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ANALYTIC SETS, BAIRE SETS AND THE STANDARD PART MAP

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The problems considered here arose in connection with the interesting use by Loeb [**8**] and Anderson [**1**], [**2**] of Loeb's measure construction [**7**] to define measures on certain topological spaces. The original problem, from which the results given here developed, was to identify precisely the family of sets on which these measures are defined.

To be precise, let $\mathscr{M}$ be a set theoretical structure and $^*\mathscr{M}$ a nonstandard extension of $\mathscr{M}$, as in the usual framework for nonstandard analysis (see [**10**]). Let X be a Hausdorff space in $\mathscr{M}$ and st_X the standard part map for X, defined on the set of nearstandard points in $^*X$. Suppose, for example, μ is an internal, finitely additive probability measure defined on the internal subsets of $^*X$. Loeb's construction defines a standard measure $^\circ\mu$ by letting $^\circ\mu(B) = \text{st}(\mu(B))$ for internal B and then extending in a unique way to a countably additive measure. This measure is defined on all sets in the σ-algebra generated by the internal sets. Loeb and Anderson use st_X as a measure preserving map to define a measure υ on X by

$$\upsilon(S) = {}^\circ\mu(\text{st}_X{}^{-1}(S)).$$

The question considered here is: for which sets S is $\upsilon(S)$ defined? That is, when is $\text{st}_X{}^{-1}(S)$ in the σ-algebra generated by the internal sets? We give a complete answer (Theorem 2) when X is completely regular (which includes all the specific cases where this construction has been used). Namely, $\text{st}_X{}^{-1}(S)$ is in the σ-algebra generated by the internal sets if and only if S is a Baire set in some (every) compactification of X. In particular, when X is compact, then the measure υ is defined exactly on the Baire sets. When X is a complete metric space, then υ is defined just on the separable Borel sets.

In general it is useful to consider the situation where the internal measure μ is only defined on subsets of an internal set A which satisfies $\text{st}_X(A) = X$. For example, A is often taken to be a *-finite set and μ to be given by a system of weights assigned to the elements of A. Thus we are led to ask when $A \cap \text{st}_X{}^{-1}(S)$ is in the σ-algebra on A generated by the internal sets. This turns out to be independent of which set A is used, as long as it satisfies $\text{st}_X(A) = X$.

This also leads to consideration of another interesting question: which subsets of X are of the form $\text{st}_X(B)$, where B is in the σ-algebra on $^*X$ generated by the internal sets? (See Theorems 1 and 4).

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In answering these questions it is necessary to introduce the Souslin operation, applied both to subsets of X and to subsets of $^*X$. We let Seq denote the set of finite sequences of positive integers and let α range over all infinite sequences of positive integers. Given any family $\mathscr{F}$ of sets, S is derived from $\mathscr{F}$ by the Souslin operation if there is a system of sets $(F_s | s \in \text{Seq})$ from $\mathscr{F}$ so that

$$S = \bigcup_\alpha \bigcap_n F_{\alpha|n}.$$

(Here $\alpha|n$ is the finite sequence $\alpha(1), \alpha(2), \ldots, \alpha(n)$.) Let $\mathscr{S}(\mathscr{F})$ denote the collection of all S which can be derived from $\mathscr{F}$ by the Souslin operation. We will make use of a number of well known facts about $\mathscr{S}(\mathscr{F})$ and the Souslin operation. For an exposition of this theory see [**3**] and [**9**]. The first of these facts is that $\mathscr{S}(\mathscr{F})$ is already closed under the Souslin operation. In particular, it is closed under countable unions and intersections. If $\mathscr{F}$ is an algebra of sets, as will be true below, then $\mathscr{S}(\mathscr{F})$ contains the σ-algebra of sets generated by $\mathscr{F}$. That is, for each set B in that σ-algebra there exist sets $(A_s | s \in \text{Seq})$ in $\mathscr{F}$ such that

$$B = \bigcup_\alpha \bigcap_n A_{\alpha|n}.$$

This direct and explicit representation of B will be of considerable importance below, where $\mathscr{F}$ is a family of internal sets.

The second general fact about the Souslin operation which is useful here is the basic separation theorem for $\mathscr{S}(\mathscr{F})$ where $\mathscr{F}$ is a semi-compact system of sets. (Theorem III.14 of [**9**].) Our application of it here is to show that a set B is in the σ-algebra generated by the internal sets if and only if both B and its complement can be represented as above using the Souslin operation applied to internal sets.

Another important separation theorem which we use here is due to Z. Frolik [**5**, Theorem 5]. It states that if X is a compact Hausdorff space and $S \subseteq X$, then S is a Baire set in X if and only if S and $X \backslash S$ are derived from the closed subsets of X by the Souslin operation.

We assume that the reader is familiar with the basic framework of nonstandard analysis (see [**10**]). For convenience we make a strong saturation assumption about $^*\mathscr{M}$, as is done in [**10**]. Namely, we assume that $^*\mathscr{M}$ is κ-saturated for some cardinal number κ which is greater than the cardinality of any set in $\mathscr{M}$. While the amount of saturation needed below varies from one result to the next, we leave to the interested reader the (easy) job of finding out in each case what is the minimum saturation assumption required for the proof to be valid. We adopt the following notation and terminology, most of which is familiar. A *space* is a completely regular, Hausdorff space. These are the only topological spaces we consider, and we assume that all spaces mentioned are in $\mathscr{M}$. Given a space X and its counterpart $^*X$ in $^*\mathscr{M}$, the standard part map for X is denoted by st_X. (When X is the space of real numbers we just write st.) Also $\text{ns}(^*X)$ denotes the set of nearstandard elements of $^*X$. Given any set

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$B \subseteq {}^*X$, $\text{st}_X(B)$ is the set $\{x \in X |$ for some $p \in B$, $\text{st}_X(p) = x\}$. For $S \subseteq X$, $\text{st}_X{}^{-1}(S)$ is the set of nearstandard points $\{p \in {}^*X | \text{st}_X(p) \in S\}$. Both of these set mappings are used extensively below.

Given a space X as above, recall that a subset of X is a Baire set if it is in the σ-algebra on X generated by sets of the form

$$\{x \in X | \quad f(x) \leqq \alpha\}$$

where f is any continuous real-valued function on X and α is any real number. When X is a metric space then every closed set can be written in this form and so the Baire sets coincide with the Borel sets.

The following Lemma gives the key property of the standard part map on which most of this paper rests. This "descending chain" property also holds for many other kinds of functions and thus many of the results proved here are true in a more general setting. For example, see [**6**, Theorem 4] for a result about Loeb measures proved using the techniques developed here. They also apply to the quotient mappings which arise in the theory of nonstandard hulls of uniform spaces or locally convex spaces.

LEMMA 1. *If $A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing chain of internal subsets of ${}^*X$, then*

$$\text{st}_X(\textstyle\bigcap_n A_n) = \textstyle\bigcap_n \text{st}_X(A_n).$$

Proof. It is clear that the left side is contained in the right side of this equation. For the other direction, suppose $x \in \text{st}_X(A_n)$ holds for all $n \in N$. Let $\mathscr{O}$ be a fundamental system of open neighborhoods of x, so that the monad of x is $\bigcap\{{}^*V | V \in \mathscr{O}\}$. Our blanket saturation assumption implies that we have $\text{card}(\mathscr{O}) < \kappa$, where ${}^*\mathscr{M}$ is κ-saturated. For each $n \in \mathbf{N}$ and $V \in \mathscr{O}$ let

$$S(n, V) = \{p \in {}^*X | \quad p \in A_n \cap {}^*V\}.$$

Given finitely many $V_1, \ldots, V_k \in \mathscr{O}$ and $n_1, \ldots, n_k \in \mathbf{N}$, we let $V = V_1 \cap \ldots \cap V_k$ and $n = \max(n_1, \ldots, n_k)$; then

$$S(n_1, V_1) \cap \ldots \cap S(n_k, V_k) \supseteq \{p \in {}^*X | \quad p \in A_n \cap {}^*V\}$$

which is nonempty since $x \in \text{st}_X(A_n)$ and V is an open neighborhood of x.

By the saturation assumption, there exists $p \in {}^*X$ such that $p \in S(n, V)$ holds for every $n \in \mathbf{N}$ and $V \in \mathscr{O}$. That is, $p \in \bigcap_n A_n$ and $p \in {}^*V$ for every $V \in \mathscr{O}$, so that $\text{st}_X(p) = x$. This shows that x is an element of the left side of the equation to be proved.

THEOREM 1. *If B is derived from the internal subsets of ${}^*X$ by the Souslin operation, then $\text{st}_X(B)$ is derived from the closed subsets of X by the Souslin operation. In particular, this is true if B is in the σ-algebra on ${}^*X$ generated by the internal sets.*

Proof. Let $(E_s | \quad s \in \text{Seq})$ be a family of internal subsets of ${}^*X$ such that

$$B = \textstyle\bigcup_\alpha \bigcap_n E_{\alpha|n}.$$

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We may assume that $E_s \supseteq E_t$ whenever $s \leqq t$. Then

$$\mathrm{st}_X(B) = \bigcup_\alpha \mathrm{st}_X(\bigcap_n E_{\alpha|n}) = \bigcup_\alpha \bigcap_n \mathrm{st}_X(E_{\alpha|n})$$

by Lemma 1. Thus $\mathrm{st}_X(B)$ is derived from the sets $(\mathrm{st}_X(E_s)| \quad s \in \mathrm{Seq})$ by the Souslin operation. Luxemburg has shown that, under our saturation assumptions, $\mathrm{st}_X(A)$ is closed in X whenever A is an internal subset of $^*X$. (*Proof.* Suppose $x \notin \mathrm{st}_X(A)$ and let $\mathscr{O}$ be a fundamental system of open neighborhoods of x in X. Now A does not intersect the set $\bigcap\{^*V| \quad V \in \mathscr{O}\}$. Hence, by our saturation assumption, there exists $V \in \mathscr{O}$ so that $A \cap {}^*V = \emptyset$. But then A is disjoint from the monad of any $y \in V$ and so V is an open neighborhood of x which is disjoint from $\mathrm{st}_X(A)$). Therefore, $\mathrm{st}_X(B)$ is derived from the closed subsets of X by the Souslin operation.

We remark that Lemma 1 and Theorem 1 are valid for any Hausdorff topological space X. However the results which follow below seem to require stronger hypotheses. Thus we have made the general restriction here only to consider completely regular, Hausdorff spaces.

Our next result gives a complete answer to the question of which sets S have $\mathrm{st}_X^{-1}(S)$ in the σ-algebra generated by the internal sets. For a compact space X these are just the Baire sets; for a general space the situation is a little more complicated.

THEOREM 2. *Let A be an internal subset of $^*X$ which satisfies* $\mathrm{st}_X(A) = X$. *For each $S \subseteq X$ the following conditions are equivalent:*

(i) *$A \cap \mathrm{st}_X^{-1}(S)$ is in the σ-algebra on A generated by the internal sets.*

(ii) *For every space $Y \supseteq X$ in which X is dense, S is a Baire set in Y.*

(iii) *For some compact space $Y \supseteq X$, S is a Baire set in Y.*

Proof. First we consider a space Y which contains X as a dense subspace. Note that if $p \in {}^*X$ and $x \in X$, then $\mathrm{st}_Y(p) = x$ holds if and only if $\mathrm{st}_X(p) = x$. Thus for any set $Z \subseteq X$, $\mathrm{st}_Y^{-1}(Z) = {}^*X \cap \mathrm{st}_X^{-1}(Z)$. Also, we have

$$X = \mathrm{st}_X(A) \subseteq \mathrm{st}_Y(A);$$

since $\mathrm{st}_Y(A)$ is closed in Y, we conclude that $\mathrm{st}_Y(A) = Y$.

Now we prove (i) implies (ii). Let $B = A \cap \mathrm{st}_X^{-1}(S)$, which equals $A \cap \mathrm{st}_Y^{-1}(S)$ by the argument above. Also this shows $S = \mathrm{st}_Y(B)$ and $Y \backslash S = \mathrm{st}_Y(A \backslash B)$. Since both B and $A \backslash B$ are in the σ-algebra on A generated by internal sets, Theorem 1 implies that S and $Y \backslash S$ are both derived from the closed subsets of Y by the Souslin operation. This holds whenever Y contains X as a dense subspace.

Consider such an extension Y of X and let $\bar{Y}$ be any compact space containing Y as a dense subspace. The argument above shows that S and $\bar{Y} \backslash S$ are derived from the closed subsets of Y by the Souslin operation. Therefore by the Baire Separation Theorem due to Z. Frolik [**5**, Theorems 3 and 5] S is a Baire set in $\bar{Y}$. It follows that S is a Baire set in Y, proving (ii).

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Evidently (ii) implies (iii). To show that (iii) implies (i), let Y be a compact space which contains X and in which S is a Baire set. We may assume that X is dense in Y, passing to the closure of X in Y if necessary. Now the set of subsets of X which satisfy (i) is evidently a σ-algebra on X. Thus we need only show that (i) holds for S of the form

$$S = \{x \in X| \quad f(x) = 0\}$$

where $f : Y \to \mathbf{R}$ is continuous. But for this S,

$$\begin{aligned} A \cap \mathrm{st}_X{}^{-1}(S) &= A \cap \mathrm{st}_Y{}^{-1}(S) \\ &= \{p \in A| \quad \mathrm{st}(^*f(p)) = 0\} \\ &= \cup_n\{p \in A| \quad |^*f(p)| < 1/n\} \end{aligned}$$

which is a union of internal subsets of A.

This last calculation and others of a related kind occur first in [**7**]. The observation that when X is a compact space, then $\mathrm{st}_X{}^{-1}(S)$ is in the σ-algebra generated by the internal sets at least when S is a Baire set, is used repeatedly in the work of Loeb and Anderson. Note that the assumption that $A \cap \mathrm{st}_X{}^{-1}(S)$ is in the σ-algebra generated by internal sets turns out to be independent of A (as long as A is internal and $\mathrm{st}_X(A) = X$). That is, any such condition implies the apparently stronger condition that $\mathrm{st}_X{}^{-1}(S)$ is in that σ-algebra. Moreover, the proof of Theorem 2 shows that when such a condition is true of S, then $\mathrm{st}_X{}^{-1}(S)$ is actually in the σ-algebra on $^*X$ generated by standard sets $^*Z$, where Z is the zero-set of some continuous function $f : X \to \mathbf{R}$.

Frolik [**5**] has called a space X *bianalytic* if there exists a compact space Y containing X such that X is a Baire set in Y. (See also the exposition in [**4**, Chapter 9] where these spaces are called *absolute Baire Spaces*.) Evidently any set S satisfying the conditions in Theorem 2 is bianalytic.

COROLLARY 2.1. *Let A be an internal subset of $^*X$ such that* $\mathrm{st}_X(A) = X$. *Then* $A \cap \mathrm{ns}(^*X)$ *is in the σ-algebra on A generated by the internal sets if and only if X is a bianalytic space.*

Proof. This is immediate from the equivalence between (i) and (iii) in Theorem 2 (for $S = X$).

All bianalytic spaces are Lindelöf and therefore a metrizable bianalytic space is separable [**5**]. These spaces are exactly the separable, metrizable spaces X with the property that if Y is metrizable and has X as a subspace, then X is a Borel set in Y. Such spaces are called *absolute Borel* spaces in [**4**], where an exposition of their properties can be found. For our purposes, the essential feature of the metric case is that the "dense subspace" requirement in condition (ii) of Theorem 2 can be dropped. That is, if Y is a metrizable space and X is a bianalytic subspace of Y, then X is a Borel set in Y, not just in the closure of X. Thus Theorem 2 can be restated in the metric case as follows:

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COROLLARY 2.2. *Let X be metrizable and let A be an internal subset of $^*X$ which satisfies* $\mathrm{st}_X(A) = X$. *For any $S \subseteq X$, $A \cap \mathrm{st}_X{}^{-1}(S)$ is in the σ-algebra generated by the internal subsets of A if and only if S is separable and is an absolute Borel space.*

We remark that a separable, complete metrizable space is an absolute Borel space. Moreover, if Y is an absolute Borel space and X is a Borel subset of Y, then X is also an absolute Borel space.

A space X is *analytic* if there exists a compact space $Y \supseteq X$ such that X is derived from the closed subsets of Y by the Souslin operation. This implies that for any space $Y \supseteq X$, the same derivation of X from closed subsets of Y is possible. (See [**5**] for an exposition and references. We have presented a definition best suited to the point of view taken here.)

COROLLARY 2.3. *Let A be an internal subset of X which satisfies* $\mathrm{st}_X(A) = X$. *If $S \subseteq X$ and $A \cap \mathrm{st}_X{}^{-1}(S)$ is derived from internal subsets of A by the Souslin operation, then S is analytic.*

Proof. Since X is completely regular, it is dense in some compact space Y. Also, as shown in the proof of Theorem 2, $\mathrm{st}_Y(A) = Y$ and $A \cap \mathrm{st}_Y{}^{-1}(S) = A \cap \mathrm{st}_X{}^{-1}(S)$. Therefore S equals $\mathrm{st}_Y(A \cap \mathrm{st}_Y{}^{-1}(S))$ and Theorem 1 implies that S is derived from the closed subsets of Y by the Souslin operation. Hence S is analytic.

In many cases the converse to Corollary 2.3 is true. For example, it is true if X is a compact space in which every closed set is a G_δ (the *perfectly normal* compact spaces). In that case, if C is a closed set in X then $C = \bigcap_n V_n$ for some open sets V_n and therefore $\mathrm{st}_X{}^{-1}(C) = \bigcap_n {}^*V_n$. Moreover, if $S = \bigcup_\alpha \bigcap_n C_{\alpha|n}$ is an analytic set in X then

$$\mathrm{st}_X{}^{-1}(S) = \bigcap_\alpha \bigcup_n \mathrm{st}_X{}^{-1}(C_{\alpha|n})$$

which is therefore derived from the internal subsets of X by the Souslin operation.

This also shows that the converse to Corollary 2.3 is valid for any space X which has a perfectly normal compactification. For example, it is true when X is a separable metric space. However, the converse to Corollary 2.3 is false in general and we do not know exactly which topological condition (if any) is equivalent to the nonstandard condition on S given there. The following result suggests that the correct condition may be that S can be derived from the Baire sets by the Souslin operation (in every extension of X).

THEOREM 3. *Let A be an internal subset of $^*X$ such that* $\mathrm{st}_X(A) = X$. *If S is a compact subset of X and $A \cap \mathrm{st}_X{}^{-1}(S)$ can be derived from the internal subsets of A by the Souslin operation, then S is a G_δ set.*

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Proof. Let $\mathcal{O}$ be the family of all open sets in X which contain S. Since S is compact

$$\mathrm{st}_X^{-1}(S) = \bigcap\{*V| \quad V \in \mathcal{O}\} \quad \text{and so}$$
$$A \cap \mathrm{st}_X^{-1}(S) = \bigcap\{A \cap *V| V \in \mathcal{O}\}.$$

This set is therefore the intersection of fewer than κ internal sets, where $*\mathcal{M}$ is κ-saturated. It follows that if $(B_n| \, n \in \mathbf{N})$ are internal sets and $A \cap \mathrm{st}_X^{-1}(S) \cap \bigcap_n B_n = \emptyset$, then there exist $V_1, \ldots, V_n \in \mathcal{O}$ and $k \in \mathbf{N}$ such that $A \cap *V_1 \cap \ldots \cap *V_n \cap B_1 \cap \ldots \cap B_k = \emptyset$.

Now suppose $A \cap \mathrm{st}_X^{-1}(S)$ is derived from internal subsets of A using the Souslin operation. Let $\mathcal{A}_0$ be the countable algebra of subsets of A generated by the internal sets used. All sets in $\mathcal{A}_0$ are internal. Let $\mathcal{A}_0' = \{B \in \mathcal{A}_0| \, A \cap \mathrm{st}_X^{-1}(S) \subseteq B\}$; we claim that $A \cap \mathrm{st}_X^{-1}(S) = \bigcap \mathcal{A}_0'$. If not, there exists $p \in \bigcap \mathcal{A}_0'$ with $p \notin A \cap \mathrm{st}_X^{-1}(S)$. Let $\mathcal{A}_0'' = \{B \in \mathcal{A}_0| \quad p \in B\}$ and let $P = \bigcap \mathcal{A}_0''$. The set P has the property that, for each $B \in \mathcal{A}_0$, either $P \subseteq B$ or $P \cap B = \emptyset$. Since $A \cap \mathrm{st}_X^{-1}(S)$ is derived from sets in $\mathcal{A}_0$ by the Souslin operation, it follows that $P \subseteq A \cap \mathrm{st}_X^{-1}(S)$ or $P \cap A \cap \mathrm{st}_X^{-1}(S) = \emptyset$. Since $p \in P$ the first is impossible. Therefore, by the saturation argument given above, there exists $B \in \mathcal{A}_0''$ such that $B \cap A \cap \mathrm{st}_X^{-1}(S) = \emptyset$. But then $A \backslash B$ is in $\mathcal{A}_0'$. This implies $p \in A \backslash B$ and $p \in B$, which is impossible.

Thus we have shown that there exist internal sets $(B_n| \quad n \in \mathbf{N})$ such that

$$\bigcap\{A \cap *V| \quad V \in \mathcal{O}\} = A \cap \mathrm{st}_X^{-1}(S) = \bigcap_n B_n.$$

By a familiar saturation argument it follows that there exist $V_1 \supseteq V_2 \supseteq \ldots$ in $\mathcal{O}$ so that $A \cap \mathrm{st}_X^{-1}(S) = \bigcap_n (A \cap *V_n)$. Also, since S is compact we may assume that V_n contains the closure of V_{n+1} for every $n \in \mathbf{N}$. Then applying the standard part function shows (Lemma 1) that

$$S = \bigcap_n V_n,$$

which completes the proof.

The method used in proving Theorem 3 actually proves the following, more general result, which fits with the methods of [**6**]: Let $\mathcal{F}$ be a filter of subsets of X in $\mathcal{M}$ and let $\mathcal{A}_0$ be a countable algebra of internal subsets of $*X$; then the filter monad of $\mathcal{F}$ $(= \bigcap\{*S| \quad S \in \mathcal{F}\})$ is in the complete Boolean algebra of subsets of $*X$ generated by $\mathcal{A}_0$ only if $\mathcal{F}$ has a countable basis.

For completeness we include here a strong converse to Theorem 1 for spaces without isolated points. It shows that the introduction of the Souslin operation was necessary for solving our original problems. The requirement that X have no isolated points may not be needed at all in this result. We know that the same result can be proved under the hypothesis that X has only countably many isolated points, but we have chosen to omit the extra details. We do not know what happens in general if there are uncountably many isolated points.

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THEOREM 4. *Let X be a space with no isolated points and let A be an internal subset of $^*X$ such that* $\mathrm{st}_X(A) = X$. *For each $S \subseteq X$ which is derived from the closed sets in X by the Souslin operation, there exist internal subsets of A $(A_{mn}| m, n \in \mathbf{N})$ such that* $S = \mathrm{st}_X(\bigcap_m \bigcup_n A_{mn})$.

Proof. First we introduce some notation. Given a set $B \subseteq {}^*X$, write $\mathrm{st}_X(B) \equiv T$ to mean that $\mathrm{st}_X(B) = T$ and for each $x \in T$ there exist infinitely many $p \in B$ which satisfy $\mathrm{st}_X(p) = x$.

Note that if A and X are as stated, then $\mathrm{st}_X(A) \equiv X$ must hold. (Indeed, given $x \in X$ let $\mathscr{O}$ be a fundamental system of open neighborhoods of x. Since x is not an isolated point, for each $V \in \mathscr{O}$ the set $A \cap {}^*V$ must be infinite. Using the saturation assumption it follows that

$$A \cap \mathrm{st}_X^{-1}(x) = A \cap \bigcap\{{}^*V| \quad V \in \mathscr{O}\}$$

is also infinite, as claimed).

Now we will prove the following technical fact:

LEMMA 2. *Assume B is internal,* $\mathrm{st}_X(B) \equiv T_1$ *and T_2 is a closed subset of T_1. Then there exist disjoint internal sets B_1, B_2 contained in B such that* $\mathrm{st}_X(B_j) \equiv T_j$ *for $j = 1$ and 2.*

Proof. The proof is a straightforward saturation argument. First choose disjoint sets D_1, D_2 contained in B such that $\mathrm{st}_X(D_j) = T_j$ and for each $x \in T_j$ there are exactly $\aleph_0$ elements in $D_j \cap \mathrm{st}_X^{-1}(x)$ $(j = 1, 2)$. Note that ${}^*\mathscr{M}$ is κ-saturated for a cardinal $\kappa > \mathrm{card}(D_1 \cup D_2)$.

Let $\mathscr{O}$ be the family of all open sets V whose closure is disjoint from T_2. Since X is completely regular, $X \backslash T_2 = \bigcup\{V| \quad V \in \mathscr{O}\}$. Since $\mathscr{O}$ is a standard family in $\mathscr{M}$, ${}^*\mathscr{M}$ is κ-saturated for a cardinal $\kappa > \mathrm{card}(\mathscr{O})$. Note that for each $x \in X$, if $x \in T_2$ then the monad of x, $\mathrm{st}_X^{-1}(x)$ is disjoint from ${}^*V$ for every $V \in \mathscr{O}$; if $x \notin T_2$ then the monad of x is contained in ${}^*V$ for some $V \in \mathscr{O}$.

Now we will show that there exists an internal set $B_2 \subseteq B$ such that $D_2 \subseteq B_2$, $D_1 \cap B_2 = \emptyset$ and $B_2 \cap {}^*V = \emptyset$ for every $V \in \mathscr{O}$. Using the saturation principle, it suffices to consider finitely many objects, $a_1, \ldots, a_m \in D_1$, $b_1, \ldots, b_n \in D_2$ and $V_1, \ldots, V_k \in \mathscr{O}$, and to find an internal set $B' \subseteq B$ such that $b_1, \ldots, b_n \in B'$, $a_1, \ldots, a_m \notin B'$ and $B' \cap ({}^*V_1 \cup \ldots \cup {}^*V_k) = \emptyset$. But $B' = B\backslash({}^*V_1 \cup \ldots \cup {}^*V_k \cup \{a_1, \ldots, a_m\})$ satisfies these conditions. (Since each b_j is in the monad of a point in T_2, it is outside of ${}^*V$ for every $V \in \mathscr{O}$. Since $D_1 \cap D_2 = \emptyset$, no b_j is equal to any a_i. Thus $b_1, \ldots, b_n$ are all in B'. The other requirements are obviously met.) Thus such a set B_2 exists; we let $B_1 = B\backslash B_2$. Then B_1, B_2 are disjoint and $D_j \subseteq B_j$ for $j = 1, 2$. Thus $\mathrm{st}_X(B_j) \supseteq \mathrm{st}_X(D_j) = T_j$ for each j. Moreover, $B_1 \subseteq B$ so we have $\mathrm{st}_X(B_1) \equiv T_1$. Also since B_2 is disjoint from ${}^*V$ for each $V \in \mathscr{O}$, it follows that $\mathrm{st}_X(B_2)$ is disjoint from V. Therefore $\mathrm{st}_X(B_2) \subseteq T_2$, and hence $\mathrm{st}_X(B_2) \equiv T_2$. This completes the proof that B_1, B_2 exist as claimed.

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This lemma can be strengthened in the following way: if B is internal and $\mathrm{st}_X(B) \equiv T$, then for each sequence $(T_n \mid n \in \mathbf{N})$ of closed subsets of T there exists a sequence $(B_n \mid n \in \mathbf{N})$ of pairwise disjoint, internal subsets of B such that $\mathrm{st}_X(B_n) \equiv T_n$ for all $n \in \mathbf{N}$. The sets B_n are obtained inductively; at the nth stage we have sets $B_1, \ldots, B_{n-1}$ as desired together with an internal B_n' which is disjoint from each of $B_1, \ldots, B_{n-1}$, contained in B and satisfies $\mathrm{st}_X(B_n') \equiv T$. Using the fact just proved, B_n' is split into two internal sets B_n and B'_{n+1} such that $\mathrm{st}_X(B_n) \equiv T_n$ and $\mathrm{st}_X(B'_{n+1}) \equiv T$.

Now we are ready to prove the theorem. Let S be a set derived from the closed subsets of X by the Souslin operation. That is, there is a family $(E_s \mid s \in \mathrm{Seq})$ of closed subsets of X such that

$$S = \bigcup_\alpha \bigcap_n E_{\alpha|n}.$$

We may assume that $E_s \supseteq E_t$ whenever $s \leqq t$ and that $E_\phi = X$.

Using the fact proved above we now construct a family $(B_s \mid s \in \mathrm{Seq})$ of internal subsets of A with the following properties:

(i) $\mathrm{st}_X(B_s) \equiv E_s$ for all $s \in \mathrm{Seq}$.

(ii) $B_s \supseteq B_{sk}$ for all $s \in \mathrm{Seq}$, $k \in \mathbf{N}$.

(iii) $B_{sk} \cap B_{sm} = \emptyset$ for all $s \in \mathrm{Seq}$ and all distinct $k, m \in \mathbf{N}$. (We construct the sets B_s by induction on the length of the sequence s, first setting $B_\phi = A$. Given B_s, use the fact proved above to obtain the sequence $(B_{sk} \mid k \in \mathbf{N})$).

Now by Lemma 1,

$$S = \mathrm{st}_X\left(\bigcup_\alpha \bigcap_n B_{\alpha|n}\right).$$

So it suffices to show that $\bigcup_\alpha \bigcap_n B_{\alpha|n}$ is of the form $\bigcap_m \bigcup_n A_{mn}$ for some internal sets $(A_{mn} \mid m, n \in \mathbf{N})$. But the conditions (ii) and (iii) imply that p is an element of $\bigcup_\alpha \bigcap_n B_{\alpha|n}$ if and only if (a) $p \in B_n$ for some $n \in \mathbf{N}$, and (b) for each $s \in \mathrm{Seq}$, if $p \in B_s$, then $p \in B_{sk}$ for some $k \in \mathbf{N}$. That is,

$$\bigcup_\alpha \bigcap_n B_{\alpha|n} = \left(\bigcup_n B_n\right) \cap \bigcap_{s \in \mathrm{Seq}} \left(A \backslash B_s\right) \cup \bigcup_k B_{sk}).$$

This set is of the desired form, completing the proof of Theorem 4.

We conclude this paper with a few methodological remarks. The key to our proof of Theorem 2 was the representation by the Souslin operation of sets in the σ-algebra generated by the internal sets. This has the advantage of giving a specific representation in terms of countably many internal sets instead of the usual inductive procedure for generating sets in the σ-algebra. It is therefore useful to know just how much is lost by using this Souslin representation. The next result shows that nothing is lost, as long as we simultaneously represent a set and its complement in this way.

THEOREM 5. *Let A be an internal set and S a subset of A. Then S is in the σ-algebra on A generated by the internal sets if and only if S and $A \backslash S$ are derived from the internal subsets of A by the Souslin operation.*

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Proof. This is an immediate consequence of a separation theorem for the Souslin operation applied to families of sets which are semicompact (see [**9**, Theorem III.14]). When $^*\mathscr{M}$ is $\aleph_1$-saturated, any family of internal sets is semi-compact.

We remark that a more precise version of Theorem 5 is sometimes useful. Suppose $(B_n| \quad n \in \mathbf{N})$ are internal subsets of A from which S and $A \backslash S$ are derived using the Souslin operation. Then [**9**, Theorem III.14] actually shows that S is in the σ-algebra on A generated by the sets $(B_n| \quad n \in \mathbf{N})$.

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