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On weaving Hilbert space frames and Riesz bases

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Abstract. **Two frames $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ are said to be weaving frames, if for every $\sigma \subset \mathbb{N}$, $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ is a frame for $\mathcal{H}$. Weaving frames are proved to be very useful in many areas, such as, distributed processing, wireless sensor networks, packet encoding and many more. Inspired by the work of Bemrose et al.[11], this paper delves into the properties and characterizations of weaving frames and weaving Riesz bases.**

1 Introduction

The concept of Hilbert space frames was first introduced by Duffin and Schaeffer [1] in 1952. After several decades, in 1986, the importance of frame theory was popularized by the groundbreaking work by Daubechies, Grossman and Meyer [2]. Since then frame theory has been widely used by mathematicians and engineers in various fields of mathematics and engineering, namely, operator theory [3], harmonic analysis [4], wavelet analysis [5], signal processing [6], image processing [7], sensor network [8], data analysis [9], Retro Banach Frame [10], etc.

Frame theory literature became richer through several generalizations - fusion frame (frames of subspaces) [12, 13] , G-frame (generalized frames) [14], K-frame (atomic systems) [15], K-fusion frame (atomic subspaces) [16], etc. - and these generalizations have been proved to be useful in various applications.

Over the years, weaving frames have been explored in various contexts (see [18, 19, 20]), yet their characterization through the structure of the associated kernel of the corresponding weaving synthesis operator remains largely unexplored. To bridge this gap, we undertake a comprehensive study of weaving frames and Riesz bases, focusing on their intrinsic properties and fundamental characterizations. Our approach provides a deeper understanding of their structural aspects, highlighting their stability and robustness in different settings. By analyzing their interplay with synthesis operators, we establish new theoretical insights into their behavior. This work aims to contribute significantly to the ongoing research in frame theory and its applications in mathematical analysis.

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and $\mathcal{L}(\mathcal{H})$ for $\mathcal{L}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, we denote $D(T)$, $N(T)$ and $R(T)$ for domain, null space and range of T, respectively, and $\mathcal{I}$ is the identity operator.

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2 Preliminaries

Before diving into the main sections, throughout this section we recall basic definitions and results needed in this paper. For a detailed discussion regarding frames, Riesz bases we refer to [17].

2.1 Frame

A collection $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \;\le\; \sum_{i\in I} |\langle f, f_i\rangle|^2 \;\le\; B\|f\|^2, \tag{2.1}$$

for all $f \in \mathcal{H}$. The numbers A, B are called *frame bounds*. The supremum over all A's and infimum over all B's satisfying above inequality are called the *optimal frame bounds*. If a collection satisfies only the right inequality in (2.1), it is called a *Bessel sequence*.

Given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, the *pre-frame operator* or *synthesis operator* is a bounded linear operator $T : l^2(I) \to \mathcal{H}$ and is defined by $T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i$. The adjoint of T, $T^* : \mathcal{H} \to l^2(I)$, given by $T^*f = \{\langle f, f_i\rangle\}_{i\in I}$, is called the *analysis operator*. The *frame operator*, $S = TT^* : \mathcal{H} \to \mathcal{H}$, is defined by

$$Sf = TT^*f = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

It is well-known that the frame operator is bounded, positive, self adjoint and invertible.

Here we present the definition of weaving frames, which serves as the foundation for our study. This paper primarily focuses on examining the structural properties and theoretical aspects of weaving frames.

Definition 2.1 [11] Two frames $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ are said to be weaving frames, if for every $\sigma \subset \mathbb{N}$, $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ is a frame for $\mathcal{H}$. i.e. for every $f \in \mathcal{H}$ and for every $\sigma \subset \mathbb{N}$, there exist universal bounds $\alpha \le \beta$ so that we have the following inequality:

$$\alpha\|f\|^2 \le \sum_{n\in\sigma} |\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, g_n\rangle|^2 \le \beta\|f\|^2.$$

3 Main Results

In this section we discuss various properties and characterizations of weaving frames and weaving Riesz bases.

Proposition 3.1. *Let $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$ and $\mathcal{G} = \{g_n\}_{n=1}^{\infty}$ be frames for $\mathcal{H}$ with bounds α_1, β_1 and α_2, β_2, and the associated synthesis operators $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ respectively. Suppose $0 < \gamma < 1$ so that $\gamma(\|T_{\mathcal{F}}\| + \|T_{\mathcal{G}}\|) \le \frac{\alpha_1}{2}$. Further, if for every $\{c_n\}_{n=1}^{\infty} \in \ell^2$ we have,*

$$\|\sum_{n=1}^{\infty} c_n(f_n - g_n)\| \le \gamma\|\{c_n\}\|. \tag{3.1}$$

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Then $\mathcal{F}$ and $\mathcal{G}$ are weaving frames.

Proof 1. *For every $\sigma \subset \mathbb{N}$, let $T_{\mathcal{F}_\sigma}(\{c_n\}_{n=1}^{\infty}) = \sum_{n\in\sigma} c_n f_n$ and $T_{\mathcal{G}_\sigma}(\{c_n\}_{n=1}^{\infty}) = \sum_{n\in\sigma} c_n g_n$.*

Then applying [Theorem 6.1, [11]] we have $\|T_{\mathcal{F}_\sigma}\| \leq \|T_{\mathcal{F}}\|$ and $\|T_{\mathcal{G}_\sigma}\| \leq \|T_{\mathcal{G}}\|$. Using equation (3.1) we have,

$$\|T_{\mathcal{F}_\sigma} - T_{\mathcal{G}_\sigma}\| \leq \|T_{\mathcal{F}} - T_{\mathcal{G}}\| < \gamma.$$

Therefore, for every $f \in \mathcal{H}$ we have,

$$\begin{aligned}\|(T_{\mathcal{F}_\sigma}T^*_{\mathcal{F}_\sigma} - T_{\mathcal{G}_\sigma}T^*_{\mathcal{G}_\sigma})f\| &\leq \|T_{\mathcal{F}_\sigma}\|\|T^*_{\mathcal{F}_\sigma} - T^*_{\mathcal{G}_\sigma}\|\|f\| + \|T^*_{\mathcal{G}_\sigma}\|\|T_{\mathcal{F}_\sigma} - T_{\mathcal{G}_\sigma}\|\|f\| \\ &\leq (\|T_{\mathcal{F}}\|\|T_{\mathcal{F}} - T_{\mathcal{G}}\| + \|T_{\mathcal{G}}\|\|T_{\mathcal{F}} - T_{\mathcal{G}}\|)\|f\| \\ &\leq \gamma(\|T_{\mathcal{F}}\| + \|T_{\mathcal{G}}\|)\|f\| \\ &\leq \frac{\alpha_1}{2}\|f\|.\end{aligned}$$

Thus for every $\sigma \subset \mathbb{N}$ and $f \in \mathcal{H}$ we obtain,

$$\begin{aligned}&\left\|\sum_{n\in\sigma}|\langle f, g_n\rangle|^2 + \sum_{n\in\sigma^c}|\langle f, f_n\rangle|^2\right\| \\ &= \left\|\sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2 + \left(\sum_{n\in\sigma}|\langle f, g_n\rangle|^2 - \sum_{n\in\sigma}|\langle f, f_n\rangle|^2\right)\right\| \\ &\geq \left\|\sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2\right\| - \left\|\sum_{n\in\sigma}|\langle f, g_n\rangle|^2 - \sum_{n\in\sigma}|\langle f, f_n\rangle|^2\right\| \\ &\geq \alpha_1\|f\|^2 - \langle(T_{\mathcal{G}_\sigma}T^*_{\mathcal{G}_\sigma} - T_{\mathcal{F}_\sigma}T^*_{\mathcal{F}_\sigma})f, f\rangle \\ &\geq \alpha_1\|f\|^2 - \frac{\alpha_1}{2}\|f\|^2 \\ &= \frac{\alpha_1}{2}\|f\|^2.\end{aligned}$$

On the other hand, the right-hand inequality will hold automatically. Consequently, $\mathcal{F}$ and $\mathcal{G}$ are weaving frames.

The following result provides necessary and sufficient conditions for the wovenness of the images of given weaving frames under a bounded linear operator.

***Theorem* 3.1** *$\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be two weaving frames for $\mathcal{H}$ with the universal bounds α, β. If $\mathcal{T}$ is a bounded linear operator in $\mathcal{H}$. Then the following are equivalent:*

(1) For every $f \in \mathcal{H}$, there exists $\lambda > 0$ so that $\|\mathcal{T}^* f\|^2 \geq \lambda\|f\|^2$.
(2) $\{\mathcal{T} f_n\}_{n=1}^{\infty}$ and $\{\mathcal{T} g_n\}_{n=1}^{\infty}$ are weaving frames for $\mathcal{H}$.

Proof 2. *($1 \implies 2$) Since $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are weaving frames for $\mathcal{H}$ with the universal bounds α, β, then for every $f \in \mathcal{H}$ and $\sigma \subset \mathbb{N}$ we have,*

$$\alpha\|f\|^2 \leq \sum_{n\in\sigma}|\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c}|\langle f, g_n\rangle|^2 \leq \beta\|f\|^2. \tag{3.2}$$

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Again for every $f \in \mathcal{H}$, *there exists* $\lambda = \inf_{\|f\|=1} \|\mathcal{T}^* f\|^2 > 0$ *for which we have* $\|\mathcal{T}^* f\|^2 \geq \lambda \|f\|^2$. *Therefore, applying equation (3.2), for every* $f \in \mathcal{H}$ *and* $\sigma \subset \mathbb{N}$ *we have,*

$$\begin{aligned}\sum_{n\in\sigma} |\langle f, \mathcal{T} f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, \mathcal{T} g_n\rangle|^2 &= \sum_{n\in\sigma} |\langle \mathcal{T}^* f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle \mathcal{T}^* f, g_n\rangle|^2 \\ &\geq \alpha \|\mathcal{T}^* f\|^2 \\ &\geq \lambda\, \alpha \|f\|^2.\end{aligned}$$

Furthermore, applying equation (3.2), for every $f \in \mathcal{H}$ *and* $\sigma \subset \mathbb{N}$ *we have,*

$$\begin{aligned}\sum_{n\in\sigma} |\langle f, \mathcal{T} f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, \mathcal{T} g_n\rangle|^2 &= \sum_{n\in\sigma} |\langle \mathcal{T}^* f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle \mathcal{T}^* f, g_n\rangle|^2 \\ &\leq \beta \|\mathcal{T}^* f\|^2 \\ &\leq \beta \|\mathcal{T}\|^2 \|f\|^2.\end{aligned}$$

Thus $\{\mathcal{T} f_n\}_{n=1}^\infty$ *and* $\{\mathcal{T} g_n\}_{n=1}^\infty$ *are weaving frames for* $\mathcal{H}$.

(2 $\implies$ 1) This implication is very straightforward and follows directly from the given conditions.

The following result establishes a characterization of weaving frames using the associated hyperplane.

Theorem 3.2 *Let* $\{f_n\}_{n=1}^\infty$ *and* $\{g_n\}_{n=1}^\infty$ *be frames for* $\mathcal{H}$. *Then the following are equivalent:*

(1) $\{f_n\}_{n=1}^\infty$ *and* $\{g_n\}_{n=1}^\infty$ *are weaving frames for* $\mathcal{H}$ *with the universal bounds* α, β.
(2) For every $\sigma \subset \mathbb{N}$ *and every hyperplane* $\mathcal{B} \subset \mathcal{H}$ *we have,*

$$\alpha \leq \sum_{n\in\sigma} \|P_{\mathcal{B}^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}^\perp} g_n\|^2 \leq \beta.$$

(3) For every $\sigma \subset \mathbb{N}$ *and every subspace* $\mathcal{B}_m$ *with codimension* m *in* $\mathcal{H}$ *we have,*

$$m\alpha \leq \sum_{n\in\sigma} \|P_{\mathcal{B}_m^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}_m^\perp} g_n\|^2 \leq m\beta.$$

Proof 3. *(1 $\implies$ 2) Let* $\{f_n\}_{n=1}^\infty$ *and* $\{g_n\}_{n=1}^\infty$ *be weaving frames for* $\mathcal{H}$ *with the universal bounds* α, β. *Then for every* $f \in \mathcal{H}$ *and* $\sigma \subset \mathbb{N}$ *we have,*

$$\alpha \|f\|^2 \leq \sum_{n\in\sigma} |\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, g_n\rangle|^2 \leq \beta \|f\|^2. \tag{3.3}$$

For every $f \in \mathcal{H}$ *with* $\|f\| = 1$, *let us assume that* $\mathcal{B} = (span\{f\})^\perp$. *Therefore, for every* n *we obtain,* $|\langle f, f_n\rangle|^2 = \|P_{\mathcal{B}^\perp} f_n\|^2$ *and* $|\langle f, g_n\rangle|^2 = \|P_{\mathcal{B}^\perp} g_n\|^2$. *Thus we have,*

$\alpha \geq \inf\left\{\sum_{n\in\sigma} \|P_{\mathcal{B}^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}^\perp} g_n\|^2 : \mathcal{B} \text{ is a hyperplane in } \mathcal{H}\right\}$ *and* $\beta \leq \sup\left\{\sum_{n\in\sigma} \|P_{\mathcal{B}^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}^\perp} g_n\|^2 : \mathcal{B} \text{ is a hyperplane in } \mathcal{H}\right\}$.

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Applying equation (3.3) we obtain,

$$\alpha \le \sum_{n\in\sigma} \|P_{\mathcal{B}^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}^\perp} g_n\|^2 \le \beta.$$

($\underline{2 \implies 1}$) *Suppose $\mathcal{B}$ is a hyperplane in $\mathcal{H}$. Then for every $f \in \mathcal{B}^\perp$ with $\|f\| = 1$ and every n we have, $|\langle f, f_n\rangle|^2 = \|P_{\mathcal{B}^\perp} f_n\|^2$ and $|\langle f, g_n\rangle|^2 = \|P_{\mathcal{B}^\perp} g_n\|^2$. Thus it is straightforward to verify that the assumed condition directly ensures that $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are weaving frames for $\mathcal{H}$ with the universal bounds α, β.*

($\underline{2 \implies 3}$) *Suppose for every $\sigma \subset \mathbb{N}$ and every hyperplane $\mathcal{B} \subset \mathcal{H}$ we have,*

$$\alpha \le \sum_{n\in\sigma} \|P_{\mathcal{B}^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}^\perp} g_n\|^2 \le \beta. \tag{3.4}$$

Let $\mathcal{B}_m$ be a subspace of $\mathcal{H}$ with codimension m. Then $\dim \mathcal{B}_m{}^\perp = m$. Suppose $\{e_1, e_2, \cdots, e_m\}$ is an orthonormal basis in $\mathcal{B}_m{}^\perp$ and let us define $\mathcal{B}_i{}^\perp = span\{e_i\}$, for every $i = 1, 2, \cdots, m$. Then using equation (3.4) with the hyperplane $\mathcal{B}_i$ for every $i = 1, 2, \cdots, m$ we have,

$$\alpha \le \sum_{n\in\sigma} \|P_{\mathcal{B}_i^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}_i^\perp} g_n\|^2 \le \beta. \tag{3.5}$$

Furthermore, we have $P_{\mathcal{B}_m^\perp} = \sum_{i=1}^m P_{\mathcal{B}_i^\perp}$. Thus we obtain,

$$\begin{aligned}
&\sum_{i=1}^m \left(\sum_{n\in\sigma} \|P_{\mathcal{B}_i^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}_i^\perp} g_n\|^2 \right) \\
&= \sum_{n\in\sigma} \sum_{i=1}^m \|P_{\mathcal{B}_i^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \sum_{i=1}^m \|P_{\mathcal{B}_i^\perp} g_n\|^2 \\
&= \sum_{n\in\sigma} \|P_{\mathcal{B}_m^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}_m^\perp} g_n\|^2.
\end{aligned}$$

Hence applying equation (3.5) we have,

$$m\alpha \le \sum_{n\in\sigma} \|P_{\mathcal{B}_m^\perp} f_n\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{B}_m^\perp} g_n\|^2 \le m\beta.$$

($\underline{3 \implies 2}$) *This implication is straightforward.*

The following result provides a characterization of weaving frames through the associated weaving synthesis operator. In this theorem, we establish a necessary and sufficient condition for two frames to form weaving frames. This characterization is based on the structure of the image of the standard orthonormal basis in ℓ^2 under the corresponding weaving synthesis operator. This characterization is achieved by examining how the frame vectors contribute to the structure of this image.

***Theorem* 3.3** *Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be two frames for $\mathcal{H}$. Then the following are equivalent:*

(1) $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are weaving frames for $\mathcal{H}$.

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(2) For every $\sigma \subset \mathbb{N}$, if T_σ is the associated weaving synthesis operator then we have,

$$T_\sigma e_n = \begin{cases} f_n : n \in \sigma \\ g_n : n \in \sigma^c \end{cases},$$

where $\{e_n\}_{n=1}^\infty$ is the canonical orthonormal basis in ℓ^2.

Proof 4. *(1 $\implies$ 2) Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be weaving frames for $\mathcal{H}$ with the universal bounds $0 < \alpha \leq \beta < \infty$. Then for every $\sigma \subset \mathbb{N}$ and for every $f \in \mathcal{H}$ we have,*

$$\alpha\|f\|^2 \leq \sum_{n\in\sigma} |\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, g_n\rangle|^2 \leq \beta\|f\|^2. \tag{3.6}$$

Therefore, the operator $T_\sigma^* : \mathcal{H} \to \ell^2$ defined as

$$T_\sigma^* f = \{\langle f, f_n\rangle\}_{n\in\sigma} \cup \{\langle f, g_n\rangle\}_{n\in\sigma^c}$$

is bounded. Thus for every $f \in \mathcal{H}$ we have $\|T_\sigma^* f\|^2 \geq \alpha\|f\|^2$. Since for every $f \in \mathcal{H}$ we have,

$$\langle f, T_\sigma e_n\rangle = \langle T_\sigma^* f, e_n\rangle = \begin{cases} \langle f, f_n\rangle : n \in \sigma \\ \langle f, g_n\rangle : n \in \sigma^c \end{cases}$$

Therefore, we obtain $T_\sigma e_n = \begin{cases} f_n : n \in \sigma \\ g_n : n \in \sigma^c. \end{cases}$

(2 $\implies$ 1) Let T_σ be the associated weaving synthesis operator so that we have $T_\sigma e_n = \begin{cases} f_n : n \in \sigma \\ g_n : n \in \sigma^c. \end{cases}$

Therefore, for every $f \in \mathcal{H}$ we have,

$$T_\sigma^* f = \sum_{n=1}^\infty \langle T_\sigma^* f, e_n\rangle e_n = \sum_{n\in\sigma} \langle f, f_n\rangle e_n + \sum_{n\in\sigma^c} \langle f, g_n\rangle e_n.$$

Thus for every $f \in \mathcal{H}$ we have,

$$\sum_{n\in\sigma} |\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, g_n\rangle|^2 = \|T_\sigma^* f\|^2 \leq \|T_\sigma^*\|^2\|f\|^2 \leq \beta\|f\|^2,$$

where $\beta = \sup_\sigma \|T_\sigma^*\|^2$. Again since for every $\sigma \subset \mathbb{N}$, the associated synthesis operator T_σ is onto then T_σ^* is one-one with closed range (see [21], p. 487). Therefore, it is bounded below on the unit sphere in $\mathcal{H}$ and hence there exists $\alpha > 0$ so that for every $\sigma \subset \mathbb{N}$ and for every $f \in \mathcal{H}$ we obtain,

$$\sum_{n\in\sigma} |\langle f, f_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle f, g_n\rangle|^2 = \|T_\sigma^* f\|^2 \geq \alpha\|f\|^2.$$

Consequently, $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are weaving frames for $\mathcal{H}$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces. It is well-known that, if $\{f_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}_1$, $\{g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}_2$, and $Q : \mathcal{H}_1 \to \mathcal{H}_2$ is an isomorphism so that $Qf_n = g_n$ for every n, then the frames $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are equivalent (see [22]).

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Thus equivalent frames preserve the same topological properties in the sense that their synthesis and analysis operators are related through the isomorphism Q, as their structural behavior remains invariant under equivalence relation. This concept is illustrated through the following example:

The frames $\{(1,0),(0,1)\}$ and $\{(2,0),(0,3)\}$ in $\mathbb{R}^2$ are equivalent through $Q = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Let us consider an orthogonal projection $P : \ell^2 \to X$, where X is a closed subspace of ℓ^2. Then $\{Pe_n\}$ is a frame. If for every $\sigma \subset \mathbb{N}$, T_σ is the associated synthesis operator of two weaving frames $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$, and $P : \ell^2 \to (KerT_\sigma)^\perp$ is the orthogonal projection onto $(KerT_\sigma)^\perp$ then applying Theorem 3.3 we have,

$$T_\sigma(Pe_n) = \begin{cases} f_n : n \in \sigma \\ g_n : n \in \sigma^c. \end{cases}$$

Thus for every $\sigma \subset \mathbb{N}$, $T_\sigma : (KerT_\sigma)^\perp \to \mathcal{H}$ is an isometric onto map and hence we conclude the following remark.

Remark 3.1 Suppose $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are weaving frames and for every $\sigma \subset \mathbb{N}$, if T_σ is the associated weaving synthesis operator then the frames $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ in $\mathcal{H}$ and $\{Pe_n\}_{n=1}^\infty$ in $(KerT_\sigma)^\perp$ are equivalent with respect to the induced map T_σ.

Proof 5. *Since $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are weaving frames, then applying Theorem 3.3, for every $\sigma \subset \mathbb{N}$ we have,*

$$T_\sigma(Pe_n) = \begin{cases} f_n : n \in \sigma \\ g_n : n \in \sigma^c, \end{cases}$$

where P is the orthogonal projection from ℓ^2 onto $(KerT_\sigma)^\perp$. Thus T_σ maps $(KerT_\sigma)^\perp$ isometrically onto $\mathcal{H}$.

Consequently, the frame $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ in $\mathcal{H}$ is equivalent to the frame $\{Pe_n\}_{n=1}^\infty$ in $(KerT_\sigma)^\perp$ (see [[22], p. 68]).

***Definition* 3.1** [23] A sequence $\{f_n\}_{n=1}^\infty$ is said to be a Besselian frame if it is a frame for $\mathcal{H}$, and after the removal of a finite number of vectors, the resulting sequence forms a Riesz basis for $\mathcal{H}$.

In this context it is to be noted that for a Besselian frame $\{f_n\}_{n=1}^\infty$ in $\mathcal{H}$, $\sum\limits_{n=1}^{\infty} a_n f_n$ converges in $\mathcal{H}$ if and only if $\{a_n\}_{n=1}^\infty \in \ell^2$.

Example 3.1. *If $\{e_n\}_{n=1}^\infty$ is an orthonormal basis in ℓ^2, then*
$\{e_1, e_1, e_2, e_3, e_4, \cdots\}$ is a Besselian frame.

***Definition* 3.2** Two Besselian frames in $\mathcal{H}$ are said to be weaving Besselian frames, if every weaving of them is a Besselian frame for $\mathcal{H}$.

Example 3.2. *If $\{e_n\}_{n=1}^\infty$ is an orthonormal basis in ℓ^2, then it is easy to verify that $\{e_1, e_1, e_2, e_3, e_4, \cdots\}$ and $\{e_1, e_1, e_2, e_2, e_3, e_4, \cdots\}$ are weaving Besselian frames.*

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Proposition 3.2. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be weaving Besselian frames for $\mathcal{H}$ with the associated weaving synthesis operator $T_\sigma : \ell^2 \to \mathcal{H}$, for every $\sigma \subset \mathbb{N}$. Then the dimension of $Ker(T_\sigma)$ is finite.*

Proof 6. *For every $\sigma \subset \mathbb{N}$, let us consider the orthogonal projection $P : \ell^2 \to (KerT_\sigma)^\perp$. Then applying Remark 3.1, $\{Pe_n\}_{n=1}^{\infty}$ is a frame for $(KerT_\sigma)^\perp$ and hence it is equivalent to the Besselian frame $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$. Thus applying [Theorem 2.3, [23]], the dimension of $Ker(T_\sigma)$ is finite.*

Motivated by the notion of Riesz basis, we discuss the concept of near Riesz basis in a similar framework. For detailed discussion on the same, we refer to the relevant literature cited herein [17, 22].

***Definition* 3.3** Let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. If there is a finite set γ so that $\{f_n\}_{n\notin\gamma}$ is a Riesz basis in $\mathcal{H}$, then $\{f_n\}_{n=1}^{\infty}$ is said to be a near Riesz basis.

We define near weaving Riesz bases in analogy with the notion of weaving Besselian frames.

***Definition* 3.4** Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be a two near Riesz bases in $\mathcal{H}$. If there exists a finite subset γ in $\mathbb{N}$ so that for every $\sigma \subset \mathbb{N} \setminus \gamma$, $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ is a near Riesz basis in $\mathcal{H}$, then $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are said to be near weaving Riesz bases.

Remark 3.2 It is to be noted that the concepts of Besselian frames and near Riesz bases are equivalent.

The following result provides a characterization of near weaving Riesz bases through the kernel of the associated weaving synthesis operator. This characterization offers insights into the interplay between the synthesis operator and the weaving property of near Riesz bases.

***Theorem* 3.4** *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be weaving frames for $\mathcal{H}$ with the associated weaving synthesis operator $T_\sigma : \ell^2 \to \mathcal{H}$, for every $\sigma \subset \mathbb{N}$. Then the following are equivalent:*

(1) $KerT_\sigma$ is finite dimensional.
(2) $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are near weaving Riesz bases in $\mathcal{H}$.

Proof 7. *($\underline{1 \implies 2}$) If for every $\sigma \subset \mathbb{N}$, P is the orthogonal projection from ℓ^2 onto $(KerT_\sigma)^\perp$, then applying Remark 3.1 the frames $\{Pe_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ are equivalent. Therefore, $\{f_n\}_{n\in\sigma} \cup \{g_n\}_{n\in\sigma^c}$ is a near Riesz basis and hence $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are near weaving Riesz bases if and only if $\{Pe_n\}_{n=1}^{\infty}$ is a near Riesz basis in $(KerT_\sigma)^\perp = R(P)$.*

Let $KerT_\sigma$ is finite dimensional. Then $(\mathcal{I} - P)$ represents the orthogonal projection onto $KerT_\sigma$ and hence we have $\sum_{n=1}^{\infty} \|(\mathcal{I} - P)e_n\|^2 < \infty$ (see [24]). Therefore, there exists a positive integer k so that $\sum_{n=k+1}^{\infty} \|e_n - Pe_n\|^2 < 1$.

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Let us consider a sequence $\{x_n\}_{n=1}^{\infty}$ *where* $x_n = \begin{cases} e_n : n = 1, 2, \cdots, k \\ Pe_n : otherwise. \end{cases}$ *Then we have* $\sum_{n=1}^{\infty} \|e_n - x_n\|^2 < 1$ *and hence* $\{x_n\}_{n=1}^{\infty}$ *forms a basis in* ℓ^2*, which is equivalent to* $\{e_n\}_{n=1}^{\infty}$ *(see [25]). Therefore,* $\{x_n\}_{n=1}^{\infty}$ *is a Riesz basis in* ℓ^2 *and hence* $\{Pe_n\}_{n=k+1}^{\infty}$ *is a Riesz basis in* $\overline{span}\{Pe_n\}_{n=k+1}^{\infty}$ *inside* ℓ^2*.*

Furthermore, since $\overline{span}\{Pe_n\}_{n=1}^{\infty} = (KerT_{\sigma})^{\perp}$*, then for every* $\sigma \subset \mathbb{N}$*,* $\{Pe_n\}_{n=k+1}^{\infty}$ *can be extended to a Riesz basis in* $(KerT_{\sigma})^{\perp}$*. Hence there exists* $\gamma \subset \{1, 2, \cdots, k\}$ *for which* $\{Pe_n\}_{n\notin\gamma}$ *is a Riesz basis in* $(KerT_{\sigma})^{\perp}$*. Thus* $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *are near weaving Riesz bases in* $\mathcal{H}$*.*

(2 $\implies$ 1) If $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *are near weaving Riesz bases, then the removal of finite number of elements from every weaving ensures that the remaining sequence forms a Riesz basis. Thus it is easy to verify that* $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *are Besselian weaving frames.*

Therefore, applying Proposition 3.2 for every $\sigma \subset \mathbb{N}$*,* $KerT_{\sigma}$ *is finite dimensional.*

References

[1] R. Duffin and A. C. Schaeffer, A class of non harmonic Fourier series. Trans. Amer. Math. Soc., **72** (1952), 341–366.

[2] I. Daubechies, A. Grossmann and Y. Meyer, Painless non orthogonal expansions, J. Math. Phys., **27** (1986), 1271–1283.

[3] D. Han and D. R. Larson, Frames, Bases and Group Representations, Amer. Math. Soc., 2000.

[4] K. Gröchenig, Foundations of Time-frequency Analysis, New York(NY), Birkhäuser, 2001.

[5] A. Gumber and N. K. Shukla, Uncertainty principle corresponding to an orthonormal wavelet system, Applicable Analysis, **97** (2018), 486–498.

[6] P. Ferreira, Mathematics for multimedia signal processing II: discrete finite frames and signal reconstruction, Signal Processing for Multimedia. J. S. Byrnes (Ed.), 1999, 35–54.

[7] A. Arefijamaal and E. Zekaee, Image processing by alternate dual Gabor frames, Bull. Iran. Math. Soc., **42** (2016), 1305–1314.

[8] P. G. Casazza, G. Kutyniok, S. Li and C. J. Rozell, Modeling sensor networks with fusion frames, Proc. SPIE. 2007;6701:67011M-1–67011M-11.

[9] P. G. Casazza and G. Kutyniok, Finite Frames: Theory and Applications, Boston: Birkhäuser, Applied and Numerical Harmonic Analysis, 2012.

[10] G. Verma, L. K. Vashisht and M. Singh, On Excess of Retro Banach Frames, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), **54** (2019), 143–146.

[11] T. Bemrose, P. G. Casazza, K. Gröchenig, M.C. Lammers and R. G. Lynch, Weaving frames, Operators and Matrices, **10** (2016), 1093–1116.

[12] P. G. Casazza and G. Kutyniok, Frames of subspaces, Contemporary Math, AMS, **345** (2014), 87–114.

[13] P. G. Casazza, G. Kutyniok and S. Li, Fusion frames and distributed processing, Appl. Comput. Harmon. Anal., **25** (2008), 114–132.

[14] W. Sun, G-frames and G-Riesz bases, J. Math. Anal. Appl., **322** (2006), 437–452.

[15] L. Găvruţa, Frames for operators, Appl. Comput. Harmon. Anal., **32** (2012), 139–144.

[16] A. Bhandari and S. Mukherjee, Atomic subspaces for operators, Indian J. Pure Appl. Math., **51** (2020), 1039–1052.

[17] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[18] Deepshikha and L.K. Vashisht, Weaving K-Frames in Hilbert Spaces, Results Math., **73**, 81 (2018).

[19] Y. Xu and J. Leng, New inequalities for weaving frames in Hilbert spaces, Journal of Inequalities and Applications, **188** (2021).

[20] P. G. Casazza , D. Freeman and R. G. Lynch, Weaving Schauder frames, Journal of Approximation Theory, **211** (2016), 42–60.

https://doi.org/10.4153/S0008439525100891 Published online by Cambridge University Press

[21] N. Dunford and J. Schwartz, Linear Operators, I, Interscience, New York, 1963.
[22] I. Singer, Bases in Banach spaces, I, Springer-Verlag, Berlin, 1970.
[23] J. R. Holub, Pre-Frame Operators, Besselian Frames, and Near-Riesz Bases in Hilbert Spaces, Proc. AMS, **122** (1994), 779–785.
[24] J. Schatten, Norm ideals of completely continuous operators, Springer-Verlag, Berlin, 1970.
[25] J. Retherford and J. Holub, The stability of bases in Banach and Hilbert spaces, J. Reine Angew. Math., **246** (1994), 136–146.

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