_{k=1}^{2n-2} \frac{d_k}{\alpha_k} > 0\,.$$

By this lemma we have

PROPOSITION 1. *Let $p_1, p_2, \cdots, p_n, \mu_1, \mu_2, \cdots, \mu_n, \nu_1, \nu_2, \cdots, \nu_n$ be non-negative numbers. Then*

$$\lambda(x) = \left(\sum_{j=1}^{n} \left(\frac{\mu_j}{p_j + ix} + \frac{\nu_j}{p_j - ix}\right)\right)^{-1} \tag{3}$$

is negative-definite.

Proof. If $\sum_{j=1}^{n} \mu_j \neq \sum_{j=1}^{n} \nu_j$, we have by (1)

$$\lambda(x) = iax + b + \sum_{k=1}^{2n-1} \frac{c_k}{ix - \alpha_k}$$

with real $a \neq 0$, $b > 0$, $\alpha_k c_k > 0$. If $\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \nu_j$ $(\neq 0)$,

$$\lambda(x) = -ax^2 + ibx + c + \sum_{k=1}^{2n-2} \frac{d_k}{ix - \alpha_k}$$

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with real b, $a > 0$, $c < 0$, $\alpha_k d_k > 0$. Therefore, in any case, $\lambda(x)$ is negative-definite.

§3. Fractional powers of Heaviside kernel.

Heaviside kernel H is the convolution kernel on $\boldsymbol{R}$ induced by the function

$$H(x) = \begin{cases} 1 , & x > 0 \\ 0 , & x \leqq 0 \end{cases} .$$

Given a non-negative Lebesgue measurable function f, its Heaviside potential $Hf(x)$ is

$$H*f(x) = \int_{-\infty}^{x} f(y)dy .$$

The kernel H has a unique positive resolvent $\{H_p\}$ $(p \geqq 0)$ such that

$$H_p(x) = \begin{cases} e^{-px} , & x > 0 \\ 0 & x \leqq 0 \end{cases} ,$$

and H satisfies the complete maximum principle. For a positive number α smaller than 1, the *fractional power* H^α of H is the convolution kernel defined by

$$\frac{\sin \alpha\pi}{\pi} \int_0^\infty p^{-\alpha} H_p dp ,$$

which is equal to the one induced by

$$H^\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} , & x > 0 \\ 0 \quad , & x \leqq 0 \end{cases} ,$$

In this section we shall show that for any $0 < \alpha,\ \beta < 1$, the sum of H^α and $\check{H}^\beta$ satisfies also the complete maximum principle, $\check{H}^\beta$ being the *adjoint kernel* of H^β, namely, $\check{H}^\beta(x) = H^\beta(-x)$. Notice that

$$\check{H}^\beta(x) = \begin{cases} \dfrac{|x|^{\beta-1}}{\Gamma(\beta)} = \dfrac{\sin \beta\pi}{\pi} \displaystyle\int_0^\infty p^{-\beta} e^{px} dp , & x < 0 \\ 0 \quad , & x \geqq 0 \end{cases} .$$

Let μ and ν be positive Radon measures on $(0, \infty)$. We exclude the case $\mu = \nu = 0$, and put

$$(*)\qquad K(x)=\begin{cases}\int_0^\infty e^{-px}d\mu(p)\,, & x>0\\ \int_0^\infty e^{px}d\nu(p)\quad, & x<0\end{cases}.$$

We assume that $K(x)$ is locally Lebesgue integrable. Then it induces a positive convolution kernel K. Our aim is to prove the complete maximum principle for K. First we restrict ourselves to the case that μ and ν are measures on a bounded closed interval $[a, b]$ $(0 < a < b < \infty)$. The Fourier transform of K, $\hat{K}(y)=\int_{-\infty}^{\infty}K(x)e^{-2\pi ixy}dx$, is equal to

$$\int_a^b\frac{d\mu(p)}{p+2\pi iy}+\int_a^b\frac{d\nu(p)}{p-2\pi iy}\,.$$

Hence $\lambda(y)=(\hat{K}(y))^{-1}$ is continuous on $\boldsymbol{R}$, and

$$\lambda(0)=\left(\int_a^b\frac{d\mu(p)}{p}+\int_a^b\frac{d\nu(p)}{p}\right)^{-1}>0\,,\qquad \lambda(-y)=\overline{\lambda(y)}\,.$$

LEMMA 2. *$\lambda(y)$ is negative-definite.*

Proof. We fix n points $y_1, y_2, \cdots, y_n$ on the real line and n complex numbers $\xi_1, \xi_2, \cdots, \xi_n$ such that $\sum_{j=1}^{n}\xi_j=0$. Given an arbitrary positive number ε, we can take a division of $[a, b]$, $a=p_0<p_1<\cdots<p_N=b$, such that

$$|\lambda(y_j-y_k)-\lambda_\varepsilon(y_j-y_k)|<\varepsilon\ (j, k=1, 2, \cdots, n)\,,$$

where $\lambda_\varepsilon(y)=\left(\sum_{l=1}^{N}\left(\frac{\mu_l}{p_l+2\pi iy}+\frac{\nu_l}{p_l-2\pi iy}\right)\right)^{-1}$, and μ_l and ν_l denote respectively μ-and ν-masses of l-th subinterval of the division. Then it holds that

$$\Big|\sum_{j,k}\lambda(y_j-y_k)\xi_j\bar{\xi}_k-\sum_{j,k}\lambda_\varepsilon(y_j-y_k)\xi_j\bar{\xi}_k\Big|\leqq\varepsilon\Big(\sum_j|\xi_j|\Big)^2\,,$$

and that, $\lambda_\varepsilon(y)$ being negative-definite by Proposition 1, $\sum_{j,k}\lambda(y_j-y_k)\xi_j\bar{\xi}_k\leqq 0$.

This lemma leads us to the following

PROPOSITION 2. *If μ and ν are positive Radon measures on $[a, b]$ $(0<a<b<\infty)$, then the convolution kernel K defined by*

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$$K(x) = \begin{cases} \int_a^b e^{-px} d\mu(p) , & x > 0 \\ \int_a^b e^{px} d\nu(p) , & x < 0 \end{cases}$$

satisfies the complete maximum principle.

Proof. It suffices to show the existence of a positive submarkovian resolvent $\{K_q\}$ $(q \geqq 0)$ such that $K_0 = K$. By the above proposition, $q + \lambda(y)$ is a negative-definite continuous function and $0 < \lambda(0) = \left(\int_a^b \frac{d\mu(p)}{p} + \int_a^b \frac{d\nu(p)}{p}\right)^{-1}$ $\leqq \operatorname{Re} \lambda(y)$. Hence $(q + \lambda(y))^{-1}$ is a positive-definite continuous functions and by Bochner's theorem it is the Fourier transform of a positive measure K_q,

$$(q + \lambda(y))^{-1} = \hat{K}_q(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} dK_q(x) .$$

Notice that $|yK(y)|$ is bounded and hence every K_q is a square integrable function. Therefore we have

$$q\widehat{K_0 * K_q} = q\hat{K}_0\hat{K}_q = \frac{q}{\lambda(y)(q + \lambda(y))} = \frac{1}{\lambda(y)} - \frac{1}{q + \lambda(y)} = \hat{K}_0 - \hat{K}_q ,$$

and hence $qK_0 * K_q = K_0 - K_q$. We have also $q\hat{K}_q(0) = \dfrac{q}{q + \lambda(0)} < 1$.

By this proposition we have

THEOREM. *Let μ and ν be positive Radon measures on $(0, \infty)$ such that the function K defined by $(*)$ is locally Lebesgue integrable. Then K satisfies the complete maximum principle.*

Proof. K is the limiting kernel of an increasing sequence of kernels discussed above. Hence our theorem follows immediately from the following lemma.

LEMMA 3. *Let $\{K_n\}$ be an increasing sequence of positive kernels satisfying the complete maximum principle. If $K = \lim K_n$ defines a positive kernel, it satisfies the complete maximum principle.*

Poof. Let f and g be essentially bounded positive Lebesgue measurable functions vanishing outside a compact set such that $Kf \leqq Kg + 1$ on $Sf = \{x \in \boldsymbol{R}; f(x) > 0\}$. We shall prove the inequality $Kf \leqq Kg + 1$

almost everywhere in $\boldsymbol{R}$. Without loss of generality, we may suppose that $Kf < Kg + 1$ on Sf. Let f_n be the restriction of f to the set $\{x \in Sf\,; Kf \leqq K_n g + 1\}$. Then $\{f_n\}$ increases to f. Since K_n satisfies the complete maximum principle, we have $K_n f_n \leqq K_n g + 1 \leqq Kg + 1$ almost everywhere. Noticing that $\{K_n f\}$ increases to Kf, we conclude that $Kf = \lim K_n f_n$, and $Kf \leqq Kg + 1$.

Setting $\mu = \dfrac{\sin \alpha\pi}{\pi} p^{-\alpha}$ and $\nu = \dfrac{\sin \beta\pi}{\pi} p^{-\beta}$, we have

COROLLARY. *$H^\alpha + \check{H}^\beta$ satisfies the complete maximum principle, for any* $0 < \alpha,\ \beta < 1$.

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