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Hodge and Laplace–Beltrami Operators for Bicovariant Differential Calculi on Quantum Groups

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Abstract. For bicovariant differential calculi on quantum matrix groups a generalisation of classical notions such as metric tensor, Hodge operator, codifferential and Laplace–Beltrami operator for arbitrary k-forms is given. Under some technical assumptions it is proved that Woronowicz' external algebra of left-invariant differential forms either contains a unique form of maximal degree or it is infinite-dimensional. Using Jucys–Murphy elements of the Hecke algebra, the eigenvalues of the Laplace–Beltrami operator for the Hopf algebra $\mathcal{O}(\mathrm{SL}_q(N))$ are computed.

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1. Introduction

About ten years ago, S. L. Woronowicz introduced the concept of bicovariant differential calculus on arbitrary Hopf algebras and developed a general theory of such calculi [11]. One of the most interesting parts of this theory is his definition of external algebras and higher-order calculi by using a braiding map instead of the flip operator in the corresponding classical constructions. The higher-order differential calculus defined in this manner becomes then an $\mathbb{N}_0$-graded differential super Hopf algebra ([1], [2]; see [7] for a complete proof). However, applying Woronowicz's construction of higher-order calculi to quantum matrix groups leads to a number of difficulties and phenomena that do not occur in the classical (commutative) case. Firstly, the vector space $(\Gamma^\wedge)_l$ of left-invariant differential forms endowed with the canonical (wedge) product does not form a Grassmann algebra in general. Secondly, it may happen that the dimensions of the spaces $(\Gamma^{\wedge k})_l$ of left-invariant k-forms do not vanish as $k \to +\infty$ (see [6]). For the irreducible N^2-dimensional bicovariant first order differential calculi on the coordinate Hopf algebra $\mathcal{O}(\mathrm{SL}_q(N))$ of the quantum group $\mathrm{SL}_q(N)$, $N \geqslant 2$, a detailed description of the higher order differential calculi $\Gamma^\wedge$ was given by A. Schüler [9]. In this important case it is proved in [9] that for transcendental values of the parameter q the dimension of the vector space of left-invariant k-forms is $\binom{N^2}{k}$ just as in the classical situation.

In 'ordinary' differential geometry the Laplace–Beltrami operator Δ acting on differential forms plays a central role. In its construction a metric tensor, the Hodge star and the codifferential operators are essentially used. The aim of this paper is to give a definition of invariant Laplace–Beltrami operators Δ for inner bicovariant differential calculi on arbitrary Hopf algebras. It will be a generalisation of the classical concept and works also in the case when the higher order calculus is infinite dimensional. The existence of Δ is shown for coquasitriangular Hopf algebras and irreducible differential calculi defined by generalised l-functionals. As tools we use σ-metrics (a generalisation of the concept of a metric tensor in the commutative case), Hodge star operators (in a special case) and codifferentials.

In Section 2 we introduce σ-metrics for a pair of bicovariant bimodules. In Section 3 we give examples for these structures. In Section 4 further basic notions like contractions with forms (see also [3]) and σ-metrics on higher order forms of Woronowicz's external algebra are introduced and a number of useful properties of these mappings are developed. Section 5 is concerned with Hodge operators and codifferential operators. For their definitions we require two assumptions. The first one is that the Hopf algebra is 'connected' (i.e. it has only one one-dimensional corepresentation), and the second assumption is satisfied (for instance) if the left-invariant part of the external algebra is finite dimensional. In Theorem 5.2 it is proved that if there is a left-covariant σ-metric on the external algebra then there exists a unique (up to a complex multiple) left-invariant differential form of maximal degree. For the proof of Theorem 5.2 (and its Corollary 5.3) we don't need the assumption that the Hopf algebra is 'connected'. Further we define Hodge star and codifferential operators and prove some of their properties. One of the formulas for the codifferential operator is independent of the Hodge star and will be taken as a definition in the next section. In Section 6 the invariant Laplace–Beltrami operator is defined and a number of results on this operator are derived. Among others, it is shown (Theorem 6.3) that there is a duality between the differential and codifferential as in the classical case. In Section 7 the eigenvalues of the Laplace–Beltrami operator for the quantum group $\mathrm{SL}_q(N)$, $N \geqslant 2$ are determined.

In this paper we shall use the convention to sum over repeated indices belonging to different terms. Throughout, $\mathcal{A}$ denotes a Hopf algebra over the complex field with comultiplication $\varDelta$ and invertible antipode S. The symbol $\otimes_{\mathcal{A}}$ means the algebraic tensor product over the Hopf algebra $\mathcal{A}$, while $\varDelta_{\mathrm{L}}$ and $\varDelta_{\mathrm{R}}$ denote left and right coactions on a bicovariant $\mathcal{A}$-bimodule, respectively. If u and v are corepresentations of $\mathcal{A}$, then we write $\mathrm{Mor}(u, v)$ for the set of intertwiners of u and v. We set $\mathrm{Mor}(u) = \mathrm{Mor}(u, u)$. Throughout the paper we freely use basic facts from the theory of bicovariant differential calculi (see [11] or [7], Chapter 14).

I want to thank Prof. Schmüdgen for posing the problem and for motivating discussions.

2. σ-Metrics

Let $\mathcal{A}$ be an arbitrary Hopf algebra and let Γ_+ and Γ_- be two finite dimensional bicovariant $\mathcal{A}$-bimodules. Recall that any bicovariant bimodule Γ is a free left and right $\mathcal{A}$-module and there are bases of Γ consisting of left- and right-invariant elements respectively. In what follows we use the symbols $(\Gamma)_{\mathrm{l}}$, $(\Gamma)_{\mathrm{r}}$ and $(\Gamma)_{\mathrm{lr}}$ to denote the vector spaces of left-, right- and biinvariant (i.e. both left- and right-invariant) elements in a bicovariant bimodule Γ. Further, there is a canonical braiding $\sigma : \Gamma_\tau \otimes_{\mathcal{A}} \Gamma_{\tau'} \to \Gamma_{\tau'} \otimes_{\mathcal{A}} \Gamma_\tau$ (defined by Woronowicz [11]) for each $\tau, \tau' \in \{+, -\}$ which is an invertible homomorphism of bicovariant bimodules. We shall write σ^+ for σ and σ^- for σ^{-1}.

DEFINITION 2.1. A linear mapping $g : \Gamma_+ \otimes_{\mathcal{A}} \Gamma_- + \Gamma_- \otimes_{\mathcal{A}} \Gamma_+ \to \mathcal{A}$ is called a *σ-metric of the* (not ordered) *pair* (Γ_+, Γ_-) if it satisfies the following conditions:

- g is a homomorphism of $\mathcal{A}$-bimodules,
- g is nondegenerate, (i.e. for $\xi \in \Gamma_\tau$ both $g(\xi \otimes_{\mathcal{A}} \xi') = 0$ for any $\xi' \in \Gamma_{-\tau}$ and $g(\xi' \otimes_{\mathcal{A}} \xi) = 0$ for any $\xi' \in \Gamma_{-\tau}$ imply $\xi = 0$)
- $g \circ \sigma = g$ (σ-symmetry),
- the following diagrams commute ($\tau, \tau' \in \{+, -\}$):

$$\begin{array}{ccc} \Gamma_\tau \otimes_{\mathcal{A}} \Gamma_{\tau'} \otimes_{\mathcal{A}} \Gamma_{-\tau} & \xrightarrow{\sigma_{23}^{\pm}} & \Gamma_\tau \otimes_{\mathcal{A}} \Gamma_{-\tau} \otimes_{\mathcal{A}} \Gamma_{\tau'} \\ \sigma_{12}^{\mp}\big\downarrow & & \big\downarrow g_{12} \\ \Gamma_{\tau'} \otimes_{\mathcal{A}} \Gamma_\tau \otimes_{\mathcal{A}} \Gamma_{-\tau} & \xrightarrow{g_{23}} & \Gamma_{\tau'} \end{array} \tag{1}$$

The σ-metric of the pair (Γ_+, Γ_-) is said to be *left-covariant* resp. *right-covariant* if

$$\Delta \circ g = (\mathrm{id} \otimes g)\Delta_L \quad \text{resp.} \tag{2}$$

$$\Delta \circ g = (g \otimes \mathrm{id})\Delta_{\mathrm{R}} \tag{3}$$

on $\Gamma_+ \otimes_{\mathcal{A}} \Gamma_- + \Gamma_- \otimes_{\mathcal{A}} \Gamma_+$. We call it *bicovariant* if it is both left- and right-covariant.

If no ambiguity can arise then we use the symbol ‘,’ in order to separate the two arguments of g. Recall that by definition we still have $g(\xi a, \rho) = g(\xi, a\rho)$ for any $a \in \mathcal{A}$, $\xi \in \Gamma_\tau$ and $\rho \in \Gamma_{-\tau}$, $\tau \in \{+, -\}$.

If g is a homomorphism of the $\mathcal{A}$-bimodules $\Gamma_\tau \otimes_{\mathcal{A}} \Gamma_{-\tau}$ and $\mathcal{A}$, $\tau \in \{+, -\}$, (e.g. if g is a σ-metric of the pair (Γ_+, Γ_-)) then on the tensor product $\bigotimes_{m=1}^{k} \Gamma_{\tau_m}$, $\tau_m \in \{+, -\}$ the equation

$$g_{i,i+1} \circ g_{j,j+1} = g_{j-2,j-1} \circ g_{i,i+1} \quad \text{for } k > j > i+1 \tag{4}$$

holds. One can check that the only conditions on the above map to be well defined is

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$\tau_{i+1} = -\tau_i$ and $\tau_{j+1} = -\tau_j$. The formulas

$$g_{i,i+1} \circ \sigma_{j,j+1} = \sigma_{j-2,j-1} \circ g_{i,i+1} \quad \text{and} \quad \sigma_{i,i+1} \circ g_{j,j+1} = g_{j,j+1} \circ \sigma_{i,i+1}, \quad j > i+1, \tag{5}$$

should be clear as well.

Let now g be a σ-metric of the pair (Γ_+, Γ_-). Then on the tensor product $\Gamma_\tau^\otimes \otimes_\mathcal{A} \Gamma_{-\tau}^\otimes$, $\tau \in \{+, -\}$ we define a map $\tilde{g}$ recursively by setting

$$\begin{aligned} &\tilde{g}(\xi, a) := \xi a, \quad \tilde{g}(a, \zeta) := a\zeta, \\ &\tilde{g}(\xi \otimes_\mathcal{A} \xi_1, \zeta_1 \otimes_\mathcal{A} \zeta) := \tilde{g}(\xi g(\xi_1, \zeta_1), \zeta) \end{aligned} \tag{6}$$

for all $\xi \in \Gamma_\tau^\otimes$, $\xi_1 \in \Gamma_\tau$, $\zeta \in \Gamma_{-\tau}^\otimes$, $\zeta_1 \in \Gamma_{-\tau}$ and $a \in \mathcal{A}$.

Since g is a homomorphism of $\mathcal{A}$-bimodules, the map $\tilde{g}$ is well defined and it is a homomorphism of bimodules. Note that $\tilde{g}$ is left-, right- or bicovariant if g is. The next lemma is crucial in what follows.

LEMMA 2.1. *For a σ-metric g of the pair (Γ_+, Γ_-) and arbitrary integers i, k, l such that $1 \leqslant i < k, l$, we have*

$$\tilde{g} \circ (\sigma^\pm_{k-i,k-i+1}, \mathrm{id}^{\otimes l}) = \tilde{g} \circ (\mathrm{id}^{\otimes k}, \sigma^\pm_{i,i+1}) \tag{7}$$

on the bimodule $\Gamma_\tau^{\otimes k} \otimes_\mathcal{A} \Gamma_{-\tau}^{\otimes l}$.

Proof. Because of (6) it suffices to show the assertion for $i = 1$ and $k = l = 2$. But in this case we have $\tilde{g} = g_{12} \circ g_{23}$ and it suffices to apply the fourth condition on the σ-metric g (see (1) in Definition 2.1) twice. We obtain

$$\begin{aligned} \tilde{g}(\sigma^\pm(\xi_1 \otimes_\mathcal{A} \xi_2), \zeta_1 \otimes_\mathcal{A} \zeta_2) &= g_{12} \circ g_{23} \circ \sigma^\pm_{12}(\xi_1 \otimes_\mathcal{A} \xi_2 \otimes_\mathcal{A} \zeta_1 \otimes_\mathcal{A} \zeta_2) \\ &= g_{12} \circ g_{12} \circ \sigma^\mp_{23}(\xi_1 \otimes_\mathcal{A} \xi_2 \otimes_\mathcal{A} \zeta_1 \otimes_\mathcal{A} \zeta_2) \\ &= g_{12} \circ g_{34} \circ \sigma^\mp_{23}(\xi_1 \otimes_\mathcal{A} \xi_2 \otimes_\mathcal{A} \zeta_1 \otimes_\mathcal{A} \zeta_2) \\ &= g_{12} \circ g_{23} \circ \sigma^\pm_{34}(\xi_1 \otimes_\mathcal{A} \xi_2 \otimes_\mathcal{A} \zeta_1 \otimes_\mathcal{A} \zeta_2) \\ &= \tilde{g}(\xi_1 \otimes_\mathcal{A} \xi_2, \sigma^\pm(\zeta_1 \otimes_\mathcal{A} \zeta_2)), \end{aligned}$$

where the third equation follows from (4). □

Let g be a homomorphism of the bicovariant bimodules $\Gamma_+ \otimes_\mathcal{A} \Gamma_- + \Gamma_- \otimes_\mathcal{A} \Gamma_+$ and $\mathcal{A}$. The general theory of bicovariant bimodules assures that g is nondegenerate whenever the matrix of g with respect to one fixed basis of $(\Gamma_+)_\mathrm{l}$ and one fixed basis of $(\Gamma_-)_\mathrm{l}$ is invertible. Conversely, if g is left-covariant (i. e. (2) is fulfilled) then the matrix G of g with respect to any basis of $(\Gamma_+)_\mathrm{l}$ and $(\Gamma_-)_\mathrm{l}$ has complex entries and the nondegeneracy of g implies the invertibility of the matrix G. In this case we easily conclude that the following assertions are equivalent:

(i) g is nondegenerate,

(ii) the restriction of g onto the subspace $(\Gamma_+ \otimes_\mathcal{A} \Gamma_-)_\mathrm{l} + (\Gamma_- \otimes_\mathcal{A} \Gamma_+)_\mathrm{l}$ is nondegenerate,

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(iii) the matrix G of g with respect to one (and then any) basis of $(\Gamma_+)_{\rm l}$ and $(\Gamma_-)_{\rm l}$ is invertible.

Obviously, this holds for left-covariant σ-metrics as well. In what follows most of the σ-metrics will be left-covariant.

3. Examples

Let $\mathcal{A}$ be a coquasitriangular Hopf algebra (see for example [7], Section 10.1) with universal r-form $\mathbf{r}$ and let $u = (u^i_j)_{i,j=1,\dots,d}$ be a corepresentation of $\mathcal{A}$. Then $u^{\rm c} = ((u^{\rm c})^i_j)_{i,j=1,\dots,d}$, $(u^{\rm c})^i_j = S(u^j_i)$ is the contragredient corepresentation of u and u and $u^{\rm c}$ determine two bicovariant $\mathcal{A}$-bimodules Γ_+ and Γ_-, respectively. They are given by fixing the bases $\{\omega_{ij}\,|\,i,j=1,\dots,d\}$ and $\{\theta_{ij}\,|\,i,j=1,\dots,d\}$ of left-invariant forms of Γ_+ resp. Γ_- and defining the right coactions $\Delta_{\rm R}$ and right $\mathcal{A}$-actions $\xi \triangleleft a = S(a_{(1)})\xi a_{(2)}$, $\xi \in (\Gamma_\tau)_{\rm l}$, $\tau \in \{+,-\}$, $a \in \mathcal{A}$, by the formulas

$$\Delta_{\rm R}(\omega_{ij}) = \omega_{kl} \otimes (uu^{\rm c})^{kl}_{ij}, \quad \Delta_{\rm R}(\theta_{ij}) = \theta_{kl} \otimes (u^{\rm cc}u^{\rm c})^{kl}_{ij}, \tag{8}$$

$$\omega_{ij} \triangleleft a = S(l^{-k}_i)l^{+j}_l(a)\omega_{kl} = \mathbf{r}(u^k_i, a_{(1)})\mathbf{r}(a_{(2)}, u^j_l)\omega_{kl}, \tag{9}$$

$$\theta_{ij} \triangleleft a = l^{+i}_k S(l^{-l}_j)(a)\theta_{kl} = \mathbf{r}(a_{(1)}, S(u^k_i))\mathbf{r}(S(u^j_l), a_{(2)})\theta_{kl}. \tag{10}$$

Note that the 1-forms $\omega := \sum_{i=1}^d \omega_{ii} \in \Gamma_+$ and $\theta := (f \circ S)(u^i_j)\theta_{ij} \in \Gamma_-$, where $f(a) = \mathbf{r}(a_{(1)}, S(a_{(2)}))$, are biinvariant.

Assume for a moment that the corepresentations u and $u^{\rm c}$ are equivalent ($u \cong u^{\rm c}$) and let $T = (T^i_j)_{i,j=1,\dots,d}$ be an invertible morphism $T \in \mathrm{Mor}(u, u^{\rm c})$. Clearly we have $T^{-1} \in \mathrm{Mor}(u^{\rm c}, u)$. Then the mapping

$$\theta_{ij} \mapsto \mathbf{r}(u^k_r, u^s_l)(T^{-1})^r_j T^i_s \omega_{kl} \tag{11}$$

extends uniquely to a homomorphism of the bicovariant bimodules Γ_- and Γ_+. Moreover, this mapping is invertible and its inverse is given by

$$\omega_{ij} \mapsto \mathbf{r}(u^r_i, S(u^j_s))T^l_r(T^{-1})^s_k\theta_{kl}. \tag{12}$$

We also see easily that this isomorphism maps θ into ω.

Let now u be an arbitrary corepresentation and let $F_1 \in \mathrm{Mor}(u^{\rm cc}, u)$, $F_2 \in \mathrm{Mor}(u, u^{\rm cc})$ and $G_1, G_2 \in \mathrm{Mor}(u)$ be invertible morphisms. Then we define linear maps $g' : \Gamma_+ \otimes_{\mathcal{A}} \Gamma_- \to \mathcal{A}$ and $g'' : \Gamma_- \otimes_{\mathcal{A}} \Gamma_+ \to \mathcal{A}$ by

$$g'(a\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl}) = aF_1{}^j_k F_2{}^l_i \quad \text{and} \quad g''(a\theta_{ij} \otimes_{\mathcal{A}} \omega_{kl}) = aG_1{}^j_k G_2{}^l_i. \tag{13}$$

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LEMMA 3.1. *The mappings* $g' : \Gamma_+ \otimes_{\mathcal{A}} \Gamma_- \to \mathcal{A}$ *and* $g'' : \Gamma_- \otimes_{\mathcal{A}} \Gamma_+ \to \mathcal{A}$ *are homomorphisms of bicovariant bimodules. Moreover, as bilinear forms they are nondegenerate.*

Proof. Firstly let us show that $g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl} a) = g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})a$. For this we compute

$$\begin{aligned}
g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl} a) &= g'(a_{(1)}\mathbf{r}(u^r_i, a_{(2)})\mathbf{r}(a_{(3)}, u^j_s)\mathbf{r}(a_{(4)}, S(u^p_k))\mathbf{r}(S(u^l_n), a_{(5)})\omega_{rs} \otimes_{\mathcal{A}} \theta_{pn})\\
&= a_{(1)}\mathbf{r}(u^r_i, a_{(2)})\mathbf{r}(a_{(3)}, u^j_s)\mathbf{r}(a_{(4)}, S(u^p_k))\mathbf{r}(S(u^l_n), a_{(5)}){F_1}^s_p{F_2}^n_r\\
&= a_{(1)}\mathbf{r}({F_2}^n_r u^r_i, a_{(2)})\mathbf{r}(a_{(3)}, S(u^p_k)u^j_s {F_1}^s_p)\mathbf{r}(S(u^l_n), a_{(4)})\\
&= a_{(1)}\mathbf{r}({F_2}^n_r u^r_i, a_{(2)})\mathbf{r}(a_{(3)}, S(u^p_k){F_1}^j_s S^2(u^s_p))\mathbf{r}(S(u^l_n), a_{(4)})\\
&= {F_1}^j_s a_{(1)}\mathbf{r}({F_2}^n_r u^r_i, a_{(2)})\mathbf{r}(a_{(3)}, S(S(u^s_p)u^p_k))\mathbf{r}(S(u^l_n), a_{(4)})\\
&= {F_1}^j_k a_{(1)}\mathbf{r}({F_2}^n_r u^r_i, a_{(2)})\mathbf{r}(S(u^l_n), a_{(3)})\\
&= {F_1}^j_k a_{(1)}\mathbf{r}(S^2(u^n_r){F_2}^r_i S(u^l_n), a_{(2)})\\
&= {F_1}^j_k {F_2}^r_i a_{(1)}\mathbf{r}(S(u^l_n S(u^n_r)), a_{(2)})\\
&= {F_1}^j_k {F_2}^l_i a = g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})a.
\end{aligned}$$

Secondly we prove the covariance of g', that is

$$(\mathrm{id} \otimes g')\Delta_{\mathrm{L}} = \Delta \circ g' \quad \text{and} \quad (g' \otimes \mathrm{id})\Delta_{\mathrm{R}} = \Delta \circ g' \tag{14}$$

as a mapping from $\Gamma_+ \otimes_{\mathcal{A}} \Gamma_-$ to $\mathcal{A} \otimes \mathcal{A}$. Similarly to the proof of Lemma 2.1 in [5] one can show that the equations (14) are equivalent to $g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl}) \in \mathbb{C}$ and $g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})(uu^{\mathrm{c}}u^{\mathrm{cc}}u^{\mathrm{c}})^{ijkl}_{mnrs} = g'(\omega_{mn} \otimes_{\mathcal{A}} \theta_{rs})$. The first one is trivial. For the second we compute

$$\begin{aligned}
g'(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})(uu^{\mathrm{c}}u^{\mathrm{cc}}u^{\mathrm{c}})^{ijkl}_{mnrs} &= {F_1}^j_k {F_2}^l_i u^i_m S(u^n_j)S^2(u^k_r)S(u^s_l)\\
&= {F_2}^l_i u^i_m S(u^n_j)u^j_k {F_1}^k_r S(u^s_l) = {F_1}^n_r S^2(u^l_i){F_2}^i_m S(u^s_l)\\
&= {F_1}^n_r {F_2}^i_m S(u^s_l S(u^l_i)) = {F_1}^n_r {F_2}^s_m = g'(\omega_{mn} \otimes_{\mathcal{A}} \theta_{rs}).
\end{aligned}$$

Hence the assertion follows.

Thirdly we have to prove the nondegeneracy of g'. We shall carry out the proof only for the second argument of g'. Let ρ be an arbitrary element of Γ_-. Then there are elements $a_{ij} \in \mathcal{A}$ such that $\rho = \theta_{ij}a_{ij}$. Assume that $g'(\rho' \otimes_{\mathcal{A}} \rho) = 0$ for all $\rho' \in \Gamma_+$. Inserting $\rho' = \omega_{kl}$, $k, l = 1, \ldots, d$ and using that g' is a right $\mathcal{A}$-linear mapping, we obtain ${F_1}^l_i {F_2}^j_k a_{ij} = 0$ for all k, l and the invertibility of F_1 and F_2 gives $a_{ij} = 0$. Hence $\rho = 0$. □

Assume for a moment that the corepresentations u and u^{c} are equivalent and let us identify Γ_- and Γ_+ via the isomorphism (11).

LEMMA 3.2. *Suppose that the corepresentations u and u^{c} are equivalent. Then the homomorphisms g' and $g'' : \Gamma_+ \otimes_{\mathcal{A}} \Gamma_+ \to \mathcal{A}$ in Lemma 3.1 coincide if and only if there*

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is a nonzero complex number c such that

$$F_1{}^i_j = c(T^{-1})^i_r G_2{}^s_r T^j_s \quad \text{and} \quad F_2{}^i_j = c^{-1}(T^{-1})^r_i G_1{}^s_r T^s_j. \tag{15}$$

Proof. Since g' and g'' are homomorphisms of $\mathcal{A}$-bimodules it suffices to prove the assertion on the vector space $(\Gamma_+)_l \otimes (\Gamma_+)_l$. Inserting (12) into the definition of g' and g'', it follows that $g'(\omega_{ij} \otimes_{\mathcal{A}} \omega_{kl}) = g''(\omega_{ij} \otimes_{\mathcal{A}} \omega_{kl})$ if and only if

$$T^y_r(T^{-1})^s_x \mathbf{r}(u^r_k, S(u^l_s))F_1{}^j_x F_2{}^y_i = \mathbf{r}(u^r_i, S(u^j_s))T^y_r(T^{-1})^s_x G_1{}^y_k G_2{}^l_x$$

for any i, j, k, l. For the right hand side we compute

$$\begin{aligned}\mathbf{r}(T^y_r u^r_i, S(u^j_s(T^{-1})^s_x))G_1{}^y_k G_2{}^l_x &= \mathbf{r}(S(u^r_y)T^r_i, S((T^{-1})^j_s S(u^x_s)))G_1{}^y_k G_2{}^l_x \\ &= T^r_i(T^{-1})^j_s \mathbf{r}(u^r_y G_1{}^y_k, S(G_2{}^l_x u^x_s)) = T^r_i(T^{-1})^j_s \mathbf{r}(G_1{}^r_y u^y_k, S(u^l_x G_2{}^x_s))\end{aligned}$$

and hence the lemma is valid if and only if

$$T^y_r(T^{-1})^s_x \mathbf{r}(u^r_k, S(u^l_s))F_1{}^j_x F_2{}^y_i = T^r_i(T^{-1})^j_s \mathbf{r}(G_1{}^r_y u^y_k, S(u^l_x G_2{}^x_s)).$$

Multiplying this equation by $\mathbf{r}(u^k_z, u^t_l)(T^{-1})^z_n T^m_t$ we obtain the equivalent condition

$$F_1{}^j_m F_2{}^n_i = \big((T^{-1})^j_s G_2{}^t_s T^m_t\big)\big((T^{-1})^z_n G_1{}^r_z T^r_i\big)$$

for any i, j, m, n, from which the assertion follows. □

Now let u be an arbitrary corepresentation of $\mathcal{A}$ and let g be the homomorphism from $\Gamma_+ \otimes_{\mathcal{A}} \Gamma_- + \Gamma_- \otimes_{\mathcal{A}} \Gamma_+$ to $\mathcal{A}$ given by g' and g''. To prove the third and fourth conditions of Definition 2.1 for g let us recall the following explicit formulas for the braiding σ (see [7], Section 13.1):

$$\begin{aligned}\sigma(\omega_{ij} \otimes_{\mathcal{A}} \omega_{kl}) &= \mathbf{r}(u^r_t, S(u^y_n))\mathbf{r}(u^t_i, u^m_x)\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(u^x_k, u^j_z)\omega_{mn} \otimes_{\mathcal{A}} \omega_{rs}, \\ \sigma(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl}) &= \mathbf{r}(u^r_t, S(u^y_n))\mathbf{r}(u^t_i, S^2(u^m_x))\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(S^2(u^x_k), u^j_z)\theta_{mn} \otimes_{\mathcal{A}} \omega_{rs}, \\ \sigma(\theta_{ij} \otimes_{\mathcal{A}} \omega_{kl}) &= \mathbf{r}(u^y_n, u^r_t)\mathbf{r}(u^m_x, S(u^t_i))\mathbf{r}(u^z_s, u^l_y)\mathbf{r}(S(u^j_z), u^x_k)\omega_{mn} \otimes_{\mathcal{A}} \theta_{rs}, \\ \sigma(\theta_{ij} \otimes_{\mathcal{A}} \theta_{kl}) &= \mathbf{r}(u^y_n, u^r_t)\mathbf{r}(S(u^m_x), u^t_i)\mathbf{r}(u^z_s, u^l_y)\mathbf{r}(u^j_z, S(u^x_k))\theta_{mn} \otimes_{\mathcal{A}} \theta_{rs}.\end{aligned}$$

The inverses σ^{-1} of these braidings take the form

$$\begin{aligned}\sigma^{-1}(\omega_{ij} \otimes_{\mathcal{A}} \omega_{kl}) &= \mathbf{r}(u^r_t, S(u^y_n))\mathbf{r}(S(u^m_x), u^t_i)\mathbf{r}(u^z_s, u^l_y)\mathbf{r}(u^x_k, u^j_z)\omega_{mn} \otimes_{\mathcal{A}} \omega_{rs}, \\ \sigma^{-1}(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl}) &= \mathbf{r}(S^2(u^y_n), u^r_t)\mathbf{r}(u^t_i, S^2(u^m_x))\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(u^j_z, S(u^x_k))\theta_{mn} \otimes_{\mathcal{A}} \omega_{rs}, \\ \sigma^{-1}(\theta_{ij} \otimes_{\mathcal{A}} \omega_{kl}) &= \mathbf{r}(S(u^r_t), u^y_n)\mathbf{r}(u^m_x, S(u^t_i))\mathbf{r}(u^z_s, u^l_y)\mathbf{r}(u^x_k, u^j_z)\omega_{mn} \otimes_{\mathcal{A}} \theta_{rs}, \\ \sigma^{-1}(\theta_{ij} \otimes_{\mathcal{A}} \theta_{kl}) &= \mathbf{r}(u^y_n, u^r_t)\mathbf{r}(u^t_i, u^m_x)\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(u^j_z, S(u^x_k))\theta_{mn} \otimes_{\mathcal{A}} \theta_{rs}.\end{aligned}$$

PROPOSITION 3.3. *Let* $F_1 \in \mathrm{Mor}(u^{\mathrm{cc}}, u)$*,* $F_2 \in \mathrm{Mor}(u, u^{\mathrm{cc}})$ *and* $G_1, G_2 \in \mathrm{Mor}(u)$ *be arbitrary invertible morphisms. Then the bilinear map* $g : \Gamma_+ \otimes_{\mathcal{A}} \Gamma_- + \Gamma_- \otimes_{\mathcal{A}} \Gamma_+$

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$\to \mathcal{A}$ given by g' and g'' in Lemma 3.1 satisfies the fourth condition

$$g_{12}\sigma_{23}^{\pm}(\xi_1 \otimes_{\mathcal{A}} \xi_2 \otimes_{\mathcal{A}} \xi_3) = g_{23}\sigma_{12}^{\mp}(\xi_1 \otimes_{\mathcal{A}} \xi_2 \otimes_{\mathcal{A}} \xi_3),$$

$\xi_1 \in \Gamma_\tau$, $\xi_2 \in \Gamma_{\tau'}$, $\xi_3 \in \Gamma_{-\tau}$, $\tau, \tau' \in \{+, -\}$, *of Definition 2.1.*

Proof. Since g is a homomorphism of $\mathcal{A}$-bimodules (see Lemma 3.1) it suffices to prove the assertion on the vector spaces $(\Gamma_\tau)_l \otimes (\Gamma_{\tau'})_l \otimes (\Gamma_{-\tau})_l$, $\tau, \tau' \in \{+, -\}$. We have to consider four cases which correspond to the possible values of τ and τ'. Since the proofs are very similar, we only show the assertion $g_{12}\sigma_{23}^{-1} = g_{23}\sigma_{12}$ for $\tau = +$ and $\tau' = -$. We will only use the formula $\mathbf{r}(S(a), S(b)) = \mathbf{r}(a, b)$ for any $a, b \in \mathcal{A}$ and the properties of F_i and G_i, $i = 1, 2$.

$$\begin{aligned}
&g_{23}\big(\sigma(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl}) \otimes_{\mathcal{A}} \theta_{ab}\big)\\
&\quad= \mathbf{r}(u^r_t, S(u^y_n))\mathbf{r}(u^t_i, S^2(u^m_x))\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(S^2(u^x_k), u^j_z)\theta_{mn}{F_1}^s_a{F_2}^b_r\\
&\quad= \mathbf{r}(S^2(u^b_r), S(u^y_n))\mathbf{r}(u^t_i, S^2(u^m_x))\mathbf{r}(S(u^l_y), u^z_s)\mathbf{r}(S^2(u^x_k), u^j_z)\theta_{mn}{F_1}^s_a{F_2}^r_t\\
&\quad= \mathbf{r}(S(u^b_r), u^y_n)\mathbf{r}(u^t_i, S^2(u^m_x))\mathbf{r}(S(u^l_y), S^2(u^s_a))\mathbf{r}(S^2(u^x_k), u^j_z)\theta_{mn}{F_1}^z_s{F_2}^r_t\\
&\quad= \mathbf{r}(S(u^b_r), u^y_n)\mathbf{r}(S^2(u^r_t), S^2(u^m_x))\mathbf{r}(u^l_y, S(u^s_a))\mathbf{r}(S^2(u^x_k), S^2(u^z_s))\theta_{mn}{F_1}^j_z{F_2}^t_i\\
&\quad= \mathbf{r}(u^r_t, u^m_x)\mathbf{r}(u^x_k, u^z_s)\mathbf{r}(S(u^b_r), u^y_n)\mathbf{r}(u^l_y, S(u^s_a))\theta_{mn}{F_1}^j_z{F_2}^t_i\\
&\quad= g_{12}\big(\omega_{ij} \otimes_{\mathcal{A}} \sigma^{-1}(\theta_{kl} \otimes_{\mathcal{A}} \theta_{ab})\big).
\end{aligned}$$

□

Let us introduce the functional $f : \mathcal{A} \to \mathbb{C}$ (see [7], Proposition 10.3) defined by $f(a) = \mathbf{r}(a_{(1)}, S(a_{(2)}))$ and let $\bar{f}$ denote the convolution inverse of f, i.e. $\bar{f}(a) = \mathbf{r}(S^2(a_{(1)}), a_{(2)})$.

PROPOSITION 3.4. *Let g be as in Proposition 3.3. Then the bilinear map g is σ-symmetric if and only if there are complex numbers c and z such that $f(S(u^i_j)) = z\bar{f}(u^i_j)$ and*

$${F_1}^i_j = c{G_2}^i_k f(u^k_j) \qquad \text{and} \qquad {F_2}^i_j = c^{-1}\bar{f}(u^i_k){G_1}^k_j$$

for $i, j = 1, \ldots, d$.

Proof. Firstly let us suppose that $g\sigma = g$. From the equation $g(\sigma(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})) = g(\omega_{ij} \otimes_{\mathcal{A}} \theta_{kl})$ we conclude that there is a nonzero complex number c' such that

$${F_1}^i_j = c'{G_2}^i_k\bar{f}(S(u^k_j)) \quad \text{and} \quad {F_2}^i_j = c'^{-1}f(S(u^i_k)){G_1}^k_j. \tag{$*$}$$

Further, $g(\sigma(\theta_{ij} \otimes_{\mathcal{A}} \omega_{kl})) = g(\theta_{ij} \otimes_{\mathcal{A}} \omega_{kl})$ gives ${G_1}^i_j = cf(u^i_k){F_2}^k_j$ and ${G_2}^i_j = c^{-1}{F_1}^i_k\bar{f}(u^k_j)$ for some nonzero complex number c. Inserting this into $(*)$ we obtain ${F_2}^i_j = c'^{-1}cf(S(u^i_k))f(u^k_l){F_2}^l_j$ for any i, j. Multiplying by $(F_2{}^{-1})^j_m\bar{f}(u^m_n)$ and summing up over j we obtain $\bar{f}(u^i_n) = c'^{-1}cf(S(u^i_n))$. Inverting this equation we also get $f(u^i_j) = c'c^{-1}\bar{f}(S(u^i_j))$. Let us set $z = c'c^{-1}$. Then $(*)$ gives the assertion. The converse direction is an easy computation. □

4. Contractions

Let Γ_+ and Γ_- be two bicovariant $\mathcal{A}$-bimodules over the Hopf algebra $\mathcal{A}$. Let $\Gamma_\tau^\wedge = \bigoplus_{k=0}^\infty \Gamma_\tau^{\wedge k}$, $\tau \in \{+,-\}$ denote the external algebra for Γ_τ as constructed by Woronowicz [11]. This means that there is an antisymmetrizer $A_k : \Gamma_\tau^{\otimes k} \to \Gamma_\tau^{\otimes k}$ for each $k \geqslant 0$ ($A_0 = A_1 = \mathrm{id}$) which is a homomorphism of bicovariant bimodules and $\Gamma_\tau^{\wedge k} = \Gamma_\tau^{\otimes k}/\ker A_k$. Let us recall some properties of A_k. Because of the general theory there are bimodule homomorphisms $A_{i,j}, B_{i,j} : \Gamma_\tau^{\otimes i+j} \to \Gamma_\tau^{\otimes i+j}$, $i,j \geqslant 0$ such that

$$A_{i+j} = A_{i,j}(A_i \otimes_{\mathcal{A}} A_j),\ A_{i+j} = (A_i \otimes_{\mathcal{A}} A_j)B_{i,j}. \tag{16}$$

In particular we have

$$A_i = \prod_{k=0}^{i-1}(A_{i-k-1,1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes k}) = \prod_{k=0}^{i-1}(\mathrm{id}^{\otimes k} \otimes_{\mathcal{A}} A_{1,i-k-1}) \tag{17}$$

$$A_i = \prod_{k=0}^{i-1}(B_{k,1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes i-k-1}) = \prod_{k=0}^{i-1}(\mathrm{id}^{\otimes i-k-1} \otimes_{\mathcal{A}} B_{1,k}), \tag{18}$$

where $A_{0,0} = A_{1,0} = A_{0,1} = \mathrm{id}$, $B_{0,0} = B_{1,0} = B_{0,1} = \mathrm{id}$ and

$$A_{1,i} = \mathrm{id} - \sigma_{12} + \sigma_{23}\sigma_{12} - \ldots + (-1)^i \sigma_{i,i+1} \cdots \sigma_{12}, \tag{19}$$

$$A_{i,1} = \mathrm{id} - \sigma_{i,i+1} + \sigma_{i-1,i}\sigma_{i,i+1} - \ldots + (-1)^i \sigma_{12} \cdots \sigma_{i,i+1}, \tag{20}$$

$$B_{1,i} = \mathrm{id} - \sigma_{12} + \sigma_{12}\sigma_{23} - \ldots + (-1)^i \sigma_{12} \cdots \sigma_{i,i+1}, \tag{21}$$

$$B_{i,1} = \mathrm{id} - \sigma_{i,i+1} + \sigma_{i,i+1}\sigma_{i-1,i} - \ldots + (-1)^i \sigma_{i,i+1} \cdots \sigma_{12}. \tag{22}$$

It is easy to see that

$$A_{1,i} = \mathrm{id} - (\mathrm{id} \otimes_{\mathcal{A}} A_{1,i-1})\sigma_{12}, \qquad A_{i,1} = \mathrm{id} - (A_{i-1,1} \otimes_{\mathcal{A}} \mathrm{id})\sigma_{i,i+1}, \tag{23}$$

$$B_{1,i} = \mathrm{id} - \sigma_{12}(\mathrm{id} \otimes_{\mathcal{A}} B_{1,i-1}), \quad B_{i,1} = \mathrm{id} - \sigma_{i,i+1}(B_{i-1,1} \otimes_{\mathcal{A}} \mathrm{id}) \tag{24}$$

for $i > 0$. One could also take (19) and (17) for the definition of A_k.

The preceding properties hold for any $\mathcal{A}$-bimodule isomorphism σ which satisfies the braid relation. Therefore, replacing everywhere σ by σ^{-1} the above works as well. In what follows we will use both kinds of operators and write A_k^τ, $A_{i,j}^\tau$ and $B_{i,j}^\tau$ whenever we are dealing with σ^τ ($\tau \in \{+,-\}$).

Let us introduce some operators in $\mathrm{End}(\bigotimes_{i=1}^m \Gamma_{\tau_i})$, $m \geqslant 1$ and $1 \leqslant j,k \leqslant m$ (they can be associated to the permutations $(j, j+1, \ldots, k)$, $(k, k-1, \ldots, j)$, $(1,m)(2,m-1)(3,m-2)\cdots$ and $(1,2,\ldots,j+1)(2,3,\ldots,j+2)\cdots(k,k+1,\ldots$

$j+k)$):

$$\sigma^{\pm}_{[j\to k]} := \sigma^{\pm}_{j,j+1}\sigma^{\pm}_{j+1,j+2}\cdots\sigma^{\pm}_{k-1,k} \quad \text{for } j<k, \quad \sigma^{\pm}_{[j\to k]} = \mathrm{id} \quad \text{for } j \geqslant k, \tag{25}$$

$$\sigma^{\pm}_{[j\leftarrow k]} := \sigma^{\pm}_{k-1,k}\sigma^{\pm}_{k-2,k-1}\cdots\sigma^{\pm}_{j,j+1} \quad \text{for } j<k, \quad \sigma^{\pm}_{[j\leftarrow k]} = \mathrm{id} \quad \text{for } j \geqslant k, \tag{26}$$

$$\sigma^{\pm}_{(m)} := \sigma^{\pm}_{[1\leftarrow 1]}\sigma^{\pm}_{[1\leftarrow 2]}\cdots\sigma^{\pm}_{[1\leftarrow m]} \quad \text{for } m \geqslant 1, \quad \sigma^{\pm}_{(0)} = \mathrm{id}, \tag{27}$$

$$\sigma^{\pm}_{(j,k)} := \sigma^{\pm}_{[k\to j+k]}\sigma^{\pm}_{[k-1\to j+k-1]}\cdots\sigma^{\pm}_{[1\to j+1]} \quad \text{for } m=j+k. \tag{28}$$

The verification of the following equations needs only braid group techniques and is left to the reader. We have

$$\sigma^{\pm}_{(k)} = \sigma_{[1\to k]}(\sigma^{\pm}_{(k-1)} \otimes_{\mathcal{A}} \mathrm{id}) = \sigma^{\pm}_{[1\leftarrow k]}(\mathrm{id} \otimes_{\mathcal{A}} \sigma^{\pm}_{(k-1)}), \tag{29}$$

$$\sigma^{\pm}_{[1\to k]}\sigma^{\pm}_{[1\leftarrow k-1]} = \sigma^{\pm}_{[1\leftarrow k]}\sigma^{\pm}_{[2\to k]} \tag{30}$$

for $k \geqslant 2$ and

$$\sigma^{\pm}_{[1\to k]}(b_{k-1} \otimes_{\mathcal{A}} \mathrm{id}) = (\mathrm{id} \otimes_{\mathcal{A}} b_{k-1})\sigma^{\pm}_{[1\to k]}, \tag{31}$$

$$\sigma^{\pm}_{[1\leftarrow k]}(\mathrm{id} \otimes_{\mathcal{A}} b_{k-1}) = (b_{k-1} \otimes_{\mathcal{A}} \mathrm{id})\sigma^{\pm}_{[1\leftarrow k]}, \tag{32}$$

$$\sigma^{\pm}_{(j,k)} = \sigma^{\pm}_{[1\leftarrow k+1]}\sigma^{\pm}_{[2\leftarrow k+2]}\cdots\sigma^{\pm}_{[j\leftarrow k+j]} \tag{33}$$

for $k \geqslant 1$, where b_k is an arbitrary expression of the complex algebra generated by $\sigma_{12}, \ldots, \sigma_{k-1,k}$ and their inverses. Observe that $A^{\pm}_k \sigma^{\mp}_{(k)} = \sigma^{\mp}_{(k)} A^{\pm}_k = (-1)^{k(k-1)/2} A^{\mp}_k$ (see [11], p. 157). Hence, in particular we have $\ker A^{+}_k = \ker A^{-}_k$.

Now let g be a σ-metric of the pair (Γ_+, Γ_-). The next formulas follow from the fourth condition on the σ-metric by induction over k:

$$g_{12}\sigma^{\pm}_{[2\to k]}\sigma^{\pm}_{[1\to k-1]} = g_{k-1,k}, \quad g_{k-1,k}\sigma^{\pm}_{[1\leftarrow k-1]}\sigma^{\pm}_{[2\leftarrow k]} = g_{12} \quad \text{for } k \geqslant 2. \tag{34}$$

Next we define *contractions* $\langle\cdot,\cdot\rangle_{\pm} : \Gamma^{\otimes k}_{\tau} \otimes_{\mathcal{A}} \Gamma^{\otimes l}_{-\tau} \to \Gamma^{\otimes |k-l|}_{\tau'}$, $\tau \in \{+,-\}$, $\tau' = \tau$ for $k \geqslant l$, otherwise $\tau' = -\tau$, by

$$\begin{aligned} \langle \xi, \xi' \rangle_{\pm} &:= \tilde{g}(B^{\pm}_{k-l,l}\xi, A^{\pm}_{l}\xi') \quad \text{for } k \geqslant l, \\ \langle \xi, \xi' \rangle_{\pm} &:= \tilde{g}(A^{\pm}_{k}\xi, B^{\pm}_{k,l-k}\xi') \quad \text{for } k < l. \end{aligned} \tag{35}$$

This maps are homomorphisms of $\mathcal{A}$-bimodules and inherit all covariance properties of g. If both k and l are less than two, then the contraction doesn't depend on the sign $\pm$ and we sometimes omit it: $\langle \xi, \xi' \rangle_{+} = \langle \xi, \xi' \rangle - =: \langle \xi, \xi' \rangle$.

Next we prove a generalisation of Lemma 2.1.

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LEMMA 4.1. *Let g be a σ-metric of the pair* (Γ_+, Γ_-) *and let* $\tilde{g}$ *be the map defined by (6). Then we have for all nonnegative integers* $i, j, k, l, 1 \leqslant i+j \leqslant k, l,$

$$\tilde{g} \circ \big((\mathrm{id}^{\otimes k-i-j} \otimes_{\mathcal{A}} A_i^{\pm} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes j}), \mathrm{id}^{\otimes l}\big) = \tilde{g} \circ \big(\mathrm{id}^{\otimes k}, (\mathrm{id}^{\otimes j} \otimes_{\mathcal{A}} A_i^{\pm} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes l-i-j})\big).$$

Proof. Using Lemma 2.1 one checks that

$$\tilde{g} \circ (\sigma^{\pm}_{[k+1-t' \rightleftarrows k+1-t]}, \mathrm{id}^{\otimes l}) = \tilde{g} \circ (\mathrm{id}^{\otimes k}, \sigma^{\pm}_{[t \rightleftarrows t']}) \tag{36}$$

for $1 \leqslant t \leqslant t' \leqslant k, l$. From this and Equations (22) and (19) we obtain

$$\begin{aligned}
&\tilde{g} \circ \big((\mathrm{id}^{\otimes k-r-s} \otimes_{\mathcal{A}} A^{\pm}_{1,s-1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes r}), \mathrm{id}^{\otimes l}\big) \\
&\quad = \tilde{g} \circ \left(\mathrm{id}^{\otimes k-r-s} \otimes_{\mathcal{A}} \left(\sum_{t=1}^{s} (-1)^{t+1} \sigma^{\pm}_{[1 \leftarrow t]}\right) \otimes_{\mathcal{A}} \mathrm{id}^{\otimes r}, \mathrm{id}^{\otimes l}\right) \\
&\quad = \tilde{g} \circ \left(\left(\sum_{t=1}^{s} (-1)^{t+1} \sigma^{\pm}_{[k+1-r-s \leftarrow k-r-s+t]}\right), \mathrm{id}^{\otimes l}\right) \\
&\quad = \tilde{g} \circ \left(\mathrm{id}^{\otimes k}, \left(\sum_{t=1}^{s} (-1)^{t+1} \sigma^{\pm}_{[r+s-t+1 \leftarrow r+s]}\right)\right) \\
&\quad = \tilde{g} \circ \left(\mathrm{id}^{\otimes k}, \mathrm{id}^{\otimes r} \otimes_{\mathcal{A}} \left(\sum_{t=1}^{s} (-1)^{t+1} \sigma^{\pm}_{[s-t+1 \leftarrow s]}\right) \otimes_{\mathcal{A}} \mathrm{id}^{\otimes l-r-s}\right) \\
&\quad = \tilde{g} \circ \big(\mathrm{id}^{\otimes k}, (\mathrm{id}^{\otimes r} \otimes_{\mathcal{A}} B^{\pm}_{s-1,1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes l-r-s})\big)
\end{aligned}$$

for all r, s with $0 \leqslant r$, $1 \leqslant s$, $r+s \leqslant k$, $r+s \leqslant l$. Using this result together with (17) and (18) similar computations give the assertion of the lemma. □

LEMMA 4.2. *Let* $\tilde{g}$ *be as in Lemma 4.1 and* $\xi_k \in \Gamma_\tau^{\otimes k}$, $\xi'_l \in \Gamma_{-\tau}^{\otimes e}$, $\tau \in \{+,-\}$, $k, l \geqslant 0$. *Then* $\sigma^{\pm}_{(k,l)}(\sigma^{\pm}_{(k)}(\xi_k), \sigma^{\pm}_{(l)}(\xi'_l))$ *is an element of* $\Gamma_{-\tau}^{\otimes l} \otimes_{\mathcal{A}} \Gamma_\tau^{\otimes k}$ *and the equation*

$$\tilde{g}\big(\sigma^{\pm}_{(k,l)}(\sigma^{\pm}_{(k)}(\xi_k), \sigma^{\pm}_{(l)}(\xi'_l))\big) = \sigma^{\pm}_{(|k-l|)}(\tilde{g}(\xi_k, \xi'_l)) \tag{37}$$

holds.

Proof. We prove the case $k \geqslant l$ by induction on l. Then the assertion follows also for $k < l$ because of the formulas $\sigma^{\pm}_{(i)}\sigma^{\mp}_{(i)} = \mathrm{id}$, $\sigma^{\pm}_{(i,j)}(A_i^{\pm}, \mathrm{id}^{\otimes j}) = (\mathrm{id}^{\otimes j}, A_i^{\pm})\sigma^{\pm}_{(i,j)}$ (see also (28) and (29)) and $\sigma^{\pm}_{(i,j)}\sigma^{\mp}_{(j,i)} = \mathrm{id}$ for all $i, j \geqslant 0$.

If $l = 0$ then $\sigma^{\pm}_{(l)} = \sigma^{\pm}_{(k,l)} = \mathrm{id}$, hence the left-hand side of (37) is equal to $\tilde{g}(\sigma^{\pm}_{(k)}(\xi_k), \xi'_l) = \sigma^{\pm}_{(k)}(\xi_k)\xi'_l$. For the right hand side we obtain $\sigma^{\pm}_{(k-l)}\tilde{g}(\xi_k, \xi'_l) = \sigma^{\pm}_{(k-l)}(\xi_k\xi'_l)$. Since $\xi'_l \in \mathcal{A}$ and $\sigma^{\pm}_{(k-l)}$ is a homomorphism of $\mathcal{A}$-bimodules, the assertion of the lemma is valid.

Suppose that (37) holds for an $l \in \mathbb{N}_0$, $l \leqslant k$. Consider the map

$$\tilde{g}\sigma^{\pm}_{(k+1,l+1)}(\sigma^{\pm}_{(k+1)}, \sigma^{\pm}_{(l+1)}) : \Gamma_\tau^{\otimes k+1} \otimes_{\mathcal{A}} \Gamma_{-\tau}^{\otimes l+1} \to \Gamma_\tau^{\otimes k-e}.$$

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We compute

$$\begin{aligned}
&\tilde{g}\sigma^{\pm}_{(k+1,l+1)}(\sigma^{\pm}_{(k+1)} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l+1)})\\
&\quad = \tilde{g}g_{l+1,l+2}\sigma^{\pm}_{[l+1\to k+l+2]}(\sigma^{\pm}_{(k+1,l)} \otimes_{\mathcal{A}} \mathrm{id})(\sigma^{\pm}_{[1\to k+1]}(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id}) \otimes_{\mathcal{A}} \sigma^{\pm}_{(l+1)})\\
&\quad = \tilde{g}g_{l+1,l+2}\sigma^{\pm}_{[l+2\to k+l+2]}\sigma^{\pm}_{[k+1\to k+l+1]}(\sigma^{\pm}_{(k+1,l)} \otimes_{\mathcal{A}} \mathrm{id})(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l+1)})\\
&\quad = \tilde{g}g_{k+l+1,k+l+2}(\sigma^{\pm}_{(k+1,l)} \otimes_{\mathcal{A}} \mathrm{id})(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l+1)})\\
&\quad = \tilde{g}g_{k+l+1,k+l+2}(\sigma^{\pm}_{(k,l)} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2})\sigma^{\pm}_{[k+1\leftarrow k+l+1]}\sigma^{\pm}_{[k+2\leftarrow k+l+2]}(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l)})\\
&\quad = \tilde{g}\sigma^{\pm}_{(k,l)}g_{k+l+1,k+l+2}\sigma^{\pm}_{[k+1\leftarrow k+l+1]}\sigma^{\pm}_{[k+2\leftarrow k+l+2]}(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l)})\\
&\quad = \tilde{g}\sigma^{\pm}_{(k,l)}g_{k+1,k+2}(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l)})\\
&\quad = \tilde{g}\sigma^{\pm}_{(k,l)}(\sigma^{\pm}_{(k)} \otimes_{\mathcal{A}} g_{12} \otimes_{\mathcal{A}} \sigma^{\pm}_{(l)}) = \tilde{g}g_{k+1,k+2} = \tilde{g}
\end{aligned}$$

where we used the following formulas: (28) and (29) in the first equation, the σ-symmetry of the σ-metric, (28) and (31) in the second, (34) in the third, (33) and (29) in the fourth, (34) in the sixth, the induction assumption in the eighth and the recursive definition of $\tilde{g}$ in the last equation. □

An important consequence of Lemma 4.1 is the possibility to extend the definition of our contractions $\langle\cdot,\cdot\rangle_\pm$ to a map $\langle\cdot,\cdot\rangle_\pm : \Gamma^{\wedge k}_{\tau} \otimes_{\mathcal{A}} \Gamma^{\wedge l}_{-\tau} \to \Gamma^{\wedge|k-l|}_{\tau'}$, $\tau \in \{+,-\}$, $\tau' = \tau$ for $k \geqslant l$, otherwise $\tau' = -\tau$. To see this, we treat the case $k \geqslant l$. Let $\xi'_k \in \Gamma^{\otimes k}_{\tau}$ and $\xi_l \in \Gamma^{\otimes l}_{-\tau}$, $\tau \in \{+,-\}$. Firstly, let ξ_l be a symmetric l-form, i.e. $A^{\mp}_l(\xi_l) = 0$. Then, by definition,

$$\langle \xi'_k, \xi_l\rangle_\pm = \tilde{g}(B^{\pm}_{k-l,l}\xi'_k, A^{\mp}_l\xi_l) = 0.$$

On the other hand, if ξ'_k is a symmetric k-form, i.e. $A^{\pm}_k(\xi'_k) = 0$, then we conclude

$$A^{\pm}_{k-l}\langle \xi'_k, \xi_l\rangle_\pm = A^{\pm}_{k-l}\tilde{g}(B^{\pm}_{k-l,l}\xi'_k, A^{\mp}_l\xi_l).$$

Applying Lemma 4.1 this is equal to

$$A^{\pm}_{k-l}\tilde{g}((\mathrm{id}^{\otimes k-l} \otimes_{\mathcal{A}} A^{\mp}_l)B^{\pm}_{k-l,l}\xi'_k, \xi_l) = \tilde{g}((A^{\pm}_{k-l} \otimes_{\mathcal{A}} A^{\mp}_l)B^{\pm}_{k-l,l}\xi'_k, \xi_l).$$

Now formula (16) insures that the latter expression is zero. Hence $\langle \xi'_k, \xi_e\rangle_\pm$ is symmetric. In the case $k < l$ similar reasoning gives the desired result.

Remark. In view of Lemma 4.5 and Proposition 4.6 we should also consider the contractions for $k = l$ (composed with the Haar functional, see in Section 6) as a kind of higher rank σ-metric. □

LEMMA 4.3. *For $\xi_i \in \Gamma^{\wedge k_i}_{\tau_i}$, $i = 0, 1, 2$, $\tau_1 = \tau_2 = -\tau_0$, $k_1 + k_2 \leqslant k_0$ the contractions satisfy the following relations:*

(i) $\langle \xi_1, \langle \xi_2, \xi_0\rangle_\pm\rangle_\pm = \langle \xi_1 \wedge \xi_2, \xi_0\rangle_\pm$ and $\langle\langle \xi_0, \xi_1\rangle_\pm, \xi_2\rangle_\pm = \langle \xi_0, \xi_1 \wedge \xi_2\rangle_\pm$,

(ii) $\langle \xi_1, \langle \xi_0, \xi_2\rangle_\pm\rangle_\pm = \langle\langle \xi_1, \xi_0\rangle_\pm, \xi_2\rangle_\pm$.

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Proof. From Lemma 4.1 and formula (16) we conclude that $A^{\pm}_{k-l}\langle\zeta'_l, \zeta''_k\rangle_\pm = \tilde{g}(\zeta'_l, A^{\pm}_k\zeta''_k)$ for $k \geqslant l$, $\zeta''_k \in \Gamma^{\wedge k}_\tau$, $\zeta'_l \in \Gamma^{\wedge l}_{-\tau}$, $\tau \in \{+,-\}$. Then for the first equation of (i) and representants $\zeta_i \in \Gamma^{\otimes k_i}_{\tau_i}$ of ξ_i, $i = 0, 1, 2$ we compute

$$\begin{aligned} A^{\pm}_{k_0-k_1-k_2}(\langle\zeta_1, \langle\zeta_2, \zeta_0\rangle_\pm\rangle_\pm) &= \tilde{g}(\zeta_1, A^{\pm}_{k_0-k_2}(\langle\zeta_2, \zeta_0\rangle_\pm)) \\ &= \tilde{g}(\zeta_1, \tilde{g}(\zeta_2, A^{\pm}_{k_0}\zeta_0)) = \tilde{g}(\zeta_1 \otimes_{\mathcal{A}} \zeta_2, A^{\pm}_{k_0}\zeta_0) \\ &= A^{\pm}_{k_0-k_1-k_2}(\langle\zeta_1 \otimes_{\mathcal{A}} \zeta_2, \zeta_0\rangle_\pm). \end{aligned}$$

The second equation can be proved similarly.

To prove (ii) we use the same arguments. For the left hand side we obtain

$$\begin{aligned} A^{\pm}_{k_0-k_1-k_2}\langle\zeta_1, \langle\zeta_0, \zeta_2\rangle_\pm\rangle_\pm &= \tilde{g}(\zeta_1, A^{\pm}_{k_0-k_2}\langle\zeta_0, \zeta_2\rangle_\pm) \\ &= \tilde{g}(\zeta_1, \tilde{g}(A^{\pm}_{k_0}\zeta_0, \zeta_2)) \end{aligned}$$

and for the right-hand side

$$\begin{aligned} A^{\pm}_{k_0-k_1-k_2}\langle\langle\zeta_1, \zeta_0\rangle_\pm, \zeta_2\rangle\pm &= \tilde{g}(A^{\pm}_{k_0-k_1}\langle\zeta_1, \zeta_0\rangle_\pm, \zeta_2) \\ &= \tilde{g}(\tilde{g}(\zeta_1, A^{\pm}_{k_0}\zeta_0), \zeta_2). \end{aligned}$$

But both last expressions are equal because of the definition of $\tilde{g}$ and since $k_1 + k_2 \leqslant k_0$. □

The following lemma contains some recursion formulas which are useful in order to compute contractions.

LEMMA 4.4. *For any* $\xi_k \in \Gamma^{\wedge k}_\tau$, $\xi'_k \in \Gamma^{\wedge k}_{-\tau}$, $\rho_1 \in \Gamma_\tau$, $\rho_2 \in \Gamma_{-\tau}$, $k \geqslant 1$, $\tau \in \{+,-\}$ *the equations*

$$\langle\xi_k \wedge \rho_1, \rho_2\rangle_\pm = \xi_k\langle\rho_1, \rho_2\rangle_\pm - \langle\xi_k, \rho^{\mp}_{(1)}\rangle_\pm \wedge \rho^{\mp}_{(2)} \tag{38}$$

and

$$\langle\rho_1, \rho_2 \wedge \xi'_k\rangle_\pm = \langle\rho_1, \rho_2\rangle_\pm\xi'_k - \rho^{\mp}_{(1)} \wedge \langle\rho^{\mp}_{(2)}, \xi'_k\rangle_\pm \tag{39}$$

hold, where $\sigma^{\mp}(\rho_1 \otimes_{\mathcal{A}} \rho_2) = \rho^{\mp}_{(1)} \otimes_{\mathcal{A}} \rho^{\mp}_{(2)} \in \Gamma_{-\tau} \otimes_{\mathcal{A}} \Gamma_\tau$.

Proof. For $k = 1$ the left hand side of the first equation reads as

$$\begin{aligned} \langle\xi_1 \wedge \rho_1, \rho_2\rangle_\pm &= \tilde{g}\big((\xi_1 \otimes_{\mathcal{A}} \rho_1 - \sigma^{\pm}(\xi_1 \otimes_{\mathcal{A}} \rho_1)), \rho_2\big) \\ &= g_{23}(\xi_1 \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2) - g_{12}\sigma^{\mp}_{23}(\xi_1 \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2) \end{aligned}$$

because of the fourth condition of Definition 2.1 on the σ-metric g. Further, if $k > 1$

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we then use (24) to conclude in a similar manner that

$$\begin{aligned}\langle \xi_k \wedge \rho_1, \rho_2\rangle_\pm &= \tilde{g}(B_{k,1}(\xi_k \otimes_{\mathcal{A}} \rho_1), \rho_2)\\ &= g_{k+1,k+2}(\xi_k \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2 - \sigma^{\pm}_{k,k+1}(B_{k-1,1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2})(\xi_k \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2))\\ &= \xi_k \otimes_{\mathcal{A}} g(\rho_1 \otimes_{\mathcal{A}} \rho_2) - g_{k,k+1}\sigma^{\mp}_{k+1,k+2}(B_{k-1,1} \otimes_{\mathcal{A}} \mathrm{id}^{\otimes 2})(\xi_k \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2)\\ &= \xi_k \otimes_{\mathcal{A}} g(\rho_1 \otimes_{\mathcal{A}} \rho_2) - g_{k,k+1}(B_{k-1,1} \otimes_{\mathcal{A}} \sigma^{\mp})(\xi_k \otimes_{\mathcal{A}} \rho_1 \otimes_{\mathcal{A}} \rho_2)\\ &= \xi_k\langle \rho_1, \rho_2\rangle_\pm - \langle \xi_k, \rho^{\mp}_{(1)}\rangle_\pm \wedge \rho^{\mp}_{(2)}.\end{aligned}$$

The proof of the second equation of the lemma is analogous. □

LEMMA 4.5. *For fixed* $\tau \in \{+,-\}, k \geqslant 1$ *let* $\rho_k \in \Gamma_\tau^{\wedge k}, \rho'_k \in \Gamma_{-\tau}^{\wedge k}$ *and* $\sigma^{\pm}_{(k,k)}(\rho_k \otimes_{\mathcal{A}} \rho'_k) = \rho'^k_{(1)} \otimes_{\mathcal{A}} \rho^k_{(2)}$. *Then the contractions* $\langle\cdot,\cdot\rangle_\pm$ *satisfy the equations*

$$\langle \rho'^k_{(1)}, \rho^k_{(2)}\rangle_\pm = \langle \rho_k, (\sigma^{\mp}_{(k)})^2(\rho'_k)\rangle_\pm = \langle (\sigma^{\mp}_{(k)})^2(\rho_k), \rho'_k\rangle_\pm. \tag{40}$$

Proof. The definition (35) of $\langle\cdot,\cdot\rangle_\pm$ gives $\langle \rho'^k_{(1)}, \rho^k_{(2)}\rangle_\pm = \tilde{g}(\rho'^k_{(1)}, A^{\pm}_k \rho^k_{(2)})$. Since $(\mathrm{id}^{\otimes k} \otimes_{\mathcal{A}} A^{\pm}_k)\sigma^{\pm}_{(k,k)} = \sigma^{\pm}_{(k,k)}(A^{\pm}_k \otimes_{\mathcal{A}} \mathrm{id}^{\otimes k})$ (see (33) and (32)) and $A^{\pm}_k = (-1)^{k(k-1)/2} A^{\mp}_k \sigma^{\pm}_{(k)}$, we conclude

$$\begin{aligned}\langle \rho'^k_{(1)}, \rho^k_{(2)}\rangle_\pm &= \tilde{g}(\mathrm{id}^{\otimes k} \otimes_{\mathcal{A}} A^{\pm}_k)\sigma^{\pm}_{(k,k)}(\rho_k \otimes_{\mathcal{A}} \rho'_k)\\ &= \tilde{g}\sigma^{\pm}_{(k,k)}(A^{\pm}_k \otimes_{\mathcal{A}} \mathrm{id}^{\otimes k})(\rho_k \otimes_{\mathcal{A}} \rho'_k)\\ &= (-1)^{k(k-1)/2}\tilde{g}\sigma^{\pm}_{(k,k)}(A^{\mp}_k \sigma^{\pm}_{(k)}\rho_k \otimes_{\mathcal{A}} \rho'_k)\\ &= (-1)^{k(k-1)/2}\tilde{g}\sigma^{\pm}_{(k,k)}(\sigma^{\pm}_{(k)}\rho_k \otimes_{\mathcal{A}} A^{\mp}_{(k)}\rho'_k)\end{aligned}$$

by Lemma 4.1. Inserting $(-1)^{k(k-1)/2}A^{\mp}_k = \sigma^{\pm}_{(k)}A^{\pm}_k(\sigma^{\mp}_{(k)})^2$ and applying Lemma 4.2 we obtain

$$\begin{aligned}\langle \rho'^k_{(1)}\rho^k_{(2)}\rangle_\pm &= \tilde{g}\sigma^{\pm}_{(k,k)}\big(\sigma^{\pm}_{(k)}\rho_k, \sigma^{\pm}_{(k)}A^{\pm}_k(\sigma^{\mp}_{(k)})^2\rho'_k\big)\\ &= \tilde{g}(\rho_k, A^{\pm}_k(\sigma^{\mp}_{(k)})^2\rho'_k) = \langle \rho_k, (\sigma^{\mp}_{(k)})^2\rho'_k\rangle_\pm.\end{aligned}$$

The second equation follows similarly. □

Finally, we should say something about the nondegeneracy of $\langle\cdot,\cdot\rangle_\pm$ as a σ-metric.

PROPOSITION 4.6. *The maps* $\langle\cdot,\cdot\rangle_\pm : \Gamma_\tau^{\wedge k} \otimes_{\mathcal{A}} \Gamma_{-\tau}^{\wedge k} \to \mathcal{A}, \tau \in \{+,-\}, k \geqslant 1$ *and their restrictions to* $(\Gamma_\tau^{\wedge k})_{\mathrm{l}} \otimes (\Gamma_{-\tau}^{\wedge k})_{\mathrm{l}}$ *are nondegenerate.*

Proof. Firstly we show that $\tilde{g} : \Gamma_\tau^{\otimes k} \otimes_{\mathcal{A}} \Gamma_{-\tau}^{\otimes k} \to \mathcal{A}$ and its restriction to $(\Gamma_\tau^{\otimes k})_{\mathrm{l}} \otimes (\Gamma_{-\tau}^{\otimes k})_{\mathrm{l}}$ are nondegenerate.

For $k = 1$ this assertion is true, since g is nondegenerate by Definition 2.1 and $\tilde{g} = g$. Suppose that it is valid for some $k \geqslant 1$ and let $\xi_{k+1} \in \Gamma_\tau^{\otimes k+1}$. Then there are finitely many k-forms $\xi^i \in \Gamma_\tau^{\otimes k}$ and linearly independent 1-forms $\rho_i \in (\Gamma_\tau)_{\mathrm{l}}$ such that $\xi_{k+1} = \sum_i \rho_i \otimes_{\mathcal{A}} \xi^i$. Suppose that $\tilde{g}(\xi_{k+1}, (\xi'_k \otimes_{\mathcal{A}} \rho')) = 0$ for any $\xi'_k \in (\Gamma_{-\tau}^{\otimes k})_{\mathrm{l}}$ and $\rho' \in (\Gamma_{-\tau})_{\mathrm{l}}$. Hence by definition of $\tilde{g}, \tilde{g}((\rho_i \otimes_{\mathcal{A}} \xi^i), (\xi'_k \otimes_{\mathcal{A}} \rho')) =$

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$g(\rho_i\tilde{g}(\xi^i, \xi'_k), \rho') = 0$ for any $\rho' \in (\Gamma_{-\tau})_l$. Since g is a homomorphism of right $\mathcal{A}$-modules, the latter is also true for any $\rho' \in \Gamma_{-\tau}$. Applying the nondegeneracy of g we conclude that $\rho_i\tilde{g}(\xi^i, \xi'_k) = 0$ and since the 1-forms $\rho_i \in (\Gamma_\tau)_l$ are linearly independent we obtain $\tilde{g}(\xi^i, \xi'_k) = 0$ for any $\xi'_k \in (\Gamma_{-\tau}^{\otimes k})_l$. Now we use that $\tilde{g}$ is a homomorphism of right $\mathcal{A}$-modules and get $\tilde{g}(\xi^i, \xi'_k) = 0$ for any $\xi'_k \in \Gamma_{-\tau}^{\otimes k}$. Then the induction assumption gives $\xi^i = 0$ and hence $\xi_{k+1} = \rho_i \otimes_{\mathcal{A}} \xi^i = 0$.

Now we prove the assertion of the proposition. Let $\xi \in \Gamma_\tau^{\wedge k}$, $\xi_0 \in \Gamma_\tau^{\otimes k}$ be a representant of ξ, and let us assume that $\langle \xi, \xi'_k \rangle_\pm = 0$ for any $\xi'_k \in (\Gamma_{-\tau}^{\wedge k})_l$. This means $\tilde{g}(A_k^\pm \xi_0, \xi'_k) = 0$ for any $\xi'_k \in (\Gamma_{-\tau}^{\wedge k})_l$. Since $\tilde{g}$ is a homomorphism of right $\mathcal{A}$-modules, the latter is true for any $\xi'_k \in \Gamma_{-\tau}^{\wedge k}$. In the first part of the proof we have shown that $A_k^\pm \xi_0 = 0$. Hence ξ_0 is a symmetric k-form, so that $\xi = 0$.

Nondegeneracy in the second component of $\langle \cdot, \cdot \rangle_\pm$ can be proved similarly. □

COROLLARY 4.7. *Let g be a left-covariant σ-metric of the pair* (Γ_+, Γ_-). *Then for any* $k \geqslant 0$ *we have* $\dim(\Gamma_+^{\wedge k})_l = \dim(\Gamma_-^{\wedge k})_l$.

5. Hodge Operators

In this section we assume that

(I) the only one-dimensional corepresentation of the Hopf algebra $\mathcal{A}$ is 1 and
(II) there exists a nonzero differential form $\omega_0^\tau \in (\Gamma_\tau^{\wedge n})_l$ for some $n \in \mathbb{Z}$ and $\tau \in \{+, -\}$ such that $\omega_0^\tau \wedge \rho = 0$ for all $\rho \in \Gamma_\tau$.

The latter is in particular fulfilled if one of the vector spaces $(\Gamma_+^\wedge)_l, (\Gamma_-^\wedge)_l$ is finite dimensional. Let us fix a triple $(n_0, \tau_0, \omega_0^{\tau_0})$ as in (II) such that for any other triple $(n_1, \tau_1, \omega_1^{\tau_1})$ having the same property we have $n_1 \geqslant n_0$.

After proving some statements we will show that both + and − can occur as the value of τ_0 and for a given left-covariant σ-metric g of the pair (Γ_+, Γ_-), $\omega_0^\pm$ can be taken biinvariant and in such a manner that

$$\langle \omega_0^+, \omega_0^- \rangle_\pm = \langle \omega_0^-, \omega_0^+ \rangle_\pm = 1. \tag{41}$$

Then we also will assume this on ω_0^+ and ω_0^-.

Let g be a (not necessarily left-covariant) σ-metric of the pair (Γ_+, Γ_-).

PROPOSITION 5.1. *For any* $\xi_k \in \Gamma_{-\tau_0}^{\wedge k}$, $\xi'_l \in \Gamma_{\tau_0}^{\wedge l}$, $0 \leqslant l \leqslant k \leqslant n_0$, *we have*

$$\langle \omega_0^{\tau_0}, \xi_k \rangle_\pm \wedge \xi'_l = \langle \omega_0^{\tau_0}, \langle \xi_k, \xi'_l \rangle_\mp \rangle_\pm. \tag{42}$$

Proof. For $l = 0$ the assertion follows from the right $\mathcal{A}$-linearity of $\tilde{g}$. Let us examine first the case $k = l = 1$. Inserting $\tau = -\tau_0$ and $\xi_k = \omega_0^{\tau_0}$ into (38) and using the condition on $\omega_0^{\tau_0}$ we obtain $0 = \omega_0^{\tau_0}\langle \rho_1, \rho_2 \rangle_\pm - \langle \omega_0^{\tau_0}, \rho_{(1)}^\mp \rangle_\pm \wedge \rho_{(2)}^\mp$ for any $\rho_1 \in \Gamma_{\tau_0}$ and $\rho_2 \in \Gamma_{-\tau_0}$, where $\rho_{(1)}^\mp \otimes_{\mathcal{A}} \rho_{(2)}^\mp = \sigma^\mp(\rho_1 \otimes_{\mathcal{A}} \rho_2)$. Now we insert $\sigma^\pm(\xi_1 \otimes_{\mathcal{A}} \xi'_1)$ for $\rho_1 \otimes_{\mathcal{A}} \rho_2$ and obtain the desired result by the σ-symmetry of the σ-metric g.

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Secondly we prove the proposition for $1 = l \leqslant k \leqslant n_0$ by induction on k. The first step for this is already done. Suppose now that the assertion is true for a $k < n_0$ and let $\xi_k \in \Gamma_{-\tau_0}^{\wedge k}$, $\rho_1 \in \Gamma_{\tau_0}$ and $\rho_2 \in \Gamma_{-\tau_0}$. By (38) we obtain

$$\langle\langle\omega_0^{\tau_0}, \xi_k\rangle_\pm \wedge \rho_1, \rho_2\rangle\pm = \langle\omega_0^{\tau_0}, \xi_k\rangle_\pm\langle\rho_1, \rho_2\rangle_\pm - \langle\langle\omega_0^{\tau_0}, \xi_k\rangle_\pm, \rho_{(1)}^\mp\rangle_\pm \wedge \rho_{(2)}^\mp \qquad (*)$$

where $\rho_{(1)}^\mp \otimes_{\mathcal{A}} \rho_{(2)}^\mp = \sigma^\mp(\rho_1 \otimes_{\mathcal{A}} \rho_2)$. The induction assumption and the second equation of Lemma 4.3(i) assure that the left-hand side of the latter equation is equal to

$$\langle\langle\omega_0^{\tau_0}, \langle\xi_k, \rho_1\rangle_\mp\rangle_\pm, \rho_2\rangle_\pm = \langle\omega_0^{\tau_0}, \langle\xi_k, \rho_1\rangle_\mp \wedge \rho_2\rangle_\pm.$$

Moving this to the right hand side and the second term of the right-hand side of $(*)$ to the left we get

$$\langle\langle\omega_0^{\tau_0}, \xi_k\rangle_\pm, \rho_{(1)}^\mp\rangle_\pm \wedge \rho_{(2)}^\mp = \langle\omega_0^{\tau_0}, \xi_k\langle\rho_1, \rho_2\rangle_\mp\rangle_\pm - \langle\omega_0^{\tau_0}, \langle\xi_k, \rho_1\rangle_\mp \wedge \rho_2\rangle_\pm,$$

where we used the right $\mathcal{A}$-linearity of the contraction and the relation $\langle\rho_1, \rho_2\rangle_+ = \langle\rho_1, \rho_2\rangle_-$. Now we take arbitrary elements $\xi_1' \in \Gamma_{\tau_0}$, $\xi_1'' \in \Gamma_{-\tau_0}$. We insert $\sigma^\pm(\xi_1'' \otimes_{\mathcal{A}} \xi_1')$ for $\rho_1 \otimes_{\mathcal{A}} \rho_2$ in the above formula and use Lemma 4.3(i) (on the left hand side), the σ-symmetry of g (in the first term of the right-hand side) and (38) (on the right hand side of the latter equation). In this manner we obtain

$$\langle\omega_0^{\tau_0}, \xi_k \wedge \xi_1''\rangle_\pm \wedge \xi_1' = \langle\omega_0^{\tau_0}, \langle\xi_k \wedge \xi_1'', \xi_1'\rangle_\mp\rangle_\pm.$$

Hence the assertion of the proposition is true for $k + 1$.

Suppose now that the assertion of the proposition is valid for a fixed $l < n_0$ and for all $k > l$. For $l = 1$ this is true. Then for arbitrary $\xi'' \in \Gamma_{\tau_0}$ we apply (42) twice and conclude

$$\langle\omega_0^{\tau_0}, \xi_k\rangle_\pm \wedge \xi_l' \wedge \xi_l'' = \langle\omega_0^{\tau_0}, \langle\xi_k, \xi_l'\rangle_\mp\rangle_\pm \wedge \xi_l'' = \langle\omega_0^{\tau_0}, \langle\langle\xi_k, \xi_l'\rangle_\mp, \xi_l''\rangle_\mp\rangle_\pm.$$

Applying now Lemma 4.3(i), we get (42) for $l + 1$. □

From now on let g be a left-covariant σ-metric of the pair (Γ_+, Γ_-). A very important consequence of Proposition 5.1 is the following.

THEOREM 5.2. *If there is a left-covariant σ-metric g of the pair (Γ_+, Γ_-) then there exists a natural number n_0 such that $\dim(\Gamma_\tau^{\wedge n_0})_l = 1$ for $\tau \in \{+, -\}$ and all k-forms $\xi_k \in \Gamma_\tau^{\wedge k}$, $k > n_0$ vanish.*

Proof. Since the σ-metric $\langle\cdot,\cdot\rangle_\pm$ is nondegenerate by Proposition 4.6 and left-covariant there is a left-invariant n_0-form $\xi_{n_0} \in \Gamma_{-\tau_0}^{\wedge n_0}$ such that $\langle\omega_0^{\tau_0}, \xi_{n_0}\rangle_+ = 1$. Inserting an arbitrary ξ_l', $l = n_0$ into (42) we obtain $\xi_{n_0}' = \langle\omega_0^{\tau_0}, \xi_{n_0}\rangle_+\xi_{n_0}' = \omega_0^{\tau_0}\langle\xi_{n_0}, \xi_{n_0}'\rangle_-$. Hence we get $\Gamma_{\tau_0}^{\wedge n_0} = \omega_0^{\tau_0} \cdot \mathcal{A}$. Since $\Gamma_{\tau_0}^{\wedge k} = \Gamma_{\tau_0}^{\wedge n_0} \wedge \Gamma_{\tau_0}^{\wedge k - n_0} = \omega_0^{\tau_0} \wedge \Gamma_{\tau_0}^{\wedge k - n_0}$ for any $k > n_0$, we obtain $\Gamma_{\tau_0}^{\wedge k} = 0$. The same assertion for $-\tau_0$ follows from Corollary 4.7. □

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Remark. In the proofs of Proposition 5.1 and Theorem 5.2 the assumption that there is only one one-dimensional corepresentation of $\mathcal{A}$ was not used. □

COROLLARY 5.3. *Let $\mathcal{A}$ be an arbitrary Hopf algebra over the complex field with invertible antipode. Let Γ_+ and Γ_- be bicovariant $\mathcal{A}$-bimodules and g a left-covariant σ-metric of the pair (Γ_+, Γ_-). Then there are precisely two possibilities:*

(i) *Both Γ_+ and Γ_- contain a unique (up to a constant factor) nonzero left-invariant form of (the same) maximal degree.*
(ii) *Both Γ_+ and Γ_- are infinite dimensional and for any form $\omega \in (\Gamma_\tau^{\wedge k})_1$ there is a one-form $\rho \in \Gamma_\tau$ such that $\omega \wedge \rho \neq 0$.*

Let us fix $\xi'_{n_0} = \omega_0^{\tau_0}$ and $\xi_{n_0} \in \Gamma_{-\tau_0}^{\wedge n_0}$ such that $\langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_+ = 1$. From Proposition 5.1 we obtain $\langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_\pm \omega_0^{\tau_0} = \omega_0^{\tau_0} \langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_\mp$. Hence the numbers $\langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_\pm$ and $\langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_\mp$ coincide. Since $\dim(\Gamma_{-\tau_0}^{\wedge n_0})_1 = 1$ by Theorem 5.2, ξ_{n_0} is an eigenvector of $\sigma_{(n_0)}$. Let $\sigma_{(n_0)}\xi_{n_0} = \lambda \xi_{n_0}$. Then we conclude from the definition of the contractions and the considerations above that

$$\begin{aligned} 1 &= \langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_+ = \langle \omega_0^{\tau_0}, (-1)^{n_0(n_0-1)/2} \sigma_{(n_0)} \xi_{n_0} \rangle_- \\ &= (-1)^{n_0(n_0-1)/2} \lambda \langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_- = (-1)^{n_0(n_0-1)/2} \lambda \langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_+ \\ &= (-1)^{n_0(n_0-1)/2} \lambda \langle (-1)^{n_0(n_0-1)/2} \sigma_{(n_0)} \xi_{n_0}, \omega_0^{\tau_0} \rangle_- = \lambda^2 \langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_- \\ &= \lambda^2 \langle \omega_0^{\tau_0}, \xi_{n_0} \rangle + = \lambda^2. \end{aligned}$$

Therefore, $\sigma_{(n_0)}^2(\xi_{n_0}) = \xi_{n_0}$.

A consequence of Theorem 5.2 is that $\Delta_R(\omega_0^{\tau_0}) = \omega_0^{\tau_0} \otimes v_0$, where v_0 is a one-dimensional corepresentation of $\mathcal{A}$. By assumption (I) stated at the beginning of this section, it follows that $v_0 = 1$. Hence $\omega_0^{\tau_0}$ is biinvariant. Similarly, ξ_{n_0} is biinvariant. This implies that $\sigma^-_{(n_0,n_0)}(\omega_0^{\tau_0} \otimes_{\mathcal{A}} \xi_{n_0}) = \xi_{n_0} \otimes_{\mathcal{A}} \omega_0^{\tau_0}$. Applying Lemma 4.5 we get

$$\begin{aligned} \langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_+ &= \langle \cdot, \cdot \rangle_+ \big(\sigma^-_{(n_0,n_0)}(\omega_0^{\tau_0}, \xi_{n_0}) \big) \\ &= \langle \omega_0^{\tau_0}, \sigma^2_{(n_0)}(\xi_{n_0}) \rangle_+ = \langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_+ = 1. \end{aligned}$$

This means that $\langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_+ = 1$ and hence $\langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_- = \langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_+ = 1$ and $\langle \xi_{n_0}, \omega_0^{\tau_0} \rangle_- = \langle \omega_0^{\tau_0}, \xi_{n_0} \rangle_+ = 1$.

Further, we have $\xi_{n_0} \wedge \rho = 0$ for all $\rho \in \Gamma_{-\tau_0}$. Therefore, the triple $(n_0, -\tau_0, \xi_{n_0})$ satisfies assumption (II) at the beginning of the section as well. Now we can set $\omega_0^{-\tau_0} := \xi_{n_0}$ and so (41) is valid. In particular, we have obtained that

$$\sigma_{(n_0)} \omega_0^\pm = (-1)^{n_0(n_0-1)/2} \omega_0^\pm. \tag{43}$$

Since the triple $(n_0, -\tau_0, \omega_0^{-\tau_0})$ satisfies assumption (II), we can replace τ_0 by $-\tau_0$ and Proposition 5.1 remains true. Moreover, it follows from Theorem 5.2 that $\rho \wedge \omega_0^\pm = 0$ for all $\rho \in \Gamma_\pm^{\wedge n_0}$. Using this ansatz a similar reasoning as used in the proof of Proposition 5.1 shows the following.

PROPOSITION 5.4. *For any* $\xi_k \in \Gamma_{-\tau}^{\wedge k}$ *and* $\xi'_l \in \Gamma_\tau^{\wedge l}$, $0 \leqslant l \leqslant k \leqslant n_0$, $\tau \in \{+, -\}$ *the equations*

$$\xi'_l \wedge \langle \xi_k, \omega_0^\tau \rangle_\pm = \langle \langle \xi'_l, \xi_k \rangle_\mp, \omega_0^\tau \rangle_\pm \tag{44}$$

hold.

Let $*_{\mathrm{L}}^\pm, *_{\mathrm{R}}^\pm : \Gamma_\tau^{\wedge k} \to \Gamma_{-\tau}^{\wedge n_0 - k}$ denote the maps given by

$$*_{\mathrm{L}}^\pm(\xi) := \langle \xi, \omega_0^{-\tau} \rangle_\pm, \quad *_{\mathrm{R}}^\pm(\xi) := \langle \omega_0^{-\tau}, \xi \rangle_\pm \tag{45}$$

for any $\xi \in \Gamma_\tau^{\wedge k}$, $0 \leqslant k \leqslant n_0$, $\tau \in \{+, -\}$.

LEMMA 5.5. (i) *For any* $a \in \mathcal{A}$ *and* $\xi \in \Gamma_\tau^\wedge$, $\tau \in \{+, -\}$ *we have* $*_{\mathrm{L}}^\pm(a\xi) = a *_{\mathrm{L}}^\pm(\xi)$ *and* $*_{\mathrm{R}}^\pm(\xi a) = *_{\mathrm{R}}^\pm(\xi) a$.

(ii) $*_{\mathrm{L}}^+ *_{\mathrm{L}}^- = *_{\mathrm{L}}^- *_{\mathrm{L}}^+ = \mathrm{id}$ *and* $*_{\mathrm{R}}^+ *_{\mathrm{R}}^- = *_{\mathrm{R}}^- *_{\mathrm{R}}^+ = \mathrm{id}$. *In particular, the mappings* $*_{\mathrm{L}}^\pm$ *and* $*_{\mathrm{R}}^\pm$ *are isomorphisms of* $\Gamma_\tau^\wedge$ *and* $\Gamma_{-\tau}^\wedge$ *as left and right* $\mathcal{A}$*-modules, respectively.*

(iii) *For any* $\rho_i \in \Gamma_\tau^{\wedge k_i}$, $i = 1, 2$, $k_1 + k_2 \leqslant n_0$, $\tau \in \{+, -\}$, *we have*

$$*_{\mathrm{L}}^\pm(\rho_1 \wedge \rho_2) = \langle \rho_1, *_{\mathrm{L}}^\pm(\rho_2) \rangle_\pm, \quad *_{\mathrm{R}}^\pm(\rho_1 \wedge \rho_2) = \langle *_{\mathrm{R}}^\pm(\rho_1), \rho_2 \rangle_\pm, \tag{46}$$

$$\langle \rho_1, *_{\mathrm{R}}^\pm(\rho_2) \rangle_\pm = \langle *_{\mathrm{L}}^\pm(\rho_1), \rho_2 \rangle_\pm. \tag{47}$$

Proof. Since $\langle \cdot, \cdot \rangle_\pm$ is a homomorphism of $\mathcal{A}$-bimodules, (i) follows from (45). (ii) is obtained from Proposition 5.1 by inserting $\xi_k = \omega_0^{-\tau}$ and applying (41). Setting $\xi_0 = \omega_0^{-\tau}$ in Lemma 4.3, (46) and (47) are equivalent to the equations of Lemma 4.3(i) and 4.3(ii), respectively. □

DEFINITION 5.1. We call the mapping $*_{\mathrm{L}}^+ : \Gamma_\tau^\wedge \to \Gamma_{-\tau}^\wedge$ *left Hodge operator* and $*_{\mathrm{R}}^+ : \Gamma_\tau^\wedge \to \Gamma_{-\tau}^\wedge$ *right Hodge operator* on $\Gamma_\tau^\wedge$, $\tau \in \{+, -\}$.

Remark. The equations in Proposition 5.1 and 5.4 with $k = l$ can also be written in the familiar form

$$*_{\mathrm{R}}^\pm(\xi_k) \wedge \xi'_k = \omega_0^\tau \langle \xi_k, \xi'_k \rangle_\mp, \tag{48}$$

$$\xi'_k \wedge *_{\mathrm{L}}^\pm(\xi_k) = \langle \xi'_k, \xi_k \rangle_\mp \omega_0^\tau. \tag{49}$$

□

Up to now $\Gamma_+^\wedge$ and $\Gamma_-^\wedge$ have been only the exterior algebras over bicovariant $\mathcal{A}$-bimodules Γ_+ and Γ_-, respectively. In the remainder of this paper we assume in addition that $\Gamma_\tau^\wedge$ is an inner bicovariant differential calculus with differentiation d_τ, $\tau \in \{+, -\}$. That the differential calculus $\Gamma_\tau^\wedge$ is inner means that there exists a

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biinvariant 1-form $\eta^\tau \in \Gamma_\tau$ such that

$$d_\tau\rho = \eta^\tau \wedge \rho - (-1)^k \rho \wedge \eta^\tau \quad \rho \in \Gamma_\tau^{\wedge k}, \tau \in \{+,-\}. \tag{50}$$

Further, we assume that the corresponding σ-metrics (and hence contractions) are left-covariant.

DEFINITION 5.2. The mappings $\partial_{\mathrm{L}}^{\pm} : \Gamma_\tau^{\wedge k} \to \Gamma_\tau^{\wedge k-1}$ defined by

$$\partial_{\mathrm{L}}^{\pm}\rho := (-1)^k *_{\mathrm{L}}^{\pm} (d_{-\tau} *_{\mathrm{L}}^{\mp} (\rho)), \quad \rho \in \Gamma_\tau^{\wedge k}, 0 \leqslant k \leqslant n_0, \tau \in \{+,-\}$$

are called *(positive and negative) left codifferential operators on* $\Gamma_\tau^\wedge$. Analogously we define the *right codifferential operators* $\partial_{\mathrm{R}}^{\pm} : \Gamma_\tau^{\wedge k} \to \Gamma_\tau^{\wedge k-1}$, $0 \leqslant k \leqslant n_0, \tau \in \{+,-\}$ *on* $\Gamma_\tau^\wedge$ by $\partial_{\mathrm{R}}^{\pm}\rho := (-1)^{n_0-1+k} *_{\mathrm{R}}^{\pm} (d_{-\tau} *_{\mathrm{R}}^{\mp} (\rho))$.

LEMMA 5.6. $*_{\mathrm{L}}^{+}(\rho) = *_{\mathrm{L}}^{-}(\rho)$ *and* $*_{\mathrm{R}}^{+}(\rho) = *_{\mathrm{R}}^{-}(\rho)$ *for any* $\rho \in \Gamma_\tau^{\wedge k}$, $\tau \in \{+,-\}$, $k \in \{0, 1, n_0-1, n_0\}$.

Proof. For $k = 0$ we have $*_{\mathrm{L}}^{+}(\rho) = *_{\mathrm{L}}^{-}(\rho) = \rho\omega_0^{-\tau}$ and $*_{\mathrm{R}}^{+}(\rho) = *_{\mathrm{R}}^{-}(\rho) = \omega_0^{-\tau}\rho$ by definition. For $k = n_0$ we obtain from Theorem 5.2 that there are $a, b \in \mathcal{A}$ such that $\rho = a\omega_0^\tau = \omega_0^\tau b$. Then Lemma 5.5(i) and Equation (41) imply that $*_{\mathrm{L}}^{\pm}(a\omega_0^\tau) = a *_{\mathrm{L}}^{\pm} (\omega^{\tau_0}) = a\langle\omega_0^\tau, \omega_0^{-\tau}\rangle_\pm = a$ and $*_{\mathrm{R}}^{\pm}(\omega_0^\tau b) = *_{\mathrm{R}}^{\pm}(\omega_0^\tau)b = \langle\omega_0^{-\tau}, \omega_0^\tau\rangle_\pm b = b$.

Let now $k = n_0 - 1$. We compute

$$\begin{aligned} *_{\mathrm{L}}^{\pm}(\rho) &= \langle\rho, \omega_0^{-\tau}\rangle\pm = \tilde{g}(A_{n_0-1}^{\pm}\rho, B_{n_0-1,1}^{\pm}\omega_0^{-\tau}) \\ &= \tilde{g}\big(\rho, (A_{n_0-1}^{\pm} \otimes_{\mathcal{A}} \mathrm{id})B_{n_0-1,1}^{\pm}\omega_0^{-\tau}\big) = \tilde{g}(\rho, A_{n_0}^{\pm}\omega_0^{-\tau}) \end{aligned} \tag{$*$}$$

by using Lemma 4.1 and the second equation of (16). We also have $A_k^+ = (-1)^{k(k-1)/2} A_k^- \sigma_{(k)}^+$ for any $k \geqslant 1$. Hence (43) gives

$$\begin{aligned} A_{n_0}^{+}\omega_0^{-\tau} &= (-1)^{n_0(n_0-1)/2} A_{n_0}^{-}\sigma_{(n_0)}^{+}\omega_0^{-\tau} \\ &= (-1)^{n_0(n_0-1)/2} A_{n_0}^{-}(-1)^{n_0(n_0-1)/2}\omega_0^{-\tau} = A_{n_0}^{-}\omega_0^{-\tau}. \end{aligned}$$

From this and equation $(*)$ we conclude that $*_{\mathrm{L}}^{+}(\rho) = *_{\mathrm{L}}^{-}(\rho)$.

In the case $k = 1$ we use that the mappings $*_{\mathrm{L}}^{\pm}$ are isomorphisms of left $\mathcal{A}$-modules. Therefore there is a $\rho' \in \Gamma_\tau^{\wedge n_0-1}$ such that $\rho = *_{\mathrm{L}}^{+}(\rho')$. By the preceding we also have $\rho = *_{\mathrm{L}}^{-}(\rho')$. Hence, $*_{\mathrm{L}}^{+}(\rho) = *_{\mathrm{L}}^{+} *_{\mathrm{L}}^{-} (\rho') = \rho'$ and $*_{\mathrm{L}}^{-}(\rho) = *_{\mathrm{L}}^{-} *_{\mathrm{L}}^{+} (\rho') = \rho'$. Similarly, $*_{\mathrm{R}}^{+}(\rho) = *_{\mathrm{R}}^{-}(\rho)$ for any $\rho \in \Gamma_\tau^{\wedge k}$, $k = 1, n_0 - 1$. □

LEMMA 5.7. *For any* $\rho \in (\Gamma_\tau)_{\mathrm{r}}$, $\tau \in \{+,-\}$ *we have* $*_{\mathrm{L}}^{\pm}(\rho) = (-1)^{n_0-1} *_{\mathrm{R}}^{\pm} (\rho)$.

Proof. The n_0-form $\omega_0^{-\tau}$ is left-invariant. Hence there are left-invariant 1-forms $\rho_1, \ldots, \rho_{n_0} \in (\Gamma_{-\tau})_{\mathrm{l}}$ such that $\omega_0^{-\tau} = \rho_1 \wedge \ldots \wedge \rho_{n_0}$. Then (39) and the σ-symmetry of the σ-metric yield

$$*_{\mathrm{L}}^{+}(\rho) = \sum_{i=1}^{n_0} (-1)^{i-1} \rho_1 \wedge \ldots \wedge \rho_{i-1} \langle\rho, \rho_i\rangle \wedge \rho_{i+1} \wedge \ldots \wedge \rho_{n_0}. \tag{51}$$

The σ-symmetry of the σ-metric implies that $\langle\rho_i, \rho\rangle = \langle\rho, \rho_i\rangle$for any $i = 1, \dots, n_0$. Using this fact and Equation (38) we obtain the same formula for $(-1)^{n_0-1} *_{\mathrm{R}}^- (\rho)$. Applying Lemma 5.6 the assertion follows. □

PROPOSITION 5.8. *The codifferentials* $\partial_{\mathrm{L}}^{\tau'}$ *and* $\partial_{\mathrm{R}}^{\tau'}$, $\tau' \in \{+, -\}$, *coincide. On* $a \in \mathcal{A}$ *they act trivially:* $\partial_{\mathrm{L}}^{\pm} a = \partial_{\mathrm{R}}^{\pm} a = 0$. *For any* $\rho \in \Gamma_\tau^{\wedge k}$, $k > 0$, $\tau \in \{+, -\}$ *we have*

$$\partial_{\mathrm{L}}^{\pm}\rho = \langle\rho, \eta^{-\tau}\rangle_\pm + (-1)^k \langle\eta^{-\tau}, \rho\rangle_\pm. \tag{52}$$

Proof. Let $k > 0$ and $\rho \in \Gamma_\tau^{\wedge k}$. The definition of $\partial_{\mathrm{L}}^{\pm}$ and (50) give

$$\partial_{\mathrm{L}}^{\pm}\rho = (-1)^k *_{\mathrm{L}}^{\pm} (\mathrm{d}_{-\tau} *_{\mathrm{L}}^{\mp} (\rho)) = (-1)^k *_{\mathrm{L}}^{\pm} (\eta^{-\tau} \wedge *_{\mathrm{L}}^{\mp}(\rho) - (-1)^{n_0-k} *_{\mathrm{L}}^{\mp} (\rho) \wedge \eta^{-\tau}).$$

From the first equation of (46) and Lemma 5.5(ii) we obtain that the first summand is equal to $(-1)^k \langle\eta^{-\tau}, *_{\mathrm{L}}^{\pm}(*_{\mathrm{L}}^{\mp}(\rho))\rangle_\pm = (-1)^k \langle\eta^{-\tau}, \rho\rangle_\pm$. For the second summand we use (46) and Lemma 5.7 and obtain $(-1)^{n_0+1}\langle *_{\mathrm{L}}^{\mp}(\rho), *_{\mathrm{L}}^{\pm}(\eta^{-\tau})\rangle_\pm = \langle *_{\mathrm{L}}^{\mp}(\rho), *_{\mathrm{R}}^{\pm}(\eta^{-\tau})\rangle_\pm$. We apply now (47) and Lemma 5.5(ii) to the latter and get $\langle *_{\mathrm{L}}^{\pm}(*_{\mathrm{L}}^{\mp}(\rho)), \eta^{-\tau}\rangle \pm = \langle\rho, \eta^{-\tau}\rangle_\pm$. This proves (52) for the left codifferentials. Similar computations lead to the same expression for $\partial_{\mathrm{R}}^{\pm}\rho$. □

PROPOSITION 5.9. *For any* $\rho \in (\Gamma_\tau^{\wedge n_0-1})_{\mathrm{l}}$, $\tau \in \{+, -\}$ *we have* $\mathrm{d}_\tau\rho = 0$.

Proof. Let $\rho \in (\Gamma_\tau^{\wedge n_0-1})_{\mathrm{l}}$. Because of Lemma 5.5(ii) and the left-covariance of $*_{\mathrm{L}}^{\pm}$ there are $\rho_1^{\pm} \in (\Gamma_{-\tau})_{\mathrm{l}}$ such that $\rho = *_{\mathrm{L}}^{\pm}(\rho_1^{\pm})$. Then $\mathrm{d}_\tau\rho = 0$ is equivalent to

$$0 = *_{\mathrm{L}}^{\pm}(\mathrm{d}_\tau\rho) = *_{\mathrm{L}}^{\pm}(\mathrm{d}_\tau *_{\mathrm{L}}^{\mp} (\rho_1^{\mp})) = -\partial_{\mathrm{L}}^{\pm}\rho_1^{\mp}.$$

Since η^τ is biinvariant, $\rho_1^{\mp}$ is left-invariant and the σ-metric is σ-symmetric, we conclude from Proposition 5.8 that

$$\partial_{\mathrm{L}}^{\pm}\rho_1^{\mp} = \langle\rho_1^{\mp}, \eta^\tau\rangle_\pm - \langle\eta^\tau, \rho_1^{\mp}\rangle_\pm = \langle\rho_1^{\mp}, \eta^\tau\rangle_\pm - \langle\rho_1^{\mp}, \eta^\tau\rangle_\pm = 0.$$

□

6. Laplace–Beltrami Operators

Let $\mathcal{A}$ be again an arbitrary Hopf algebra and let Γ_+, Γ_- be two bicovariant $\mathcal{A}$-bimodules which admit a left-covariant σ-metric in the sense of Definition 2.1. Moreover, (as in the last part of Section 5,) we assume that the bicovariant $\mathcal{A}$-bimodules $\Gamma_\tau^{\wedge}$, $\tau \in \{+, -\}$ admit a differential operator d_τ such that they become inner bicovariant differential calculi on $\mathcal{A}$. Further we suppose that the σ-metrics (and hence contractions) are left-covariant.

In addition we now assume that the Hopf algebra $\mathcal{A}$ is cosemisimple ([7], Sect. 11.2), that is, there exists a linear functional h on $\mathcal{A}$, called the Haar functional, such that $h(1) = 1$ and

$$(h \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h)\Delta(a) = h(a)1 \tag{53}$$

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for all $a \in \mathcal{A}$. Further, we suppose that the Haar functional is regular, that is, both $h(ab) = 0$ for all $b \in \mathcal{A}$ and $h(ba) = 0$ for all $b \in \mathcal{A}$ imply that $a = 0$. Recall that any CQG-algebra is cosemisimple and its Haar functional is regular ([7], Proposition 11.29). By Proposition 4.6 the restriction of $\langle \cdot, \cdot \rangle_\pm$ to $\Gamma_\tau^{\wedge k} \otimes_{\mathcal{A}} \Gamma_{-\tau}^{\wedge k}$ is nondegenerate. Hence for each $\rho \in \Gamma_\tau^{\wedge k}$ there is a $\rho' \in \Gamma_{-\tau}^{\wedge k}$ such that $\mathcal{A} \ni a := \langle \rho, \rho' \rangle_\pm \neq 0$. By the regularity of the Haar functional there is a $b \in \mathcal{A}$ such that $h(ab) \neq 0$. Then we have $h\langle \rho, \rho' b \rangle_\pm = h(\langle \rho, \rho' \rangle_\pm b) = h(ab) \neq 0$. Therefore, the mapping $h \circ \langle \cdot, \cdot \rangle_\pm : \Gamma_\tau^{\wedge k} \otimes_{\mathcal{A}} \Gamma_{-\tau}^{\wedge k} \to \mathbb{C}$ is nondegenerate for all $k \geqslant 0$ and $\tau \in \{+, -\}$. We shall consider it as a generalisation of the classical notion of the metric on k-forms.

Motivated by Definition 5.2 and Proposition 5.8, we introduce the following notion.

DEFINITION 6.1. The mappings $\partial_\tau^\pm : \Gamma_\tau^{\wedge k} \to \Gamma_\tau^{\wedge k-1}$, $k \geqslant 0$, $\tau \in \{+, -\}$, defined by $\partial_\tau^\pm(a) = 0$ for $a \in \mathcal{A}$ and

$$\partial_\tau^\pm \rho = \langle \rho, \eta^{-\tau} \rangle_\pm + (-1)^k \langle \eta^{-\tau}, \rho \rangle_\pm \tag{54}$$

for $\rho \in \Gamma_\tau^{\wedge k}$, $k > 0$, are called *(positive and negative) codifferential operators on* $\Gamma_\tau^{\wedge k}$.

LEMMA 6.1. *(i)* $(\partial_\tau^\pm)^2 = 0$.

(ii) $\partial_\tau^\pm(a\rho) = a\partial_\tau^\pm \rho + (-1)^k \langle \mathrm{d}_{-\tau} a, \rho \rangle_\pm$ *for any* $a \in \mathcal{A}$, $\rho \in \Gamma_\tau^{\wedge k}$, $\tau \in \{+, -\}, k \geqslant 1$.

Proof. (i) Since $(\partial_\tau^\pm)^2(\rho) \in \Gamma_\tau^{\wedge k-2}$ for any $\rho \in \Gamma_\tau^{\wedge k}$, $k \geqslant 0$, $\tau \in \{+, -\}$, we obtain $(\partial_\tau^\pm)^2(\rho) = 0$ for $\rho \in \Gamma_\tau^{\wedge k}$, $k \leqslant 1$. For $k \geqslant 2$ we get

$$\begin{aligned}(\partial_\tau^\pm)^2(\rho) &= \partial_\tau^\pm\big(\langle \rho, \eta^{-\tau} \rangle_\pm + (-1)^k \langle \eta^{-\tau}, \rho \rangle_\pm\big)\\ &= \langle\big(\langle \rho, \eta^{-\tau} \rangle_\pm + (-1)^k \langle \eta^{-\tau}, \rho \rangle_\pm\big), \eta^{-\tau} \rangle_\pm\\ &\quad + (-1)^{k-1} \langle \eta^{-\tau}, \big(\langle \rho, \eta^{-\tau} \rangle_\pm + (-1)^k \langle \eta^{-\tau}, \rho \rangle_\pm\big)\rangle_\pm.\end{aligned}$$

Applying Lemma 4.3(i) on the first and fourth summand we obtain

$$\begin{aligned}=&\langle \rho, \eta^{-\tau} \wedge \eta^{-\tau} \rangle_\pm + (-1)^k \langle \langle \eta^{-\tau}, \rho \rangle_\pm, \eta^{-\tau} \rangle_\pm\\ &+ (-1)^{k-1} \langle \eta^{-\tau}, \langle \rho, \eta^{-\tau} \rangle_\pm \rangle_\pm - \langle \eta^{-\tau} \wedge \eta^{-\tau}, \rho \rangle_\pm.\end{aligned}$$

Since $\eta^{-\tau}$ is biinvariant, $\eta^{-\tau} \wedge \eta^{-\tau} = 0$. Using Lemma 4.3(ii) the second and third summand in the last expression also vanish.

(ii) From (50) it follows that

$$\langle \mathrm{d}_{-\tau} a, \rho \rangle_\pm = \langle \eta^{-\tau} a, \rho \rangle_\pm - \langle a\eta^{-\tau}, \rho \rangle_\pm = \langle \eta^{-\tau}, a\rho \rangle_\pm - a\langle \eta^{-\tau}, \rho \rangle_\pm.$$

Then (54) gives the assertion. □

LEMMA 6.2. *For any* $a \in \mathcal{A}$ *and* $\rho \in (\Gamma_\tau)_1$, $\rho' \in (\Gamma_{-\tau})_1$, $\tau \in \{+, -\}$ *we have*

(i) $h(\langle a\rho, \rho' \rangle_\pm) = h(\langle \rho a, \rho' \rangle_\pm) = h(a)\langle \rho, \rho' \rangle_\pm$,

(ii) $h\big(\partial_{-\tau}^\pm(a\rho')\big) = 0$.

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Proof. (i) Let $\{\theta_i | i = 1, \ldots, m\}$ be a basis of the vector space $(\Gamma_\tau)_l$. It suffices to prove the assertion for $\rho = \theta_i$. The left-invariance of the σ-metric ensures that $\langle \rho, \rho' \rangle_\pm \in \mathbb{C}$ and we conclude that $h(\langle a\rho, \rho' \rangle_\pm) = h(a\langle \rho, \rho' \rangle_\pm) = h(a)\langle \rho, \rho' \rangle_\pm$.

By the general theory [11] there are functionals f^i_j, $i, j = 1, \ldots, m$, such that $\theta_i a = a_{(1)} f^i_j(a_{(2)})\theta_j$ and $f^i_j(1) = \delta^i_j$. We have again $\langle \theta_j, \rho' \rangle_\pm \in \mathbb{C}$ and therefore

$$\begin{aligned} h(\langle \theta_i a, \rho' \rangle_\pm) &= h(a_{(1)}) f^i_j(a_{(2)}) \langle \theta_j, \rho' \rangle_\pm = f^i_j(h(a_{(1)})a_{(2)}) \langle \theta_j, \rho' \rangle_\pm \\ &= f^i_j(h(a) \cdot 1) \langle \theta_j, \rho' \rangle_\pm = h(a) \langle \theta_i, \rho' \rangle_\pm \end{aligned}$$

by (53). Hence we get (i).

(ii) Firstly we see from (54) that $\partial^\pm_{-\tau}(\rho') = \langle \rho', \eta^\tau \rangle_\pm - \langle \eta^\tau, \rho' \rangle_\pm = 0$ since the σ-metric is σ-symmetric, η^τ is bi-invariant and ρ' is left-invariant. Secondly, Lemma 6.1(ii) gives $h(\partial^\pm_{-\tau}(a\rho')) = h(a\partial^\pm_{-\tau}\rho' - \langle \mathrm{d}_\tau a, \rho' \rangle_\pm) = h\big(\langle a\eta^\tau, \rho' \rangle_\pm - \langle \eta^\tau a, \rho' \rangle_\pm\big)$. Then the assertion follows from (i). □

THEOREM 6.3. *Suppose that g is a left-invariant σ-metric of the pair* (Γ_+, Γ_-). *Let* $\langle \cdot, \cdot \rangle_\pm$ *be the corresponding contractions. Then for any* $\rho \in \Gamma^{\wedge k}_\tau$, $\rho' \in \Gamma^{\wedge k+1}_{-\tau}$, $\tau \in \{+, -\}$ *the equations*

$$h(\langle \rho, \partial^\pm_{-\tau}\rho' \rangle_\pm) = h(\langle \mathrm{d}_\tau \rho, \rho' \rangle_\pm) \quad \text{and} \tag{55}$$

$$h(\langle \partial^\pm_{-\tau}\rho', \rho \rangle_\pm) = h(\langle \rho', \mathrm{d}_\tau \rho \rangle_\pm) \tag{56}$$

hold.

Proof. Inserting the definitions (54) and (50) we obtain

$$\begin{aligned} h\big(\langle \rho, \partial^\pm_{-\tau}\rho' \rangle_\pm - \langle \mathrm{d}_\tau \rho, \rho' \rangle_\pm\big) &= h\big(\langle \rho, \big(\langle \rho', \eta^\tau \rangle_\pm + (-1)^{k+1} \langle \eta^\tau, \rho' \rangle_\pm\big)\rangle_\pm \\ &\quad - \langle (\eta^\tau \wedge \rho + (-1)^{k+1} \rho \wedge \eta^\tau), \rho' \rangle_\pm\big). \end{aligned}$$

Applying Lemma 4.3 we now substitute $\langle \rho, \langle \rho', \eta^\tau \rangle_\pm \rangle_\pm$ by $\langle \langle \rho, \rho' \rangle_\pm, \eta^\tau \rangle_\pm$, $\langle \rho, \langle \eta^\tau, \rho' \rangle_\pm \rangle_\pm$ by $\langle \rho \wedge \eta^\tau, \rho' \rangle_\pm$ and $\langle \eta^\tau \wedge \rho, \rho' \rangle_\pm$ by $\langle \eta^\tau, \langle \rho, \rho' \rangle_\pm \rangle_\pm$. Then we have

$$\begin{aligned} h(\langle \rho, \partial^\pm_{-\tau}\rho' \rangle_\pm) - h(\langle \mathrm{d}_\tau \rho, \rho' \rangle_\pm) &= h\big(\langle \langle \rho, \rho' \rangle_\pm, \eta^\tau \rangle_\pm - \langle \eta^\tau, \langle \rho, \rho' \rangle_\pm \rangle_\pm\big) \\ &= h\big(\partial^\pm_{-\tau} \langle \rho, \rho' \rangle_\pm\big). \end{aligned}$$

Since $\langle \rho, \rho' \rangle_\pm$ is an element of $\Gamma_{-\tau} = \mathcal{A}(\Gamma_{-\tau})_l$, we obtain (55) by Lemma 6.2(ii). The proof of (56) is similar. □

DEFINITION 6.2. We call the operators $\Delta^\pm_\tau : \Gamma^{\wedge k}_\tau \to \Gamma^{\wedge k}_\tau$, $\Delta^\pm_\tau := \mathrm{d}_\tau \partial^\pm_\tau + \partial^\pm_\tau \mathrm{d}_\tau$ *Laplace–Beltrami operators.*

The following properties of $\Delta^\pm_\tau$ are simple consequences of the facts that $\mathrm{d}^2 = 0$, $(\partial^\pm_\tau)^2 = 0$ and (54).

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LEMMA 6.4. *The Laplace–Beltrami operators satisfy the equations*

$$\Delta_\tau^\pm = (\mathrm{d}_\tau + \partial_\tau^\pm)^2, \tag{57}$$

$$\Delta_\tau^\pm \mathrm{d}_\tau = \mathrm{d}_\tau \Delta_\tau^\pm = \mathrm{d}_\tau \partial_\tau^\pm \mathrm{d}_\tau, \tag{58}$$

$$\Delta_\tau^\pm \partial_\tau^\pm = \partial_\tau^\pm \Delta_\tau^\pm = \partial_\tau^\pm \mathrm{d}_\tau \partial_\tau^\pm, \tag{59}$$

$$\Delta_+^{\tau'} a = \Delta_-^{\tau'} a = \langle \eta^+ a, \eta^- \rangle + \langle \eta^- a, \eta^+ \rangle - 2a\langle \eta^+, \eta^- \rangle \tag{60}$$

for any $a \in \mathcal{A}$ *and* $\tau, \tau' \in \{+, -\}$.

Remark. By (60) the Laplace–Beltrami operator on $\mathcal{A} \subset \Gamma_\pm^\wedge$ neither depends on the sign τ' of the antisymmetrizer nor on the $\mathcal{A}$-bimodule $\Gamma_\pm^\wedge$ containing $\mathcal{A}$. □

PROPOSITION 6.5. *For any* $\rho \in \Gamma_\tau^{\wedge k}$, $\rho' \in \Gamma_{-\tau}^{\wedge k}$, $\tau \in \{+, -\}$, $k \geqslant 0$ *we have*

$$h(\langle \Delta_\tau^\pm \rho, \rho' \rangle_\pm) = h(\langle \rho, \Delta_{-\tau}^\pm \rho' \rangle_\pm). \tag{61}$$

Proof. Using Theorem 6.3 we compute

$$\begin{aligned} h(\langle \Delta_\tau^\pm \rho, \rho' \rangle_\pm) &= h(\langle \mathrm{d}_\tau \partial_\tau^\pm \rho, \rho' \rangle_\pm + \langle \partial_\tau^\pm \mathrm{d}_\tau \rho, \rho' \rangle_\pm) \\ &= h(\langle \partial_\tau^\pm \rho, \partial_{-\tau}^\pm \rho' \rangle_\pm + \langle \mathrm{d}_\tau \rho, \mathrm{d}_{-\tau} \rho' \rangle_\pm) \\ &= h(\langle \rho, \mathrm{d}_{-\tau} \partial_{-\tau}^\pm \rho' \rangle_\pm + \langle \rho, \partial_{-\tau}^\pm \mathrm{d}_{-\tau} \rho' \rangle_\pm) = h(\langle \rho, \Delta_{-\tau}^\pm \rho' \rangle_\pm). \end{aligned}$$

□

7. Eigenvalues of the Laplace–Beltrami operator for $\mathrm{SL}_q(N)$

Throughout this section we assume that q is a transcendental complex number and $\mathcal{A}$ is the Hopf algebra $\mathcal{O}(\mathrm{SL}_q(N))$, $N \geqslant 2$. Then $\mathcal{A}$ is cosemisimple, i. e. any element of $\mathcal{A}$ is a finite linear combination of matrix elements of irreducible matrix corepresentations of $\mathcal{A}$ ([7], Theorem 11.22). Further, $\mathcal{A}$ is coquasitriangular and admits a universal r-form $\mathbf{r} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ defined by $\mathbf{r}(u_j^i \otimes u_l^k) = z^{-1} \hat{R}_{jl}^{ki}$, where z is a fixed complex number with $z^N = q$, and

$$\hat{R}_{kl}^{ij} = q^{\delta_j^i} \delta_l^i \delta_k^j + (i < j)(q - q^{-1}) \delta_k^i \delta_l^j. \tag{62}$$

Here the number $(i < j)$ is 1 if $i < j$ and zero otherwise. We shall write $\hat{R}^\pm$ for $\hat{R}^{\pm 1}$.

Let Γ_+ and Γ_- be the N^2-dimensional bicovariant differential calculi on $\mathcal{A}$ determined by the fundamental corepresentation u and the contragredient corepresentation u^c (see Section 3). Further, let denote F_1, F_2, G_1, G_2 the $N \times N$-matrices with entries ${F_1}_j^i = z^{-1} q^{N-2i} \delta_j^i$, ${F_2}_j^i = q^{2i} \delta_j^i$, ${G_1}_j^i = z^{-1} q^N \delta_j^i$, ${G_2}_j^i = \delta_j^i$.

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Then $F_1 \in \mathrm{Mor}(u^{cc}, u)$, $F_2 \in \mathrm{Mor}(u, u^{cc})$ and $G_1, G_2 \in \mathrm{Mor}(u)$ and they determine a bicovariant σ-metric of the pair (Γ_+, Γ_-) (see Section 3).

The Laplace–Beltrami operator Δ on $\mathcal{A}$ is given by (60). For $n \in \mathbb{Z}$ and a complex number $p \neq 0, \pm 1$ let $[n]_p$ denote the number $(p^n - p^{-n})/(p - p^{-1})$.

PROPOSITION 7.1. *The Laplace–Beltrami operator Δ on $\mathcal{A}$ is diagonalizable. Let v^λ be a fixed irreducible corepresentation of $\mathcal{A}$ corresponding to a Young diagram λ. Then the matrix elements of v^λ are eigenvectors of Δ to the eigenvalue*

$$E_\lambda := (z - z^{-1})^2 \left([m]_z^2 [N]_q + [N]_z \sum_{(i,j)\in\lambda} [N^2 - 2m + 2N(j-i)]_z \right), \tag{63}$$

where $(i,j) \in \lambda$ means that there is a box in the ith *row and* jth *column of λ and m is the number of boxes in λ.*

Proof. Using the relations $\mathbf{r}(u^i_j, S(u^k_l)) = zq^{2k-2l}\hat{R}^{-1}{}^{ik}_{lj}$ and $\mathbf{r}(S(u^i_j), u^k_l) = z\hat{R}^{-1}{}^{ik}_{lj}$, some properties of the r-form $\mathbf{r}$ and the $\hat{R}$-matrix, Equation (60), for any $m \geqslant 0$ we get

$$\Delta(u^{i_1}_{j_1} u^{i_2}_{j_2} \dots u^{i_m}_{j_m}) = q^{-N-1} u^{i_1}_{l_1} u^{i_2}_{l_2} \dots u^{i_m}_{l_m} (z^{-2m} D^+_{m+1} + z^{2m} D^-_{m+1} - 2\mathrm{id})^{l_1 \dots l_m k}_{j_1 \dots j_m n} q^{2k} \delta^n_k,$$

where

$$D^\pm_{m+1} = \hat{R}^\pm_{m,m+1} \hat{R}^\pm_{m-1,m} \dots \hat{R}^\pm_{23} \hat{R}^{\pm}_{12}{}^2 \hat{R}^\pm_{23} \dots \hat{R}^\pm_{m,m+1}, \quad m \geqslant 2 \tag{64}$$

are the so-called Jucys–Murphy operators of the Hecke algebra, $D^\pm_1 = \mathrm{id}$. Since $q^{2d-2a} \hat{R}^{bd}_{ac} \hat{R}^{ec}_{fd} = \delta^e_a \delta^b_f + q^{2N+1}(q - q^{-1}) q^{-2a} \delta^b_a \delta^e_f$ and $q^{2d-2a} \hat{R}^{-1}{}^{bd}_{ac} \hat{R}^{-1}{}^{ec}_{fd} = \delta^e_a \delta^b_f - q(q-q^{-1}) q^{-2a} \delta^b_a \delta^e_f$ we obtain

$$\sum_k q^{2k} (D^\pm_{m+1})^{l_1 l_2 \dots l_m k}_{j_1 j_2 \dots j_m k} = \left(q^{N+1} [N]_q \mathrm{id} \pm q^{N+1} q^{\pm N} (q - q^{-1}) \sum_{n=1}^m D^\pm_n \right)^{l_1 \dots l_m}_{j_1 \dots j_m}$$

and hence

$$\Delta(u^{i_1}_{j_1} \cdots u^{i_m}_{j_m}) = u^{i_1}_{l_1} \cdots u^{i_m}_{l_m} \Big((z^m - z^{-m})^2 [N]_q \mathrm{id} + \\ + (q - q^{-1}) \sum_{n=1}^m (q^N z^{-2m} D^+_n - q^{-N} z^{2m} D^-_n) \Big)^{l_1 \dots l_m}_{j_1 \dots j_m}.$$

Since q is transcendental, $\mathcal{A}$ is cosemisimple. Moreover, $\mathcal{A}$ is generated by the matrix elements of the fundamental corepresentation u of $\mathcal{A}$. Let P_λ be a projection of $u^{\otimes m}$ onto the irreducible corepresentation of $\mathcal{A}$ corresponding to the Young diagram λ. Then Proposition 4.7 and the preceding considerations in [8] imply that $\sum_{n=1}^m D^\pm_n P_\lambda = \sum_{(i,j)\in\lambda} q^{\pm(2j-2i)} P_\lambda$ and therefore $\Delta(u^{i_1}_{k_1} \cdots u^{i_m}_{k_m} P_\lambda{}^{k_1 \dots k_m}_{j_1 \dots j_m}) = E_\lambda u^{i_1}_{k_1} \cdots u^{i_m}_{k_m}$

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$P_{\lambda j_1 \ldots j_m}^{k_1 \ldots k_m}$, where

$$E_\lambda = (z^m - z^{-m})^2 [N]_q + (q - q^{-1}) \sum_{(i,j)\in\lambda} (z^{-2m} q^{N+2j-2i} - z^{2m} q - ^{N-2j+2i}).$$

Since $q = z^N$, (63) follows. □

Remarks. 1. The corepresentation v^λ of $\mathcal{A}$ with Young diagram λ corresponds to the representation of $U_q(\mathfrak{g})$ with highest weight $\lambda = \sum_{i=1}^{N-1} m_i \omega_i$ where ω_i are the fundamental weights and m_i the number of columns in λ of length i. Let $B(\cdot,\cdot)$ denote the Killing metric on the Lie algebra sl_{N-1} and let ρ_0 be the half sum of positive roots. Then the eigenvalue of the classical Laplace–Beltrami operator (with respect to the biinvariant metric) corresponding to the highest weight λ is given by the formula

$$\begin{aligned}\tilde{E}_\lambda &= \mathrm{B}(\lambda + \rho_0, \lambda + \rho_0) - \mathrm{B}(\rho_0, \rho_0)\\ &= \sum_{i=1}^{N-1} \frac{(N-i)m_i}{N} \left(i(m_i + N) + 2 \sum_{j=1}^{i-1} j m_j \right)\end{aligned} \tag{65}$$

(see [10]). For the quantum case one can check that $\lim_{q\to 1} (q - 1/q)^{-2} E_\lambda = \tilde{E}_\lambda$.

2. For $N = 2$ we have $q = z^2$ and Equation (63) reduces to the formula

$$E_{[m]} := 2(z - z^{-1})^2 [m]_z [m+2]_z. \tag{66}$$

□

PROPOSITION 7.2. *Let z be a transcendental real number and $q = z^N$.*

(i) All the eigenvalues of the Laplace–Beltrami operator $\Delta : \mathcal{A} \to \mathcal{A}$ are nonnegative.
(ii) For any $a \in \mathcal{A}$ we have $\Delta(a) = 0$ if and only if $a \in \mathbb{C}1$.
(iii) The smallest positive eigenvalue of $\Delta : \mathcal{A} \to \mathcal{A}$ is

$$\min\{\mathrm{E}_\lambda \mid \lambda = [1^{\mathrm{k}}, 0^{\mathrm{N-k}}], \mathrm{k} = 1, \ldots, \mathrm{N} - 1\}. \tag{67}$$

Proof. We prove the assertions of the Proposition in the case $z > 0$. The other cases are an easy consequence of this one.

Firstly one shows that if $\lambda = [l_1, l_2, \ldots, l_N]$, $l_1 \geqslant l_2 \geqslant \ldots \geqslant l_N \geqslant 1$, then $E_\lambda = E_{\lambda'}$, where $\lambda' = [l_1 - 1, l_2 - 1, \ldots, l_N - 1]$. Secondly, if $\lambda = [l_1, l_2, \ldots, l_k, 0^{N-k}]$, $l_k > 0$, $1 \leqslant k < N$, and $l_i > l_{i+1}$, $l_i \geqslant 2$ for some $i = 1, 2, \ldots, k$, then let λ' be the diagram $[l_1, l_2, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_k, 1, 0^{N-k-1}]$. One can prove that $E_\lambda > E_{\lambda'}$ since $[n]_z > [n - 2n']_z$ for all $n' \in \mathbb{N}$, $n \in \mathbb{Z}$. Therefore, for any $\lambda \neq [0^N]$ there exists a $\lambda' = [1^k, 0^{N-k}]$ such that $E_\lambda \geqslant E_{\lambda'}$. Obviously, $E_{[0^N]} = 0$ and because of $[m]_p > 0$ for any $m \in \mathbb{N}$, $p > 0$, we also have

$$E_{\lambda'} = (z - z^{-1})^2 \big([k]_z^2 [N]_q + [N]_z [k]_q [(N+2)(N-k-1) + 2]_z \big) > 0$$

for any $\lambda' = [1^k, 0^{N-k}]$, $1 \leqslant k < N$. Hence the assertions follow. □

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Remark. Let q be a transcendental complex number. Let $\mathcal{A}$ be one of the quantum groups $\mathcal{O}(\mathrm{Sp}_q(N))$ or $\mathcal{O}(\mathrm{O}_q(N))$, $N \geqslant 3$, and Γ_+, Γ_- as in Section 3, where u is the fundamental corepresentation of $\mathcal{A}$. Then the settings ${G_1}^i_j := \varepsilon r/2\delta^i_j$, ${G_2}^i_j := \delta^i_j$, ${F_1}^i_j := rq^{2\rho_i}/2\delta^i_j$, ${F_2}^i_j := \varepsilon q^{-2\rho_i}\delta^i_j$, where $r = \varepsilon q^{N-\varepsilon}$ (we use the notation of [4]), determine a left-covariant σ-metric of the pair (Γ_+, Γ_-). Similarly to the proof of Proposition 7.1, using (6.14) in [8] one can show that the eigenvalues of the Laplace–Beltrami operator Δ on $\mathcal{A}$ corresponding to the Young diagram λ are

$$E_\lambda = (q - q^{-1})^2 \sum_{(i,j)\in\lambda} [N - \varepsilon + 2j - 2i]_q.$$

During the computations the operators $r\sum_{k=1}^m D_k^+ - r^{-1}\sum_{k=1}^m D_k^-$ of the Birman–Wenzl–Murakami algebra appear – one can take (64) for the definition of $D_k^\pm$, where $\hat{R}^\pm$ denote the matrices

$$\hat{R}^{\pm}{}^{ij}_{kl} = q^{\pm(\delta^i_j - \delta^i_{j'})}\delta^i_l\delta^j_k \pm (\pm i < \pm l)(q - q^{-1})(\delta^i_k\delta^j_l - \varepsilon_i\varepsilon_l q^{\rho_l - \rho_i}\delta^i_{j'}\delta^k_{l'}), \tag{68}$$

which are central in the algebra $\mathrm{Mor}(u^{\otimes m+1})$. □

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