Canad. J. Math. Vol. 57 (1), 2005 pp. 17–60

On Amenability and Co-Amenability of Algebraic Quantum Groups and Their Corepresentations

Erik Bédos, Roberto Conti and Lars Tuset

Abstract. We introduce and study several notions of amenability for unitary corepresentations and $*$-representations of algebraic quantum groups, which may be used to characterize amenability and co-amenability for such quantum groups. As a background for this study, we investigate the associated tensor C^*-categories.

1 Introduction

The concept of amenability plays an important role in the theory of locally compact groups and in the theory of operator algebras (see [29] and references therein). The concept of amenability and its companion, co-amenability, have been introduced and studied by several authors in various quantum group settings (see [38, 15, 33, 1, 2, 3, 9, 26, 27, 5, 6, 7], in chronological order). We refer to [5, 6, 7] for a discussion of the relationship between these papers. Some recent works related to this topic, which we received after the first draft of the present article had been submitted for publication, are [28, 11, 10, 35, 20].

In this paper, we introduce various concepts of "amenability" for unitary corepresentations of analytic extensions of algebraic quantum groups (as defined by J. Kustermans and A. van Daele [23]): *co-amenability* (a notion inspired by results in [5, 6]), *amenability* (inspired by the concept of amenability of a unitary representation of a locally compact group introduced by M. Bekka [8]) and the *weak containment property* (inspired by the classical characterization of the amenability of a group in terms of weak containment). We present several equivalent formulations of these properties, and use them to characterize amenability and co-amenability of algebraic quantum groups. One should note that C.-K. Ng [28] has independently introduced similar notions for unitary corepresentations of Kac algebras.

This paper is organized as follows. After some preliminaries in Section 2, we include a categorical interlude in Section 3. Here we show that the category of nondegenerate $*$-representations of the universal C*-algebraic quantum group associated to an algebraic quantum group [22] and the category of unitary corepresentations of the analytic extension of the dual quantum group (with opposite co-product) are

Received by the editors January 22, 2002; revised May 26, 2003.
The first and third authors were partially supported by the Norwegian Research Council.
AMS subject classification: Primary: 46L05, 46L65; secondary: 22D10, 22D25, 43A07, 43A65, 58B32.
Keywords: quantum group, amenability.

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

naturally isomorphic as tensor C^*-categories. This result makes it possible to transfer all notions introduced for unitary corepresentations to non-degenerate $*$-representations and vice-versa. We also derive the absorbing property of the fundamental multiplicative unitary and of the regular representation. In Section 4 we introduce the conjugate corepresentation and the Hilbert-Schmidt corepresentation associated with a unitary corepresentation. Section 5 is devoted to co-amenability, Section 6 to amenability and Section 7 to the weak containment property, where we also consider briefly property (T) for algebraic quantum groups. In Section 8 we gather some remarks on the relationship between these amenability concepts. Finally, in Section 9 we specialize our study of amenability to the setting of algebraic quantum groups of discrete type, where it is possible to exploit the structural properties of these quantum groups to push our analysis further. When we revised this paper prior to its publication, we realized that a careful tuning of minor parts of the proofs of Lemmas 9.3 and 9.4 gives Theorem 9.5 without having to restrict ourselves to discrete Kac algebras. We have implemented these changes in the final version. As a noteworthy corollary of Theorem 9.5 we now obtain a new proof of the recent result due to E. Blanchard and S. Vaes [10] and to R. Tomatsu [35], stating that amenability of a discrete quantum group implies (and is therefore equivalent to) co-amenability of its dual.

Every vector space will be over the ground field $\mathbb{C}$. Given a set V, ι_V denotes the identity map on it (but we simply write ι when there is no danger of confusion). If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ (resp., $B_0(\mathcal{H})$) denotes the algebra of all bounded (resp., compact) linear operators acting on $\mathcal{H}$. If $\mathcal{B}$ is a $*$-algebra, $M(\mathcal{B})$ denotes the multiplier algebra of $\mathcal{B}$. If $\mathcal{B}$ is unital, we denote its unit by $I_{\mathcal{B}}$, or by I when this causes no confusion. In this case, we denote by $\mathcal{U}(\mathcal{B})$ the unitary group of $\mathcal{B}$. When $\mathcal{B}$ is a C^*-algebra, $S(\mathcal{B})$ denotes its state space. As usual $\otimes$ denotes tensor product; depending on the context, it may be the tensor product of vector spaces, the Hilbert space tensor product or the minimal (that is, spatial) tensor product of C^*-algebras, $\bar{\otimes}$ being used for tensor products in the von Neumann algebra setting. However, we often use $\odot$ to stress that we are dealing with an algebraic tensor product. If V, W are vector spaces, $\chi\colon V \otimes W \to W \otimes V$ is the flip map sending $v \otimes w$ to $w \otimes v$ ($v \in V, w \in W$); if $\mathcal{H}$ is a Hilbert space then Σ is the flip map on $\mathcal{H} \otimes \mathcal{H}$. We use the leg-numbering notation as introduced in [1].

2 Preliminaries

Throughout this paper, $(\mathcal{A}, \Delta)$ denotes an algebraic quantum group in the sense of [36], see also [37, 23], where $\Delta\colon \mathcal{A} \to M(\mathcal{A} \odot \mathcal{A})$ is the co-product map. We follow notation and use terminology from these papers. Hence, S denotes the antipode of $(\mathcal{A}, \Delta)$, ε its co-unit and φ a fixed faithful left Haar functional. The functional φ is not necessarily tracial. However, there is a unique bijective homomorphism $\rho\colon \mathcal{A} \to \mathcal{A}$ such that $\varphi(ab) = \varphi(b\rho(a))$, for all $a, b \in \mathcal{A}$. Moreover, $\rho(\rho(a^*)^*) = a$.

The pair $(\mathcal{A}_r, \Delta_r)$ denotes the associated analytic extension (which is a reduced locally compact quantum group in the sense of [24]); $\pi_r\colon \mathcal{A} \to \mathcal{A}_r \subset B(\mathcal{H})$ is the (left) regular representation of $\mathcal{A}$ acting on the GNS Hilbert space $\mathcal{H}$ of φ; $\Lambda\colon \mathcal{A} \to \mathcal{H}$ is the canonical injection; $W \in M(\mathcal{A}_r \otimes B_0(\mathcal{H}))$ is the associated multiplicative

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

unitary; $\mathcal{M} = \mathcal{A}_r'' = \pi_r(\mathcal{A})''$ is the von Neumann algebra generated by $\pi_r(\mathcal{A})$; and R is the anti-unitary antipode (which is defined on $\mathcal{M}$).

We denote by $(\hat{\mathcal{A}}, \hat{\Delta})$ the dual algebraic quantum group and by $(\hat{\mathcal{A}}_r, \hat{\Delta}_r)$ the associated analytic extension. We recall that $\hat{\mathcal{A}}$ is the subspace of the algebraic dual of $\mathcal{A}$, consisting of all functionals $a\varphi$, where $a \in \mathcal{A}$. Here, $(a\varphi)(b) = \varphi(ba)$, and similarly, $(\varphi a)(b) = \varphi(ab)$, $a, b \in \mathcal{A}$. Since $\varphi a = \rho(a)\varphi$, we have $\hat{\mathcal{A}} = \{\varphi a \mid a \in \mathcal{A}\}$.

A right-invariant positive linear functional $\hat{\psi}$ is defined on $\hat{\mathcal{A}}$ by setting $\hat{\psi}(\hat{a}) = \varepsilon(a)$, for all $a \in \mathcal{A}$. Here $\hat{a} = a\varphi$. Since the linear map, $\mathcal{A} \to \hat{\mathcal{A}}$, $a \mapsto \hat{a}$, is a bijection (by faithfulness of φ), the functional $\hat{\psi}$ is well defined. Further, we have $\hat{\psi}(\hat{b}^*\hat{a}) = \varphi(b^*a)$, for all $a, b \in \mathcal{A}$.

As shown in [23], one may assume that the regular representation $\hat{\pi}_r$ of $\hat{\mathcal{A}}$ also acts on $\mathcal{H}$. Accordingly, we identify $\hat{\mathcal{A}}_r$ with the C*-algebra generated by $\hat{\pi}_r(\hat{\mathcal{A}})$ and set $\hat{\mathcal{M}} = \hat{\mathcal{A}}_r''$. A useful fact is that both $\mathcal{M}$ and $\hat{\mathcal{M}}$ act standardly on $\mathcal{H}$.

We will quite often work with the "opposite" dual quantum group $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$. Note that when we add op as a subscript to a co-product map, we mean by this the opposite co-product. One reason for working with $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ is that it corresponds to the dual of $(\mathcal{A}_r, \Delta_r)$ as defined in [24]. Further, the multiplicative unitary associated to $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ is simply given by $\hat{W} = \Sigma W^* \Sigma$, which fits with the usual convention for multiplicative unitaries and their duals (*cf.* [1]).

We denote by $(\mathcal{A}_u, \Delta_u)$ the universal (locally compact) C^*-algebraic quantum group associated to $(\mathcal{A}, \Delta)$, as introduced by J. Kustermans [22]. We recall here some details of his construction.

The C^*-algebra $\mathcal{A}_u$ is the completion of $\mathcal{A}$ with respect to the C^*-norm $\|\cdot\|_u$ on $\mathcal{A}$ defined by

$$\|a\|_u = \sup\{\|\Phi(a)\| \mid \Phi \text{ is a } *\text{-homomorphism from } \mathcal{A} \text{ into some } C^*\text{-algebra}\}$$

(The non-trivial fact that this expression gives a well-defined norm on $\mathcal{A}$ is shown in [22]). The C^*-algebra $\mathcal{A}_u$ has the universal property that one can extend from $\mathcal{A}$ to $\mathcal{A}_u$ any $*$-homomorphism from $\mathcal{A}$ into some C^*-algebra.

The definition of Δ_u relies on the following proposition [22, Proposition 3.8], which we restate here as we will need it in the sequel.

Proposition 2.1 *Consider C^*-algebras C_1, C_2 and $*$-homomorphisms ϕ_1 from $\mathcal{A}$ into $M(C_1)$ and ϕ_2 from $\mathcal{A}$ into $M(C_2)$ such that $\phi_1(\mathcal{A})C_1$ is dense in C_1 and $\phi_2(\mathcal{A})C_2$ is dense in C_2. Then there exists a unique $*$-homomorphism ϕ from $\mathcal{A}$ into $M(C_1 \otimes C_2)$ such that*

$$\big(\phi_1(a_1) \otimes \phi_2(a_2)\big)\,\phi(a) = (\phi_1 \odot \phi_2)\big((a_1 \otimes a_2)\Delta(a)\big)$$

and

$$\phi(a)\big(\phi_1(a_1) \otimes \phi_2(a_2)\big) = (\phi_1 \odot \phi_2)\big(\Delta(a)(a_1 \otimes a_2)\big)$$

for every $a_1, a_2 \in \mathcal{A}$. We have moreover that $\phi(\mathcal{A})(C_1 \otimes C_2)$ is dense in $C_1 \otimes C_2$.

Now, let π_u denote the identity mapping from $\mathcal{A}$ into $\mathcal{A}_u$. Hence, π_u is an injective $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A}_u$ such that $\pi_u(\mathcal{A})$ is dense in $\mathcal{A}_u$, so $\pi_u(\mathcal{A})\mathcal{A}_u$ is dense in $\mathcal{A}_u$ (as $\mathcal{A}^2 = \mathcal{A}$). By applying the above proposition with $\phi_1 = \phi_2 = \pi_u$

and exploiting the universal property of $\mathcal{A}_u$, one obtains that there exists a unique non-degenerate $*$-homomorphism $\Delta_u \colon \mathcal{A}_u \to M(\mathcal{A}_u \otimes \mathcal{A}_u)$ such that

$$(\pi_u \odot \pi_u)(x)\Delta_u(\pi_u(a)) = (\pi_u \odot \pi_u)(x\Delta(a))$$

and

$$\Delta_u(\pi_u(a))(\pi_u \odot \pi_u)(x) = (\pi_u \odot \pi_u)(\Delta(a)x)$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{A} \odot \mathcal{A}$.

Being a $*$-homomorphism from $\mathcal{A}$ onto $\mathbb{C}$, the co-unit ε of $(\mathcal{A}, \Delta)$ extends to a $*$-homomorphism ε_u from $\mathcal{A}_u$ onto $\mathbb{C}$, which is easily seen to satisfy the co-unit property for $(\mathcal{A}_u, \Delta_u)$. Of course, we identify implicitly here $\mathcal{A}$ with its canonical copy $\pi_u(\mathcal{A})$ inside $\mathcal{A}_u$. Note that sometimes we add u as an index to denote the extension to $\mathcal{A}_u$ of a $*$-homomorphism from $\mathcal{A}$ into some C^*-algebra, and sometimes use the same symbol to denote the extension when there is no danger of confusion. For example, we get a canonical map $\pi_r \colon \mathcal{A}_u \to \mathcal{A}_r$ which is the extension of $\pi_r \colon \mathcal{A} \to \mathcal{A}_r$.

Now let $(\mathcal{A}, \Delta)$ be an algebraic quantum group of *compact* type, that is, $\mathcal{A}$ has a unit I. It is immediate that $(\mathcal{A}_r, \Delta_r)$ is a compact quantum group in the sense of Woronowicz [39, 40], with Haar state φ_r given by the restriction of the vector state $\omega_{\Lambda(I)}$ to $\mathcal{A}_r$. The unique dense Hopf $*$-subalgebra [5] of $(\mathcal{A}_r, \Delta_r)$ may be identified with $(\mathcal{A}, \Delta, \varepsilon, S)$ (via the Hopf $*$-algebra isomorphism π_r). Using this identification, we may introduce the family $(f_z)_{z\in\mathbb{C}}$ of multiplicative linear functionals on $\mathcal{A}$ constructed by Woronowicz (see [39, 40]).

Some of the properties of this family are: $f_0 = \varepsilon$; $f_z * f_{z'} = f_{z+z'}$, where $\omega * \eta = (\omega \otimes \eta)\Delta$; the maps $a \to f_z * a = (\iota \otimes f_z)\Delta(a)$ and $a \to a * f_z = (f_z \otimes \iota)\Delta(a)$ are automorphisms of $\mathcal{A}$; we have $f_z^* = f_{-\bar{z}}$ and $f_z \circ S = f_{-z}$; we have $\varphi(ab) = \varphi(b(f_1 * a * f_1))$ and $S^2(a) = f_{-1} * a * f_1$. We also recall that the following three conditions are equivalent:

$$\varphi \text{ is tracial}; \quad f_z = \varepsilon \text{ for all } z \in \mathbb{C}; \quad f_1 = \varepsilon.$$

It follows from [21, Theorem 2.12] that $M(\hat{\mathcal{A}})$, the multiplier algebra of $\hat{\mathcal{A}}$, may be concretely realized as the subspace of the algebraic dual of $\mathcal{A}$ consisting of elements θ such that $(\theta \odot \iota)\Delta(a)$ and $(\iota \odot \theta)\Delta(a)$ belong to $\mathcal{A}$ for every $a \in \mathcal{A}$. Hence, in the compact case, we have $f_z \in M(\hat{\mathcal{A}})$ for all $z \in \mathbb{C}$.

The following description of algebraic quantum groups of *discrete* type, that is, those which are dual to algebraic quantum groups of compact type, will be useful.

Proposition 2.2 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group of compact type and let $(U^\alpha)_{\alpha\in A}$ denote a complete set of pairwise inequivalent irreducible unitary corepresentations of the compact quantum group $(\mathcal{A}_r, \Delta_r)$. Note that we have $U^\alpha \in \mathcal{A} \odot M_{d_\alpha}(\mathbb{C})$ for some $d_\alpha < \infty$, when identifying $\mathcal{A}$ as the dense Hopf $*$-algebra of $\mathcal{A}_r$. Write each U^α as a matrix (u^α_{ij}) over $\mathcal{A}$ and recall that the set $\{\, u^\alpha_{ij} | 1 \le i, j \le d_\alpha, \alpha \in A \,\}$ is a linear basis for $\mathcal{A}$. Let*

$$M_\alpha = \sum_{i=1}^{d_\alpha} f_{-1}(u^\alpha_{ii}) = \sum_{i=1}^{d_\alpha} f_1(u^\alpha_{ii})$$

denote the quantum dimension of U^α. *Further, set* $\hat{A}_\alpha = \operatorname{Span}\{\hat{u}^\alpha_{ij} \mid 1 \le i, j \le d_\alpha\}$ *and define*

$$p_\alpha = M_\alpha \sum_{i,j=1}^{d_\alpha} f_1(u^\alpha_{ji})\hat{u}^\alpha_{ij} \in \hat{A}_\alpha.$$

Then

(1) (i) $\hat{u}^\alpha_{ij}\hat{u}^\beta_{kl} = \frac{\delta_{\alpha\beta}}{M_\alpha} f_{-1}(u^\alpha_{kj})\hat{u}^\alpha_{il}$,

(ii) $(\hat{u}^\alpha_{ij})^* = \hat{u}^\alpha_{ji}$,

(iii) $p_\alpha \hat{u}^\beta_{kl} = \delta_{\alpha\beta}\hat{u}^\beta_{kl} = \hat{u}^\beta_{kl} p_\alpha$, *where* $1 \le i, j \le d_\alpha, 1 \le k, l \le d_\beta, \alpha, \beta \in A$.

(2) *The set* $\{\hat{u}^\alpha_{ij} \mid 1 \le i, j \le d_\alpha\}$ *is a linear basis for* $\hat{A}_\alpha$.

(3) *Each* $\hat{A}_\alpha$ *is a* $*$*-subalgebra of* $\hat{A}$, *which is unital with unit* p_α. *As a* $*$*-algebra,* $\hat{A}_\alpha$ *is isomorphic to the matrix algebra* $M_{d_\alpha}(\mathbb{C})$.

(4) $\hat{A} = \bigoplus_\alpha \hat{A}_\alpha$ *(algebraic direct sum).*

(5) *For each* $\alpha \in A$, *let* Tr_α *denote the canonical trace on* $\hat{A}_\alpha = M_{d_\alpha}(\mathbb{C})$ *satisfying* $\operatorname{Tr}_\alpha(p_\alpha) = d_\alpha$. *Then*

$$\hat{\psi}(x) = \bigoplus_\alpha M_\alpha \operatorname{Tr}_\alpha(p_\alpha x f_{-1}), \quad x \in \hat{A}.$$

Proof This result is essentially known (see [31, p. 393–394 and Theorem 3.3] or [13, p. 722]), but we will need the explicit description presented here in the sequel. We give a proof for the sake of completeness. It relies on the so-called orthogonality relations for the U_α's established in [39, 40].

(1) With obvious index notation, we have

$$\begin{aligned}
(\hat{u}^\alpha_{ij}\hat{u}^\beta_{kl})((u^\gamma_{pq})^*) &= \sum_r \varphi((u^\gamma_{pr})^* u^\alpha_{ij})\varphi((u^\gamma_{rq})^* u^\beta_{kl}) \\
&= (1/M_\alpha)(1/M_\beta)\sum_r \delta_{\alpha\gamma} f_{-1}(u^\alpha_{ip})\delta_{rj}\delta_{\beta\gamma} f_{-1}(u^\beta_{kr})\delta_{ql} \\
&= (1/M_\alpha)(1/M_\beta)\delta_{\alpha\gamma}\delta_{\beta\gamma}\delta_{ql} f_{-1}(u^\alpha_{ip}) f_{-1}(u^\beta_{kj}) \\
&= (1/M_\alpha^2)\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{ql} f_{-1}(u^\alpha_{ip}) f_{-1}(u^\alpha_{kj}) \\
&= (1/M_\alpha)\delta_{\alpha\beta} f_{-1}(u^\alpha_{kj})\varphi((u^\gamma_{pq})^* u^\alpha_{il}) \\
&= \big((1/M_\alpha)\delta_{\alpha\beta} f_{-1}(u^\alpha_{kj})\hat{u}^\alpha_{il}\big)((u^\gamma_{pq})^*),
\end{aligned}$$

and (i) follows. Concerning (ii), we have

$$(\hat{u}^\alpha_{ij})^* = (S(u^\alpha_{ij})^*)^\wedge = \hat{u}^\alpha_{ji},$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

using [23, Lemma 7.14]. Further, using (i), we get

$$
\begin{aligned}
p_\alpha \hat{u}^\beta_{kl} &= M_\alpha \sum_{i,j} f_1(u^\alpha_{ji}) \hat{u}^\alpha_{ij} \hat{u}^\beta_{kl} \\
&= M_\alpha (1/M_\alpha) \delta_{\alpha\beta} \sum_{i,j} f_1(u^\alpha_{ji}) f_{-1}(u^\alpha_{kj}) \hat{u}^\alpha_{il} \\
&= \delta_{\alpha\beta} \sum_i \delta_{ik} \hat{u}^\alpha_{il} = \delta_{\alpha\beta} \hat{u}^\beta_{kl},
\end{aligned}
$$

which is equal to $\hat{u}^\beta_{kl} p_\alpha$ by a similar computation. Hence, (iii) is proved.

(2) As $\{\, u^\alpha_{ij} | 1 \le i, j \le d_\alpha, \alpha \in A \,\}$ is a linear basis for $\mathcal{A}$ and the map $a \to \hat{a}$ is a linear isomorphism between $\mathcal{A}$ and $\hat{\mathcal{A}}$, (2) is clear.

(3) The first sentence is an obvious consequence of (1). Now, $\hat{\mathcal{A}}_\alpha$ may be seen as a finite dimensional C^*-algebra (using the faithfulness of the $*$-homomorphism $\hat{\pi}_r$). Hence, to show that $\hat{\mathcal{A}}_\alpha$ is isomorphic to $M_{d_\alpha}(\mathbb{C})$, it is enough to show that if $b = \sum_{i,j} b_{ij} \hat{u}^\alpha_{ij}$, $b_{ij} \in \mathbb{C}$, is an element of the center of $\hat{\mathcal{A}}^\alpha$, then $b = \lambda p_\alpha$, that is,

$$
b_{ij} = \lambda f_1(u^\alpha_{ji}) \ \text{ for some } \ \lambda \in \mathbb{C}, 1 \le i, j \le d_\alpha.
$$

Now, using (1)(i), one sees immediately that $\hat{u}^\alpha_{kl} b = b \hat{u}^\alpha_{kl}$ hold for all k and l if and only if

$$
\sum_{i,j} b_{ij} f_{-1}(u^\alpha_{kj}) \hat{u}^\alpha_{il} = \sum_{i,j} b_{ij} f_{-1}(u^\alpha_{il}) \hat{u}^\alpha_{kj}
$$

for all k, l, which in turn is equivalent to

$$
(\star) \qquad \delta_{rk} \sum_i b_{is} f_{-1}(u^\alpha_{il}) = \delta_{sl} \sum_j b_{rj} f_{-1}(u^\alpha_{kj}), \quad \forall\, k, l, r, s.
$$

We introduce now the two complex matrices $B = (b_{ij})$ and $C = (c_{ij})$, where $c_{ij} = f_{-1}(u^\alpha_{ji})$. Then $(\star)$ may be rewritten as

$$
(\star\star) \qquad \delta_{rk} d_{ls} = \delta_{sl} e_{rk}, \quad \forall\, k, l, r, s,
$$

where $d_{ls} = \sum_i c_{li} b_{is}$ and $e_{rk} = \sum_j b_{rj} c_{jk}$. From $(\star\star)$, we clearly get

$$
(BC)_{sl} = 0 = (CB)_{sl}, \ s \ne l; \qquad (BC)_{ll} = (BC)_{kk},
$$

hence that $BC = CB$ is a complex multiple λ of the identity matrix. But C is invertible, with inverse $C^{-1} = (c'_{ij})$, where $c'_{ij} = f_1(u^\alpha_{ji})$. Indeed, we have

$$
\sum_j c_{ij} c'_{jk} = \sum_j f_{-1}(u^\alpha_{ji}) f_1(u^\alpha_{kj}) = \varepsilon(u^\alpha_{ki}) = \delta_{ik}.
$$

Therefore, we can conclude that $B = \lambda C^{-1}$, that is, $b_{ij} = \lambda f_1(u^\alpha_{ji})$. This establishes (3).

(4) This is an easy consequence of the previous assertions.

(5) Now fix $\alpha \in A$ and define a linear functional τ on $\hat{\mathcal{A}}_\alpha$ by

$$\tau(x) = (1/M_\alpha)\hat{\psi}(xf_1),\ x \in \hat{\mathcal{A}}_\alpha.$$

To show (5), a moment's thought makes it clear that it is enough to show that $\tau = \mathrm{Tr}_\alpha$. Owing to the uniqueness property of Tr_α, we only have to show that

(a) τ is tracial; (b) $\tau(p_\alpha) = d_\alpha$.

To show (a), we have to show

$$\text{(a}'\text{)} \qquad \hat{\psi}(xyf_1) = \hat{\psi}(yxf_1), \quad x, y \in \hat{\mathcal{A}}^\alpha.$$

Now, let $\hat{\rho}$ denote the automorphism of $\hat{\mathcal{A}}$ satisfying $\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$ for all $a, b \in \mathcal{A}$. Then we get $\hat{\psi}(yxf_1) = \hat{\psi}(xf_1\hat{\rho}(y))$, so (a$'$) follows if $yf_1 = f_1\hat{\rho}(y)$ hold for all $y \in \hat{\mathcal{A}}$, that is, if $\hat{\rho}(y) = f_{-1}yf_1,\ y \in \hat{\mathcal{A}}$. This follows from Lemma 2.3.

To show (b), we first observe that

$$\begin{aligned}(\hat{u}^\alpha_{ij}f_1)((u^\beta_{kl})^*) &= \sum_r \varphi\big((u^\beta_{kr})^* u^\alpha_{ij}\big)\, f_1\big((u^\beta_{rl})^*\big)\\ &= \delta_{\alpha\beta}(1/M_\alpha)\sum_k f_1((u^\beta_{rl})^*)\delta_{rj}f_{-1}(u^\alpha_{ik})\\ &= \delta_{\alpha\beta}(1/M_\alpha)f_1((u^\beta_{jl})^*)f_{-1}(u^\alpha_{ik}).\end{aligned}$$

Using this, we show that $p_\alpha f_1 = M_\alpha \sum_i \hat{u}^\alpha_{ii}$. Indeed, we have

$$\begin{aligned}(p_\alpha f_1)((u^\beta_{kl})^*) &= M_\alpha \sum_{i,j} f_1(u^\alpha_{ji})\big(\hat{u}^\alpha_{ij}f_1\big)\,\big((u^\beta_{kl})^*\big)\\ &= \delta_{\alpha\beta}\sum_{i,j} f_1(u^\alpha_{ji})f_1((u^\beta_{jl})^*)f_{-1}(u^\alpha_{ik})\\ &= \delta_{\alpha\beta}\sum_j \delta_{jk}f_1((u^\beta_{jl})^*) = \delta_{\alpha\beta}f_1((u^\beta_{kl})^*)\\ &= \delta_{\alpha\beta}f_1(S(u^\beta_{lk})) = \delta_{\alpha\beta}f_{-1}(u^\beta_{lk}),\end{aligned}$$

while

$$\begin{aligned}M_\alpha \sum_i \hat{u}^\alpha_{ii}((u^\beta_{kl})^*) &= M_\alpha \sum_i \varphi((u^\beta_{kl})^* u^\alpha_{ii})\\ &= \delta_{\alpha\beta}\sum_i \delta_{li}f_{-1}(u^\beta_{ik}) = \delta_{\alpha\beta}f_{-1}(u^\beta_{lk}).\end{aligned}$$

But then we get

$$\tau(p_\alpha) = (1/M_\alpha)\hat{\psi}(p_\alpha f_1) = \sum_i \hat{\psi}(\hat{u}^\alpha_{ii}) = \sum_i \varepsilon(u^\alpha_{ii}) = d_\alpha,$$

and (b) is shown. This finishes the proof of (5) and of the proposition. ∎

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Lemma 2.3 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group of compact type. Let $\hat{\rho}$ denote the automorphism of $\hat{\mathcal{A}}$ satisfying $\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$ for all $a, b \in \mathcal{A}$. Then*

$$\hat{\rho}(\hat{a}) = f_{-1}\hat{a}f_1, \quad a \in \mathcal{A}.$$

Proof Being of compact type, $(\mathcal{A}, \Delta)$ is unimodular, that is, the modular element δ of $\mathcal{A}$ is trivial. Hence, it follows from [7, Lemma 2.2] that $\hat{\rho}(\hat{a}) = (S^2(a))^\wedge$ for all $a \in \mathcal{A}$. Therefore, we have

$$\begin{aligned}\hat{\rho}(\hat{a})(b) &= (S^2(a))^\wedge(b) = \varphi(bS^2(a))\\ &= \varphi S^{-2}(bS^2(a)) = \varphi(S^{-2}(b)a) = \hat{a}(S^{-2}(b))\\ &= \hat{a}(f_1 * b * f_{-1}) = \hat{a}((f_{-1} \odot \iota \odot f_1)(\Delta \odot \iota)\Delta(b))\\ &= (f_{-1} \odot \hat{a} \odot f_1)(\Delta \odot \iota)\Delta(b) = (f_{-1}\hat{a}f_1)(b)\end{aligned}$$

for all $a, b \in \mathcal{A}$, which proves the assertion. ∎

3 Categorical Interlude

In this section we introduce $\mathrm{Rep}(\mathcal{A}_u, \Delta_u)$, $\mathrm{Rep}(\mathcal{A}, \Delta)$ and $\mathrm{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ as concrete tensor C^*-categories, and describe explicitly isomorphisms (of tensor C^*-categories) between them. We also discuss some related categories. Finally, we establish the absorbing property for the regular representation with respect to tensor product.

3.1 Tensor C^*-Categories Associated With Algebraic Quantum Groups

We refer to [17, 25] for terminology concerning tensor C^*-categories. Let $(\mathcal{A}, \Delta)$ denote an algebraic quantum group. Let us explain how the category $\mathcal{R} = \mathrm{Rep}(\mathcal{A}_u, \Delta_u)$ can be organized as a concrete tensor C^*-category with irreducible unit. The objects in $\mathcal{R}$ are the $*$-representations π of $\mathcal{A}_u$ acting on a Hilbert space H_π satisfying the non-degeneracy (denseness) condition $\overline{\pi(\mathcal{A}_u)\mathcal{H}_\pi} = \mathcal{H}_\pi$. The family of arrows (or morphisms) between two objects π and π' is given by

$$\mathrm{Mor}(\pi, \pi') = \{T \in B(\mathcal{H}_\pi, \mathcal{H}_{\pi'}) \mid T\pi(a) = \pi'(a)T,\ \forall a \in \mathcal{A}\}.$$

The element $1_\pi \in \mathrm{Mor}(\pi, \pi)$ is given by the identity on $\mathcal{H}_\pi$. The adjoint of an element $T \in \mathrm{Mor}(\pi, \pi')$ is given by its Hilbert space adjoint $T^* \in \mathrm{Mor}(\pi', \pi)$, so we clearly have $\|T^*T\| = \|T\|^2$. The tensor product $\pi \times \pi'$ of two objects π and π' is defined as $\pi \times \pi' = (\pi \otimes \pi')\Delta_u$, while on arrows we use the usual tensor product of operators. The unit in the tensor category is ε_u. Note that this unit is irreducible, since $\mathrm{Mor}(\varepsilon_u, \varepsilon_u) = \mathbb{C}$. It is clear that this category has natural subobjects and that one may form direct sums in an obvious way.

Next, we introduce the closely related category $\mathcal{R}_{\mathrm{alg}} = \mathrm{Rep}(\mathcal{A}, \Delta)$. The objects in $\mathcal{R}_{\mathrm{alg}}$ are now the $*$-representations π of $\mathcal{A}$ acting on a Hilbert space H_π satisfying the non-degeneracy condition $\overline{\pi(\mathcal{A})\mathcal{H}_\pi} = \mathcal{H}_\pi$. Arrows and adjoints are defined in

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

a similar way as above. To define the tensor product of objects, we have to appeal to Proposition 2.1. Let ϕ_1 and ϕ_2 be objects in $\mathcal{R}_{\mathrm{alg}}$, and consider ϕ_1 (resp., ϕ_2) as a $*$-homomorphism from $\mathcal{A}$ into $M(B_0(\mathcal{H}_{\phi_1}))$ (resp., $M(B_0(\mathcal{H}_{\phi_2}))$). As ϕ_1 and ϕ_2 are non-degenerate (by assumption), the proposition applies and produces a unique $*$-homomorphism $\phi_1 \times \phi_2$ from $\mathcal{A}$ into $M(B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2})) = M(B_0(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})) = B(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})$ such that

$$\big(\phi_1(a_1) \otimes \phi_2(a_2)\big)\,(\phi_1 \times \phi_2)(a) = (\phi_1 \odot \phi_2)\big((a_1 \otimes a_2)\Delta(a)\big)$$

and

$$(\phi_1 \times \phi_2)(a)\big(\phi_1(a_1) \otimes \phi_2(a_2)\big) = (\phi_1 \odot \phi_2)\big(\Delta(a)(a_1 \otimes a_2)\big)$$

for every $a_1, a_2 \in \mathcal{A}$. Moreover, we have that $(\phi_1 \times \phi_2)(\mathcal{A})\big(B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2})\big)$ is dense in $B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2})$. It follows easily that $\phi_1 \times \phi_2$, when regarded as a $*$-homomorphism from $\mathcal{A}$ into $B(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})$, satisfies the non-degeneracy condition in order to qualify as an object in $\mathcal{R}_{\mathrm{alg}}$. Finally, the unit in the tensor category is of course ε.

Not surprisingly, the following proposition holds:

Proposition 3.1 *Define $P\colon \mathcal{R} \to \mathcal{R}_{\mathrm{alg}}$ on objects by $P(\pi) = \pi \circ \pi_u$, and let P act trivially on arrows. Then P is an isomorphism of tensor C^*-categories.*

Proof The only non-trivial fact in this assertion is perhaps to show that P preserves tensor products.

Let π_1, π_2 be objects in $\mathcal{R}$, and set $\phi_1 = P(\pi_1), \phi_2 = P(\pi_2), \phi = P(\pi_1 \times \pi_2)$. We have to show that $\phi = \phi_1 \times \phi_2$.

Now, let $a, a_1, a_2 \in \mathcal{A}$. Then we have

$$\begin{aligned}
\phi(a)\big(\phi_1(a_1) \otimes \phi_2(a_2)\big) &= (\pi_1 \times \pi_2)\pi_u(a)\big(\pi_1\pi_u(a_1) \otimes \pi_2\pi_u(a_2)\big)\\
&= (\pi_1 \otimes \pi_2)(\Delta_u\pi_u(a))(\pi_1 \otimes \pi_2)\big(\pi_u(a_1) \otimes \pi_u(a_2)\big)\\
&= (\pi_1 \otimes \pi_2)\big(\Delta_u\pi_u(a)(\pi_u(a_1) \otimes \pi_u(a_2))\big)\\
&= (\pi_1 \odot \pi_2)(\pi_u \odot \pi_u)\big(\Delta(a)(a_1 \otimes a_2)\big)\\
&= (\phi_1 \odot \phi_2)\big(\Delta(a)(a_1 \otimes a_2)\big)\,.
\end{aligned}$$

In the same way, one shows that

$$\big(\phi_1(a_1) \otimes \phi_2(a_2)\big)\,\phi(a) = (\phi_1 \odot \phi_2)\big((a_1 \otimes a_2)\Delta(a)\big)\,.$$

From the uniqueness property of $\phi_1 \times \phi_2$, we may then conclude that $\phi = \phi_1 \times \phi_2$, as desired. ■

Another category $\mathcal{C} = \mathrm{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ which may be organized as a concrete tensor C^*-category is defined as follows. The objects in $\mathcal{C}$ consist of unitary elements

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

$U \in M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U))$, for some Hilbert space $\mathcal{H}_U$, which satisfy the corepresentation property

$$(\hat{\Delta}_{r,\mathrm{op}} \otimes \iota)U = U_{13}\, U_{23}.$$

For objects U and V in $\mathcal{C}$, we set

$$\mathrm{Mor}(U,V) = \{T \in B(\mathcal{H}_U, \mathcal{H}_V) \mid T(\omega\bar{\otimes}\iota)(U) = (\omega\bar{\otimes}\iota)(V)T\ ,\ \forall\, \omega \in \hat{\mathcal{M}}_*\}.$$

The element $1_U \in \mathrm{Mor}(U,U)$ is given by the identity on $\mathcal{H}_U$. The adjoint of an element $T \in \mathrm{Mor}(U,V)$ is given by its Hilbert space adjoint $T^* \in \mathrm{Mor}(V,U)$, so we have $\|T^*\, T\| = \|T\|^2$. The monoidal structure on the objects is determined by setting

$$U \times V = V_{13}\, U_{12} \in M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U) \otimes B_0(\mathcal{H}_V)) = M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U \otimes \mathcal{H}_V)).$$

(Clearly $U \times V$ is unitary. Moreover,

$$\begin{aligned}(\hat{\Delta}_{r,\mathrm{op}} \otimes \iota)(U \times V) &= (\hat{\Delta}_{r,\mathrm{op}} \otimes \iota \otimes \iota)(V_{13}\, U_{12})\\ &= (\hat{\Delta}_{r,\mathrm{op}} \otimes \iota \otimes \iota)(V_{13})(\hat{\Delta}_{r,\mathrm{op}} \otimes \iota \otimes \iota)(U_{12})\\ &= V_{14}\, V_{24}\, U_{13}\, U_{23} = V_{14}\, U_{13}\, V_{24}\, U_{23} = (U \times V)_{13}(U \times V)_{23},\end{aligned}$$

so $U \times V \in \mathcal{C}$.) The reason for "reversing" the "natural" tensor product will be evident from Theorem 3.3. Again, on the arrows, we just set $T \times S = T \otimes S$.

The unit in the tensor category is given by $I \otimes 1 \in M(\hat{\mathcal{A}}_r \otimes \mathbb{C})$, where I denotes the unit of $M(\hat{\mathcal{A}}_r)$. As we clearly have $\mathrm{Mor}(I \otimes 1, I \otimes 1) = \mathbb{C}$, this unit is irreducible.

For objects U, V in $\mathcal{C}$, we say that U is (unitarily) *equivalent* to V, and write $U \simeq V$, whenever there exists a unitary $T \in \mathrm{Mor}(U,V)$.

One may clearly introduce several related tensor C^*-categories, such as $\mathrm{Rep}(\mathcal{A}_u, \Delta_{u,\mathrm{op}})$, $\mathrm{Corep}(\mathcal{A}_u, \Delta_u)$, $\mathrm{Corep}(\mathcal{A}_r, \Delta_r)$ and $\mathrm{Corep}(\mathcal{A}_r, \Delta_{r,\mathrm{op}})$, along the same lines. Note that in the sequel we always refer to the monoidal structure defined as above. Nevertheless, it should be noted that there are relations between the possible choices of monoidal structure in these tensor categories. For example, if one considers $\mathrm{Corep}(\mathcal{A}_r, \Delta_r)$ with monoidal structure given by $X \bar{\times} Y = X_{12}Y_{13}$ and $\mathrm{Corep}(\mathcal{A}_r, \Delta_{r,\mathrm{op}})$ with $U \times V = V_{13}U_{12}$, then it is easy to check that the map $X \to X^*$ gives an isomorphism between these two tensor categories (acting trivially on arrows).

3.2 From Corepresentations to Representations and Back

We now recall some results from [7]. First, there exists an injective, not necessarily $*$-preserving, homomorphism $Q_r \colon \mathcal{A} \to \hat{\mathcal{A}}_r^*$ determined by

$$Q_r(a)(\hat{\pi}_r(\hat{b})) = \hat{b}(S^{-1}(a)) = \varphi(S^{-1}(a)b), \quad \forall\, a, b \in \mathcal{A}.$$

In fact, there exists an injective homomorphism $Q \colon \mathcal{A} \to \hat{\mathcal{M}}_*$ satisfying $Q(a)_{|\hat{\mathcal{A}}_r} = Q_r(a)$ for all $a \in \mathcal{A}$, and such that $Q(\mathcal{A})$ is norm-dense in $\hat{\mathcal{M}}_*$. For all $a \in \mathcal{A}$, we have

$Q(a) = \omega_{\Lambda(a),\Lambda(c)}$ (restricted to $\hat{\mathcal{M}}$), where $c \in \mathcal{A}$ is chosen such that $\hat{c}\,\widehat{S(a^*)} = \widehat{S(a^*)}$ (such a choice is always possible).

We will need the following lemma.

Lemma 3.2 *Let $a \in \mathcal{A}$ and assume that $c \in \mathcal{A}$ satisfies $\hat{c}\widehat{S(a^*)} = \widehat{S(a^*)}$. Then we have*

$$\Delta(a) = (\iota \odot \iota \odot \varphi)\big((I \otimes I \otimes c^*)(\iota \odot \Delta)\Delta(a)\big)\,.$$

Proof Recall from [7] that

$$\hat{c}\widehat{S(a^*)} = \sum_i \varphi(q_i)\hat{p}_i,$$

where $c \otimes S(a^*) = \sum_i \Delta(p_i)(q_i \otimes I)$ for some $p_i, q_i \in \mathcal{A}$, $(i = 1, \dots, n)$. Note also that the inversion formula from [23, p. 1024] then gives

$$\sum_i q_i \otimes p_i = \big((S^{-1} \odot \iota)\Delta S(a^*)\big)\,(c \otimes I).$$

Hence, we have

$$\begin{aligned}\big(\hat{c}\widehat{S(a^*)}\big)\,(b) &= \sum_i \varphi(q_i)\hat{p}_i(b)\\ &= \sum_i \varphi(q_i)\varphi(bp_i) = (\varphi \odot \varphi)\Big((I \otimes b)\big(\sum_i q_i \otimes p_i\big)\Big)\\ &= (\varphi b)\Big((\varphi \odot \iota)\big(\sum_i q_i \otimes p_i\big)\Big)\\ &= (\varphi b)\big((\varphi \odot \iota)(((S^{-1} \odot \iota)\Delta S(a^*))(c \otimes I))\big)\,,\end{aligned}$$

while

$$\widehat{S(a^*)}(b) = \varphi(bS(a^*)) = (\varphi b)(S(a^*))$$

for all $b \in \mathcal{A}$.

From the assumption and the fact that $\{\varphi b \mid b \in A\}$ separates points in $\mathcal{A}$ (since φ is faithful on $\mathcal{A}$), we get

$$\begin{aligned}S(a^*) = (\varphi \odot \iota)\,\big(((S^{-1} \odot \iota)\Delta S(a^*))(c \otimes I)\big) &= (c\varphi \otimes \iota)(S^{-1} \odot \iota)\Delta S(a^*)\\ &= (\iota \odot c\varphi)(S \odot \iota)\Delta(a^*) = S(\iota \odot c\varphi)\Delta(a^*),\end{aligned}$$

where we have used that $\Delta S = \chi(S \odot S)\Delta$. Therefore, we have

$$a^* = (\iota \odot c\varphi)\Delta(a^*) = (\iota \odot \varphi)(\Delta(a^*)(I \otimes c)),$$

so

$$a = (\iota \odot \varphi)\left((I \otimes c^*)\Delta(a)\right),$$

as $\iota \odot \varphi$ is $*$-preserving.

This implies that

$$\begin{aligned}\Delta(a) &= \Delta(\iota \odot \varphi)\left((I \otimes c^*)\Delta(a)\right)\\ &= (\iota \odot \iota \odot \varphi)(\Delta \odot \iota)\left((I \otimes c^*)\Delta(a)\right)\\ &= (\iota \odot \iota \odot \varphi)\left((I \otimes I \otimes c^*)(\Delta \odot \iota)\Delta(a)\right)\\ &= (\iota \odot \iota \odot \varphi)\left((I \otimes I \otimes c^*)(\iota \odot \Delta)\Delta(a)\right),\end{aligned}$$

as asserted. ■

The following result tells us that $\mathcal{C}$ and $\mathcal{R}$ are isomorphic as tensor C^*-categories.

Theorem 3.3 *Define $F\colon \mathcal{C} \to \mathcal{R}$ on objects by $F(U) = \pi_U$, where π_U is determined by*

$$\pi_U(a) = (Q_r(a) \otimes \iota)U, \quad \forall\, a \in \mathcal{A},$$

and let F act identically on arrows.

Define $G\colon \mathcal{R} \to \mathcal{C}$ on objects by $G(\pi) = U_\pi$, where U_π is determined by

$$U_\pi(\Lambda(a) \otimes \pi(b)v) = \sum_{i=1}^{n} \Lambda(a_i) \otimes \pi(b_i)v$$

for all $a, b \in \mathcal{A}, v \in H_\pi$, and the a_i's and b_i's are elements in $\mathcal{A}$ chosen as to satisfy $\Delta(a)(b \otimes I) = \sum_{i=1}^n b_i \otimes a_i$. Let G act identically on arrows.

Then F and G are covariant monoidal (tensor preserving) functors which are adjoint- and unit-preserving, and satisfy $GF = \mathrm{id}$, $FG = \mathrm{id}$.

We also have $F(\hat{W}) = \pi_r$ and $F(I \otimes 1) = \varepsilon_u$.

Proof The fact that F and G are well defined on objects is established in [7], where it is also shown that $GF = \mathrm{id}$, $FG = \mathrm{id}$, $F(\hat{W}) = \pi_r$ and $F(I \otimes 1) = \varepsilon_u$.

We now check that F and G are well defined on arrows. Let U, V be unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$, and let $T \in \mathrm{Mor}(U, V)$. Then, for all $a \in \mathcal{A}$, we have

$$T(Q_r(a) \otimes \iota)(U) = (Q_r(a) \otimes \iota)(V)T,$$

hence, $T\pi_U(a) = \pi_V(a)T$. As $\mathcal{A}$ is norm-dense in $\mathcal{A}_u$, we get $T\pi_U(x) = \pi_V(x)T$ for all $x \in \mathcal{A}_u$, that is $T \in \mathrm{Mor}(\pi_U, \pi_V)$. Conversely, if $T \in \mathrm{Mor}(\pi_U, \pi_V)$, then, using that $Q(\mathcal{A})$ is norm-dense in $\hat{\mathcal{M}}_*$, one readily sees that $T \in \mathrm{Mor}(U, V)$.

It is obvious that F and G are adjoint- and unit-preserving. To show that F and G are monoidal, that is, preserve tensor products, it is enough to show that $F(U \times V) = F(U) \times F(V)$, that is, $\pi_{U\times V} = \pi_U \times \pi_V$, where U and V are unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$.

Let $a, b, f \in \mathcal{A}, \xi \in \mathcal{H}_U, \eta \in \mathcal{H}_V$. Then we have

$$\begin{aligned}[(\pi_U \times \pi_V)(a)](\pi_U(b)\xi \otimes \pi_V(f)\eta) &= [(\pi_U \times \pi_V)(a)]\big(\pi_U(b) \odot \pi_V(f)\big)(\xi \otimes \eta)\\ &= (\pi_U \odot \pi_V)\big(\Delta(a)(b \otimes f)\big)(\xi \otimes \eta).\end{aligned}$$

On the other hand, choose $c \in \mathcal{A}$ such that $\hat{c}\widehat{S(a^*)} = \widehat{S(a^*)}$, so we have $Q_r(a) = \omega_{\Lambda(a),\Lambda(c)}$ (restricted to $\hat{\mathcal{A}}_r$). Then

$$\pi_{U\times V}(a) = (Q_r(a) \otimes \iota \otimes \iota)(U \times V) = (\omega_{\Lambda(a),\Lambda(c)} \otimes \iota \otimes \iota)(V_{13}U_{12}).$$

Now, choose $h_k, g_k \in \mathcal{A}$, $(k = 1, \ldots, n)$, such that $\Delta(a)(b \otimes I) = \sum_k h_k \otimes g_k$.

Further, for each k, choose $f_l, g_k^l \in \mathcal{A}$ $(l = 1, \ldots, n_k)$, such that

$$\Delta(g_k)(f \otimes I) = \sum_l f_l \otimes g_k^l.$$

We then get

$$\begin{aligned}&\big([(\pi_{U\times V})(a)](\pi_U(b)\xi \otimes \pi_V(f)\eta), \xi' \otimes \eta'\big)\\ &\quad= (V_{13}U_{12}(\Lambda(a) \otimes \pi_U(b)\xi \otimes \pi_V(f)\eta), \Lambda(c) \otimes \xi' \otimes \eta')\\ &\quad= \sum_k (V_{13}(\Lambda(g_k) \otimes \pi_U(h_k)\xi \otimes \pi_V(f)\eta, \Lambda(c) \otimes \xi' \otimes \eta')\\ &\qquad\qquad \text{(using the relation between } U \text{ and } \pi_U)\\ &\quad= \sum_{k,l} (\Lambda(g_k^l) \otimes \pi_U(h_k)\xi \otimes \pi_V(f_l)\eta), \Lambda(c) \otimes \xi' \otimes \eta')\\ &\qquad\qquad \text{(using the relation between } V \text{ and } \pi_V)\\ &\quad= \sum_{k,l} \varphi(c^* g_k^l)\,((\pi_U \odot \pi_V)(h_k \otimes f_l)(\xi \otimes \eta), \xi' \otimes \eta'),\end{aligned}$$

for all $\xi' \in \mathcal{H}_U, \eta' \in \mathcal{H}_V$.

Therefore, it suffices to show that

$$\Delta(a)(b \otimes f) = \sum_{k,l} \varphi(c^* g_k^l)(h_k \otimes f_l).$$

Now, we have $\Delta(a)(b \otimes f) = \Delta(a)(b \otimes I)(I \otimes f)$, while

$$\begin{aligned}\sum_{k,l} \varphi(c^* g_k^l)(h_k \otimes f_l) &= \sum_k (\iota \odot \iota \odot \varphi c^*)\Big(h_k \otimes \Big(\sum_l f_l \otimes g_k^l\Big)\Big)\\ &= \sum_k (\iota \odot \iota \odot \varphi c^*)\big(h_k \otimes \Delta(g_k)(f \otimes I)\big)\\ &= \sum_k h_k \otimes (\iota \odot \varphi)\big((I \otimes c^*)\Delta(g_k)(f \otimes I)\big)\\ &= \Big(\sum_k h_k \otimes (\iota \odot \varphi)\big((I \otimes c^*)\Delta(g_k)\big)\Big)(I \otimes f).\end{aligned}$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Hence, this reduces to showing

$$\Delta(a)(b\otimes I)=\sum_k h_k\otimes(\iota\odot\varphi)\big((I\otimes c^*)\Delta(g_k)\big)\,.$$

Now, using the previous lemma, we have

$$\begin{aligned}\Delta(a)(b\otimes I)&=\big((\iota\odot\iota\odot\varphi)((I\otimes I\otimes c^*)(\iota\odot\Delta)\Delta(a))\big)(b\otimes I)\\&=(\iota\odot\iota\odot\varphi)\big((I\otimes I\otimes c^*)(\iota\odot\Delta)\Delta(a)(b\otimes I\otimes I)\big)\\&=(\iota\odot\iota\odot\varphi)\big((I\otimes I\otimes c^*)(\iota\odot\Delta)(\Delta(a)(b\otimes I))\big)\\&=(\iota\odot\iota\odot\varphi)\Big((I\otimes I\otimes c^*)(\iota\odot\Delta)\Big(\sum_k h_k\otimes g_k\Big)\Big)\\&=(\iota\odot\iota\odot\varphi)\Big((I\otimes I\otimes c^*)\Big(\sum_k h_k\otimes\Delta(g_k)\Big)\Big)\\&=(\iota\odot\iota\odot\varphi)\Big(\sum_k h_k\otimes(I\otimes c^*)\Delta(g_k)\Big)\\&=\sum_k h_k\otimes(\iota\odot\varphi)\big((I\otimes c^*)\Delta(g_k)\big)\,,\end{aligned}$$

which finishes the proof. ■

We may dualize this result by using Pontryagin's duality for algebraic quantum groups [37, 23]. Attached to $(\hat{\mathcal{A}},\hat{\Delta})$, we can first associate an injective homomorphism $\hat{Q}_r\colon\hat{\mathcal{A}}\to\hat{\hat{\mathcal{A}}}_r^*\simeq\mathcal{A}_r^*$ which is determined by

$$\hat{Q}_r(\hat{a})\,(\pi_r(b))=\hat{a}\,(S(b))=\varphi(S(b)a)$$

for all $a,b\in\mathcal{A}$. Then we get a functor $\hat{F}\colon\ \mathrm{Corep}(\mathcal{A}_r,\Delta_r)\to\mathrm{Rep}(\hat{\mathcal{A}}_u,\hat{\Delta}_{u,\mathrm{op}})$ determined on objects by

$$\hat{F}(U)(\hat{a})=(\hat{Q}_r(\hat{a})\otimes\iota)U,\quad \hat{a}\in\hat{\mathcal{A}},$$

and acting trivially on arrows, which is an isomorphism of tensor C^*-categories and satisfies $\hat{F}(W)=\hat{\pi}_r$, $\hat{F}(I\otimes 1)=\hat{\varepsilon}_u$. We will write $\hat{\pi}_U$ for $\hat{F}(U)$ in the sequel.

3.3 The Absorbing Property for π_r and $\hat{W}$

We show that π_r and $\hat{W}$ have an absorbing property with respect to tensoring, which is analogous to Fell's classical result for the regular representation of a group [16].

Proposition 3.4 *Let U be a unitary corepresentation of $(\hat{\mathcal{A}}_r,\hat{\Delta}_{r,\mathrm{op}})$ and $I_U=I\otimes I_{\mathcal{H}_U}$ be the trivial unitary corepresentation of $(\hat{\mathcal{A}}_r,\hat{\Delta}_{r,\mathrm{op}})$. Then $U\times\hat{W}$ and $I_U\times\hat{W}$ are equivalent objects in $\mathcal{C}$.*

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Proof Set $T = \chi U^* \in M(B_0(\mathcal{H}_U) \otimes \hat{\mathcal{A}}_r) \subset B(\mathcal{H}_U \otimes \mathcal{H})$. It suffices to check that this unitary satisfies the relation

$$T(\omega\bar{\otimes}\iota)(U \times \hat{W}) = (\omega\bar{\otimes}\iota)(I_U \times \hat{W})T, \quad \forall\, \omega \in \hat{\mathcal{M}}_*.$$

After some manipulations, this relation reduces to

$$(\omega\bar{\otimes}\iota\bar{\otimes}\iota)(U_{32}^*\hat{W}_{13}U_{12}) = (\omega\bar{\otimes}\iota\bar{\otimes}\iota)(\hat{W}_{13}U_{32}^*).$$

Now, using the fact that $\hat{W}$ is a multiplicative unitary, we get

$$\hat{W}_{13}U_{12} = U_{32}\hat{W}_{13}U_{32}^* \in M\big(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U \otimes \mathcal{H})\big).$$

Thus, we have $U_{32}^*\hat{W}_{13}U_{12} = \hat{W}_{13}U_{32}^*$, and the result clearly follows. ■

Combining this result with Theorem 3.3, one gets at once that $\pi \times \pi_r$ is equivalent to $I_\pi \times \pi_r$ for every $\pi \in \mathrm{Rep}(\mathcal{A}_u, \Delta_u)$, where $I_\pi \in \mathrm{Rep}(\mathcal{A}_u, \Delta_u)$ is given by $I_\pi(a) = \varepsilon_u(a)I_{\mathcal{H}_\pi}$, $\forall\, a \in \mathcal{A}_u$.

By duality, we have a similar result for $\hat{\pi}_r$ and W.

4 Conjugate and Hilbert-Schmidt Corepresentations

In this section, we define the conjugate and the Hilbert-Schmidt corepresentations associated with a unitary corepresentation. Such objects play an important role in the classical representation theory for groups and we will need these concepts in later sections.

4.1 Conjugate Corepresentations

Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. Let $\overline{\mathcal{H}}_U$ be any Hilbert space such that there exists an anti-unitary map $J\colon \mathcal{H}_U \to \overline{\mathcal{H}}_U$. Define then $j\colon B(\mathcal{H}_U) \to B(\overline{\mathcal{H}}_U)$ by $j(x) = Jx^*J^*$, $\forall x \in B(\mathcal{H}_U)$. Then j is linear, unital, normal, isometric, $*$-preserving and anti-multiplicative, with inverse $j^{-1}(\bar{x}) = J^*\bar{x}^*J, \bar{x} \in B(\overline{\mathcal{H}}_U)$. Note that $j(B_0(\mathcal{H}_U)) = B_0(\overline{\mathcal{H}}_U)$.

Now define

$$\overline{U} = (R \otimes j)U \in M\big(\mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U)\big).$$

Proposition 4.1 *$\overline{U}$ is a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$ with $\mathcal{H}_{\overline{U}} = \overline{\mathcal{H}}_U$.*

Proof We have

$$\overline{U}^*\overline{U} = ((R \otimes j)U)^*((R \otimes j)U) = (R \otimes j)(UU^*) = (R \otimes j)(I) = I_{M(\mathcal{A}_r)} \otimes I_{\overline{\mathcal{H}}_U},$$

and similarly

$$\overline{U}\overline{U}^* = I_{M(\mathcal{A}_r)} \otimes I_{\overline{\mathcal{H}}_U}.$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Furthermore,

$$\begin{aligned}(\Delta_r \otimes \iota)\overline{U} &= (\Delta_r R \otimes j)U = (\chi(R \otimes R) \otimes j)(\Delta_r \otimes \iota)U \\ &= (\chi(R \otimes R) \otimes j)(U_{13}U_{23}) = (\chi(R \otimes R) \otimes j)(U_{23})(\chi(R \otimes R) \otimes j)(U_{13}) \\ &= ((R \otimes j)U)_{13}((R \otimes j)U)_{23} = \overline{U}_{13}\overline{U}_{23}.\end{aligned}$$

■

Remark We clearly have $\overline{\overline{U}} \simeq U$.

Assume now that $(\mathcal{A}, \Delta)$ is of compact type and let U be an irreducible unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$, which is then necessarily finite-dimensional [39, 40].

We will show that the conjugate of U, as defined above, agrees with the conjugate of U as defined by J. Roberts and L. Tuset [32]. We first recall their definition.

Let $\bar{U} \in \mathcal{A} \odot B(\mathcal{H}_U)$ be given by $\bar{U} = (* \odot \tilde{j})U$, where $\tilde{j}(x) = \tilde{J}x\tilde{J}^{-1}$ for all $x \in B(\mathcal{H}_U)$, and $\tilde{J}\colon \mathcal{H}_U \to \overline{\mathcal{H}}_U$ is any anti-linear invertible operator such that $\tilde{J}^*\tilde{J}$ intertwines U and $(S^2 \odot \iota)U$. Then $(\Delta \odot \iota)\bar{U} = \bar{U}_{13}\bar{U}_{23}$ holds as Δ is $*$-preserving and $\tilde{j}$ is multiplicative. The fact that $\bar{U}$ is unitary is shown in [32]. (Note that $(S^2 \odot \iota)U$ is the double contragradient representation; it is not unitary.)

Proposition 4.2 *Assume that $(\mathcal{A}, \Delta)$ is of compact type and let U be an irreducible unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. Let $\overline{U} = (R \otimes j)U$ denote the conjugate of U as defined before. Let $\tilde{J}\colon \mathcal{H}_U \to \overline{\mathcal{H}}_U$ be given by $\tilde{J} = ((f_{1/2} \odot j)U)J$. Then $\tilde{J}$ is an anti-linear invertible operator such that $\tilde{J}^*\tilde{J}$ intertwines U and $(S^2 \odot \iota)U$. Further, we have $\overline{U} = \bar{U}$, where $\bar{U}$ is defined as above.*

Proof Recall from [23] that $R = S\tau_{i/2} = \tau_{i/2}S$, where $\tau_{i/2}(a) = f_{1/2} * a * f_{-1/2}$. Set $V = I \otimes (f_{1/2} \odot j)(U^*)$. Then V is invertible and $(V^*)^{-1} = I \otimes (f_{-1/2} \odot j)(U^*)$. Using the corepresentation property of U, a straightforward computation gives that

$$\overline{U} = (\tau_{i/2} \odot j)(U^*) = (V^*)^{-1}(\iota \odot j)(U^*)V = (* \odot \tilde{J} \cdot \tilde{J}^{-1})U = \bar{U}.$$

To see that $\tilde{J}^*\tilde{J}$ intertwines U and $(S^2 \odot \iota)U$, observe first that

$$\tilde{J}^*\tilde{J} = J^*((f_{1/2} \odot j)U)^*((f_{1/2} \odot j)U)J = ((f_1 \odot \iota)U)^* = F_U,$$

where $F_U = (f_1 \odot \iota)U$.

Now, inserting $a = (\iota \odot \omega)U$ in $S^2(a) = f_{-1} * a * f_1$, one gets

$$(S^2 \odot \omega)U = (f_1 \odot \iota \odot f_{-1} \odot \omega)(\Delta \odot \iota \odot \iota)(\Delta \odot \iota)U,$$

for all $\omega \in B(\mathcal{H}_U)^*$. Hence $(S^2 \odot \iota)U = (I \otimes F_U)U(I \otimes F_U^{-1})$, that is, $\tilde{J}^*\tilde{J} = F_U$ intertwines U and $(S^2 \odot \iota)U$, as claimed. ■

4.2 Hilbert-Schmidt Corepresentations

Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. We introduce the Hilbert-Schmidt corepresentation U_{HS} associated with U and show that $\overline{U} \times U \simeq U_{HS}$.

We let J and j be as in the previous subsection and denote the Hilbert-Schmidt operators acting on $\mathcal{H}_U$ by $HS(\mathcal{H}_U)$. We recall that $HS(\mathcal{H}_U)$ is a Hilbert space with inner product $(x, y) = \operatorname{Tr}(y^* x)$, $x, y \in HS(\mathcal{H}_U)$, where Tr denotes the canonical trace on $B(\mathcal{H}_U)$.

We define first a unitary $\mathcal{V}\colon \overline{\mathcal{H}}_U \otimes \mathcal{H}_U \to HS(\mathcal{H}_U)$ by

$$\mathcal{V}(\eta \otimes \xi)(\xi') = (\xi', J^*\eta)_{\mathcal{H}_U} \xi = (\eta, J\xi')_{\overline{\mathcal{H}}_U}\, \xi,\ \xi, \xi' \in \mathcal{H}_U, \eta \in \overline{\mathcal{H}}_U.$$

Define then a normal unital $*$-isomorphism $\tilde{\mathcal{V}}\colon B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U) \to B(HS(\mathcal{H}_U))$ by

$$\tilde{\mathcal{V}}(X) = \mathcal{V} X \mathcal{V}^*,\ X \in B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U).$$

Note that $\tilde{\mathcal{V}}(B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)) = B_0(HS(\mathcal{H}_U))$. Further, let

$$\iota \otimes \tilde{\mathcal{V}}\colon M\big(\mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)\big) \to M\big(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U))\big)$$

denote the canonical extension of

$$\iota \otimes \tilde{\mathcal{V}}\colon \mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U) \to \mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U)) \subset M\big(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U))\big).$$

It is clear that $\iota \otimes \tilde{\mathcal{V}}$ is a unital $*$-isomorphism.

Define then $U_{HS} \in M(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U)))$ by

$$U_{HS} = (\iota \otimes \tilde{\mathcal{V}})(\overline{U} \times U).$$

Proposition 4.3 *U_{HS} is a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$, with*

$$\mathcal{H}_{U_{HS}} = HS(\mathcal{H}_U),$$

which is equivalent to $\overline{U} \times U$.

Proof U_{HS} is unitary as $\iota \otimes \tilde{\mathcal{V}}$ is a unital $*$-isomorphism and $\overline{U} \times U$ is a unitary. Moreover,

$$(\Delta_r \otimes \iota)U_{HS} = (\iota \otimes \iota \otimes \tilde{\mathcal{V}})(\Delta_r \otimes \iota)(\overline{U} \times U) = (U_{HS})_{13}(U_{HS})_{23}$$

since $\overline{U} \times U$ satisfies the corepresentation property and $\iota \otimes \iota \otimes \tilde{\mathcal{V}}$ is multiplicative. Finally, as $(\omega \bar{\otimes} \iota)U_{HS}\mathcal{V} = \mathcal{V}(\omega \bar{\otimes} \iota)(\overline{U} \times U)$, $\omega \in \mathcal{M}_*$, we see that U_{HS} is equivalent to $\overline{U} \times U$ with unitary $\mathcal{V} \in \operatorname{Mor}(U_{HS}, \overline{U} \times U)$. ■

It will be useful for us later to have another way of looking at U_{HS}.

Let l (resp., r): $B(\mathcal{H}_U) \to B(HS(\mathcal{H}_U))$ be the normal $*$-homomorphism (resp., $*$-antihomomorphism) defined by

$$l(x)(y) = xy \text{ (resp., } r(x)(y) = yx), \quad x \in B(\mathcal{H}_U), y \in HS(\mathcal{H}_U).$$

It is then straightforward to check that

$$\mathcal{V}(I \otimes x)\mathcal{V}^* = l(x), \quad x \in B(\mathcal{H}_U),$$
$$\mathcal{V}(z \otimes I)\mathcal{V}^* = r(j^{-1}(z)), \quad z \in B(\overline{\mathcal{H}}_U).$$

Using these relations, one easily gets

$$(I \otimes \mathcal{V})X_{13}(I \otimes \mathcal{V}^*) = (\iota\bar{\otimes}l)X, \quad X \in B(\mathcal{H})\bar{\otimes}B(\mathcal{H}_U),$$
$$(I \otimes \mathcal{V})Z_{12}(I \otimes \mathcal{V}^*) = (\iota\bar{\otimes}rj^{-1})Z, \quad Z \in B(\mathcal{H})\bar{\otimes}B(\overline{\mathcal{H}}_U).$$

Now, regarding $U \in \mathcal{M}\bar{\otimes}B(\mathcal{H}_U) \subset B(\mathcal{H})\bar{\otimes}B(\mathcal{H}_U)$, we have:

Proposition 4.4 $U_{HS} = (\iota\bar{\otimes}l)U(R\bar{\otimes}r)U$.

Proof Indeed,

$$\begin{aligned} U_{HS} &= (I \otimes \mathcal{V})U_{13}\overline{U}_{12}(I \otimes \mathcal{V}^*) \\ &= (I \otimes \mathcal{V})U_{13}(I \otimes \mathcal{V}^*)(I \otimes \mathcal{V})\overline{U}_{12}(I \otimes \mathcal{V}^*) \\ &= (\iota\bar{\otimes}l)U(\iota\bar{\otimes}rj^{-1})(R\bar{\otimes}j)U = (\iota\bar{\otimes}l)U(R\bar{\otimes}r)U. \end{aligned}$$

■

Remark One may also associate with U another Hilbert-Schmidt corepresentation $U_{HS'}$ of $(\mathcal{A}_r, \Delta_r)$ on $HS(\mathcal{H}_U)$ which is given by

$$U_{HS'} = (\iota \otimes \tilde{\mathcal{V}}\chi)(U \times \overline{U}),$$

where χ denotes the flip map from $B(\mathcal{H}_U \otimes \overline{\mathcal{H}}_U)$ to $B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)$. One easily checks that $U_{HS'} \simeq U \times \overline{U} \simeq (\overline{U})_{HS}$, and that $U_{HS'} = (R\bar{\otimes}r)U(\iota\bar{\otimes}l)U$. The two Hilbert-Schmidt corepresentations associated with U agree when $\mathcal{A}$ is commutative.

5 Co-amenable Unitary Corepresentations

Inspired by [5, Theorem 2.5] and [6, Theorem 4.2], we introduce the notion of co-amenability for unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$.

Definition 5.1 Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. A unitary corepresentation U of $(\mathcal{A}_r, \Delta_r)$ is said to be co-amenable if there exists $\phi \in S(\mathcal{A}_r)$ such that $(\phi \otimes \iota)U = I_{\mathcal{H}_U}$.

Note that we can equivalently require that $\phi \in S(B(\mathcal{H}))$ in this definition. The following result shows that this definition is consistent with the notion of co-amenability for algebraic quantum groups. Recall from [6, 7] (see [5] for the compact case) that an algebraic quantum group $(\mathcal{A}, \Delta)$ is *co-amenable* if its co-unit ε is bounded with respect to the norm on $\mathcal{A}$ given by $\|a\| = \|\pi_r(a)\|$, $a \in \mathcal{A}$. Equivalently, $(\mathcal{A}, \Delta)$ is co-amenable if there exists a bounded linear functional $\varepsilon_r \colon \mathcal{A}_r \to \mathbb{C}$ such that $(\iota \otimes \varepsilon_r)\Delta_r = (\varepsilon_r \otimes \iota)\Delta_r = \iota$. The map ε_r is then a $*$-homomorphism from $\mathcal{A}_r$ onto $\mathbb{C}$ and the existence of such a homomorphism characterizes the co-amenability of $(\mathcal{A}, \Delta)$.

Theorem 5.2 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. Then the following conditions are equivalent:*

(1) *$(\mathcal{A}, \Delta)$ is co-amenable;*
(2) *W is co-amenable (as a corepresentation);*
(3) *all unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are co-amenable.*

Proof The equivalence (1) $\Leftrightarrow$ (2) follows by [6, Theorem 4.2]. The implication (3) $\Rightarrow$ (2) is obvious. In order to show (1) $\Rightarrow$ (3), we set $\phi = \varepsilon_r$ and let U be a unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$. Then

$$\begin{aligned} U &= (\iota \otimes \varepsilon_r \otimes \iota)(\Delta_r \otimes \iota)U \\ &= (\iota \otimes \varepsilon_r \otimes \iota)(U_{13}U_{23}) \\ &= U(I \otimes (\varepsilon_r \otimes \iota)U) \end{aligned}$$

(here I denotes the unit of $M(\mathcal{A}_r)$). Multiplying by U^* from the left, we get $I \otimes I_{\mathcal{H}_U} = I \otimes (\varepsilon_r \otimes \iota)U$ and therefore $(\varepsilon_r \otimes \iota)U = I_{\mathcal{H}_U}$. ■

The next result may be seen as an analog of Day's classical characterization of the amenability of a group.

Proposition 5.3 *Let U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. Then the following conditions are equivalent:*

(1) *U is co-amenable;*
(2) *there exists a net (v_i) of unit vectors in $\mathcal{H}$ such that*

$$\lim_i \|U(v_i \otimes \xi) - v_i \otimes \xi\|_2 = 0, \quad \forall\, \xi \in \mathcal{H}_U.$$

Proof (2) $\Rightarrow$ (1): By weak*-compactness of $S(B(\mathcal{H}))$, the net of vector states (ω_{v_i}) has an accumulation point ϕ in $S(B(\mathcal{H}))$. Passing to a subnet of (v_i) if necessary, we may suppose that $\phi(x) = \lim_i (xv_i, v_i)$, $x \in B(\mathcal{H})$.

Now, by assumption we have $\lim_i \|U(v_i \otimes \xi) - v_i \otimes \xi\|_2 = 0$, for all $\xi \in \mathcal{H}_U$. Thus

$$\begin{aligned} \omega_\xi((\phi \otimes \iota)U) &= \phi((\iota \otimes \omega_\xi)U) = \lim_i((\iota \otimes \omega_\xi)(U)v_i, v_i) \\ &= \lim_i(U(v_i \otimes \xi), v_i \otimes \xi) = \lim_i(v_i \otimes \xi, v_i \otimes \xi) = \omega_\xi(I) \end{aligned}$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

for every $\xi \in \mathcal{H}_U$. Since the set of vector states ω_ξ separates the elements of $B(\mathcal{H})$, it follows that $(\phi \otimes \iota)U = I$.

(1) $\Rightarrow$ (2). Let $\phi \in S(B(\mathcal{H}))$ be such that $(\phi \otimes \iota)U = I$. As $\mathcal{M}$ acts standardly on $\mathcal{H}$, there exists a net (v_i) of unit vectors in $\mathcal{H}$ such that $\phi(x) = \lim_i (xv_i, v_i)$, $x \in \mathcal{M}$. Then, for all $v \in \mathcal{H}_U$,

$$\lim_i (U(v_i \otimes \xi), v_i \otimes \xi) = \lim_i ((\iota \otimes \omega_\xi)(U)v_i, v_i) = \phi((\iota \otimes \omega_\xi)U) = \omega_\xi((\phi \otimes \iota)U)$$
$$= \omega_\xi(I) = (\xi, \xi) = \lim_i (v_i \otimes \xi, v_i \otimes \xi).$$

The conclusion follows easily from this. ∎

We may use the results in Section 3.1 to transfer the notion of co-amenability from corepresentations to representations: the $*$-representation $\hat{\pi}_U$ of $\hat{\mathcal{A}}_u$ associated with a unitary corepresentation U of $(\mathcal{A}_r, \Delta_r)$ is said to be *co-amenable* if U is co-amenable. We have for the moment no intrinsic characterization of this notion.

Finally, concerning compact matrix pseudogroups [39], we mention:

Proposition 5.4 *Suppose that $(\mathcal{A}_r, \Delta_r)$ is a compact matrix pseudogroup with fundamental unitary corepresentation U (so U is finite-dimensional with matrix elements generating $\pi_r(\mathcal{A})$ as a $*$-algebra). Then U is co-amenable if and only if $(\mathcal{A}, \Delta)$ is co-amenable.*

Proof This result is merely a restatement of [5, Theorem 2.5]. A sketch of the argument is as follows. Write $U = \sum_{i,j} u_{ij} \otimes e_{ij}$, where $u_{ij} \in \mathcal{A}$ and the e_{ij}'s form a usual system of matrix units for $B(\mathcal{H}_U)$. If $\phi \in S(\mathcal{A}_r)$, then $(\phi \otimes \iota)U = I_{H_U}$ if and only if $\phi(u_{ij}) = \delta_{ij}$ for all i, j if and only if $\phi|_{\mathcal{A}} = \varepsilon$. ∎

In fact, more generally, [5, Theorem 2.5] may be restated as saying that if $(\mathcal{A}, \Delta)$ is of compact type and U is a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$ such that its matrix elements generate $\mathcal{A}_r$ as a C^*-algebra, then $(\mathcal{A}, \Delta)$ is co-amenable if and only if U is co-amenable.

6 Amenable Unitary Corepresentations

We first recall the following definition due to M. Bekka [8]. A continuous unitary representation u of a locally compact group G on a Hilbert space $\mathcal{H}_u$ is called *amenable* if there exists an invariant "mean" on $B(\mathcal{H}_u)$, that is, if there exists $m_u \in S(B(\mathcal{H}_u))$ such that

$$m_u(u_g x u_g^*) = m_u(x), \quad \forall\, x \in B(\mathcal{H}_u), \forall\, g \in G.$$

It is not clear what the direct counterpart of this definition should be in the quantum group setting. However, Bekka also introduces a notion of "topological" invariant mean whose existence is equivalent to the amenability of u, see [8, Theorem 3.5]. Inspired by this result, we introduce the following notion:

Definition 6.1 A unitary corepresentation U of $(\mathcal{A}_r, \Delta_r)$ is called left-amenable (resp., right-amenable) if there exists m_U (resp., m'_U)) $\in S(B(\mathcal{H}_U))$ such that

$$m_U((\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)) = \omega(I)m_U(x)$$
$$(\text{resp., } m'_U((\omega\bar{\otimes}\iota)(U(I \otimes x)U^*)) = \omega(I)m'_U(x))$$

for all $x \in B(\mathcal{H}_U), \omega \in \mathcal{M}_*$.

The state m_U (resp., m'_U) is called a left-invariant (resp., right-invariant) mean for U.

Remarks

(i) When U is the unitary corepresentation of $(C_c(\Gamma), \Delta)$ associated to a unitary representation u of a discrete group Γ, one easily checks that U is left-amenable (resp., U is right-amenable) if and only if u is amenable. This is a simple consequence of the fact that $(\delta_\gamma\bar{\otimes}\iota)U = u_\gamma$, where the delta function δ_γ at $\gamma \in \Gamma$ is considered as an element of $\ell^1(\Gamma)$, that is, of the predual of $\ell^\infty(\Gamma)$. Further, in this case, it is quite obvious that a left- (resp., right-) invariant mean for U is both left- and right-invariant.

(ii) We do not know whether the existence of a left-invariant mean for U is equivalent to the existence of a right-invariant one in the general situation. However, we have that U is left- (resp., right-) amenable if and only if $\overline{U}$ is right- (resp., left-) amenable.

Indeed, if m_U is a left-invariant mean for U, then $m_U \circ j^{-1}$ is a right-invariant mean for $\overline{U}$. If m'_U is a right-invariant mean for $\overline{U}$, then $m'_U \circ j$ is a left-invariant mean for U. The resp. assertions are proven similarly.

(iii) The property of left-amenability (resp., right-amenability) is clearly invariant under unitary equivalence.

(iv) By "linearizing" the concept of amenability, one gets a related, but seemingly independent, notion: a unitary corepresentation U of $(\mathcal{A}_r, \Delta_r)$ is said to be *hypertracial* if there exists $m''_U \in S(B(\mathcal{H}_U))$ such that

$$(1)\quad m''_U\left((\omega\bar{\otimes}\iota)(U(I \otimes x))\right) = m''_U\left((\omega\bar{\otimes}\iota)((I \otimes x)U)\right), \ \forall\, x \in B(\mathcal{H}_U), \forall\, \omega \in \mathcal{M}_*.$$

Actually, condition (1) is equivalent to

$$m''_U\left((\omega\bar{\otimes}\iota)(U)x\right) = m''_U(x\left(\omega\bar{\otimes}\iota)(U)\right) \quad \forall\, x \in B(\mathcal{H}_U), \forall\, \omega \in \mathcal{M}_*,$$

which in turn is equivalent to

$$m'_U(\hat{\pi}_U(a)x) = m'_U(x\hat{\pi}_U(a)), \forall x \in B(\mathcal{H}_U),\ \forall\, a \in \mathcal{A}_u.$$

Hence, hypertraciality of U is equivalent to hypertraciality of $\hat{\pi}_U$ in the sense of [4]. This hypertrace property is easily seen to correspond to left- and right-amenability in the case of a corepresentation arising from a unitary representation of a discrete group.

Now recall from [6, 7] that an algebraic quantum group $(\mathcal{A}, \Delta)$ is called *amenable* if there exists a left-invariant mean for $(\mathcal{A}, \Delta)$, that is, if there exists $m \in S(\mathcal{M})$ such that

$$m\left((\omega\bar{\otimes}\iota)\Delta_r(x)\right) = \omega(I)m(x), \quad \forall\, x \in \mathcal{M}, \forall\, \omega \in \mathcal{M}_*.$$

A right-invariant mean for $(\mathcal{A}, \Delta)$ is defined similarly. By composing with the anti-unitary antipode R (which is defined on $\mathcal{M}$, see [23, 24]), one easily sees that the existence of a left-invariant mean for $(\mathcal{A}, \Delta)$ is equivalent to the existence of a right-invariant mean for it. It is then straightforward to check that $(\mathcal{A}, \Delta_{\mathrm{op}})$ is amenable if and only if $(\mathcal{A}, \Delta)$ is amenable.

The following result is well known (see [8] for the equivalence between (3), (4) and (5); the equivalence between (1), (2) and (3) is merely classical, as explained in [5, 6]).

Theorem 6.2 *Let Γ be a discrete group and let $(\mathcal{A}, \Delta)$ be the algebraic quantum group associated with $\mathcal{A} = C_c(\Gamma)$, so $\hat{\mathcal{A}} = \mathbb{C}[\Gamma]$ is the group-algebra of Γ. Then the following are equivalent:*

(1) $(\mathcal{A}, \Delta)$ *is amenable;*
(2) $(\hat{\mathcal{A}}, \hat{\Delta})$ *is co-amenable;*
(3) Γ *is amenable;*
(4) *the (left-) regular representation λ of Γ is amenable;*
(5) *all unitary representations of Γ are amenable.*

In the case of an algebraic quantum group, it is known [6, Theorem 4.7] that (2) implies (1). The converse implication holds when $(\mathcal{A}, \Delta)$ is discrete (see Section 9). Amenability of an algebraic quantum group may be characterized through amenability of its corepresentations as follows.

Theorem 6.3 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. Then the following conditions are equivalent:*

(1) $(\mathcal{A}, \Delta)$ *is amenable;*
(2) W *is left-amenable (as a corepresentation);*
(3) $\overline{W}$ *is right-amenable (as a corepresentation);*
(4) *all unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are left-amenable.*
(5) *all unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are right-amenable.*

Proof The implications (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3) are obvious. The equivalence between (2) and (3) is just a special case of (ii) in our previous remark.

(2) $\Rightarrow$ (1). Let $m_W \in S(B(\mathcal{H}))$ be a left-invariant mean for W, and let m to be the restriction of m_W to $\mathcal{M}$. Then m is clearly a state, which is left-invariant since

$$m((\omega\bar{\otimes}\iota)\Delta_r(x)) = m_W((\omega\bar{\otimes}\iota)(W^*(I \otimes x)W)) = \omega(I)m_W(x) = \omega(I)m(x),$$

for all $x \in \mathcal{M}$, $\omega \in \mathcal{M}_*$.

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5). Let m be a left-invariant mean for $(\mathcal{A}, \Delta)$ and let U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. We pick a normal state Ω on $B(\mathcal{H}_U)$ and define $m_U \in S(B(\mathcal{H}_U))$ by

$$m_U(x) = m\left((\iota\bar{\otimes}\Omega)(U^*(I \otimes x)U)\right), \quad x \in B(\mathcal{H}_U).$$

Then we check the validity of the equation expressing the left-invariance property of m_U: the *l.h.s.* is

$$m_U(x) = m(\iota\bar{\otimes}\Omega)(U^*(I \otimes [(\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)])U)$$

while the *r.h.s.* is equal to

$$\begin{aligned}
\omega(I)m_U(x) &= \omega(I)m\big((\iota\bar{\otimes}\Omega)(U^*(I \otimes x)U)\big)\\
&= m(\omega\bar{\otimes}\iota)\Delta\big((\iota\bar{\otimes}\Omega)(U^*(I \otimes x)U)\big) \text{ (using left- invariance of } m)\\
&= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)(\Delta\bar{\otimes}\iota)(U^*(I \otimes x)U)\\
&= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)\big(((\Delta\bar{\otimes}\iota)U)^*(I \otimes I \otimes x)(\Delta\bar{\otimes}\iota)U\big)\\
&= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)\big((U_{13}U_{23})^*(I \otimes I \otimes x)U_{13}U_{23}\big)\\
&= m(\iota\bar{\otimes}\Omega)(\omega\bar{\otimes}\iota\bar{\otimes}\iota)(U_{23}^*U_{13}^*(I \otimes I \otimes x)U_{13}U_{23}).
\end{aligned}$$

Hence, the desired conclusion follows from the identity

$$U^*(I \otimes [(\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)])U = (\omega\bar{\otimes}\iota\bar{\otimes}\iota)(U_{23}^*U_{13}^*(I \otimes I \otimes x)U_{13}U_{23})$$

which is easily verified.

Similarly, we define $m'_U \in S(B(\mathcal{H}_U))$ by

$$m'_U(x) = m \circ R\big((\iota\bar{\otimes}\Omega)(U(I \otimes x)U^*)\big), \quad x \in B(\mathcal{H}_U).$$

Then, using the right-invariance of $m \circ R$, one now checks that m'_U is a right-invariant mean for U.

It clearly follows that all stated conditions are equivalent. ■

As already mentioned, co-amenability of $(\mathcal{A}, \Delta)$ implies that $(\hat{\mathcal{A}}, \hat{\Delta})$ is amenable, hence that $(\hat{\mathcal{A}}, \hat{\Delta}_{\mathrm{op}})$ is amenable. By combining this fact with Theorem 5.2 and Theorem 6.3, we see that if all the unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are co-amenable then all the unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ are (left- and right-)amenable. This lends some evidence that there might be some correspondence between co-amenable elements in $\mathrm{Corep}(\mathcal{A}_r, \Delta_r)$ and amenable elements in $\mathrm{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$.

By using Theorem 3.3, one may clearly transfer the notion of amenability to representations of algebraic quantum groups. Theorem 6.3 may then be reformulated in an obvious manner.

An analog of [8, Theorem 3.6], which characterizes the amenability of a unitary representation of a group, is as follows.

Proposition 6.4 *Let U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. Organize $TC(\mathcal{H}_U)$, the trace class operators on $\mathcal{H}_U$, as a Banach $\mathcal{M}_*$-module by means of*

$$\operatorname{Tr}((\omega \cdot s)x) = \operatorname{Tr}(s(\omega \bar{\otimes} \iota)(U^*(I \otimes x)U)),$$

$\omega \in \mathcal{M}_*, s \in TC(\mathcal{H}_U), x \in B(\mathcal{H}_U)$. Then U is left-amenable if and only if there exists a net (s_i) in $TC(\mathcal{H}_U)_1^+$ such that

$$\lim_i \|\omega \cdot s_i - s_i\|_1 = 0, \quad \forall\, \omega \in \mathcal{M}_*.$$

Proof The proof is an easy adaptation of the proof of [8, Theorem 3.6]. If (s_i) is a net as above, then a left-invariant mean for U is obtained by picking any weak*-limit point of the net $(m_i) \subset S(B(\mathcal{H}_U))$ given by $m_i(\,\cdot\,) = \operatorname{Tr}(s_i \,\cdot\,)$. Conversely, assume that m_U is a left-invariant mean for U. As the normal states are weak*-dense in $S(B(\mathcal{H}_U))$, we may pick a net $(t_i) \subset TC(\mathcal{H}_U)_1^+$ such that m_U is weak*-limit point of the net $(\operatorname{Tr}(t_i \,\cdot\,)) \subset S(B(\mathcal{H}_U))$. Namioka's classical argument [29] gives then the existence of a net (s_i) with the required properties. ∎

One may clearly also obtain a similar characterization of right-amenability for unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$.

To illustrate the notion of invariant mean for corepresentations, we now consider the case where $(\mathcal{A}, \Delta)$ is of compact type. Then let U be a finite-dimensional unitary representation of $(\mathcal{A}_r, \Delta_r)$. As $(\mathcal{A}, \Delta)$ is amenable, (see the paragraph preceding Theorem 4.7 in [6]) we deduce from Theorem 6.3 that all unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are left- (and right-) amenable. We shall now describe somewhat more explicitly a left-invariant mean m_U for U, following the construction given in the proof of Theorem 6.3.

Identifying $\mathcal{A}$ with the dense Hopf $*$-subalgebra $\pi_r(\mathcal{A})$ of $(\mathcal{A}_r, \Delta_r)$, we may write $U = \sum_i a_i \otimes b_i \in \mathcal{A} \odot B(\mathcal{H}_U)$ for some $a_1, \dots, a_N \in \mathcal{A}, b_1, \dots, b_N \in B(\mathcal{H}_U)$. Recall that a left-invariant mean for U is provided by

$$m_U(x) = \varphi_r\big((\iota \bar{\otimes} \Omega)(U^*(I \otimes x)U)\big), \quad x \in B(\mathcal{H}),$$

where we have the freedom to choose any $\Omega \in B(\mathcal{H}_U)_{*,1}^+$. Set $d_U = \dim \mathcal{H}_U$ and let $\tau = 1/d_U \operatorname{Tr}$ denote the normalized trace on $B(\mathcal{H}_U)$. Plugging in $\Omega = \tau$, we get

$$\begin{aligned} m_U(x) = \varphi_r\left((\iota \bar{\otimes} \tau)(U^*(I \otimes x)U)\right) &= \sum_{i,j} \varphi\big((\iota \otimes \tau)(a_i^* a_j \otimes b_i^* x b_j)\big) \\ &= \sum_{i,j} \varphi(a_i^* a_j)\tau(b_i^* x b_j) = \frac{1}{d_U} \operatorname{Tr}\Big(\sum_{i,j} \varphi(a_i^* a_j) b_j b_i^* x\Big) \end{aligned}$$

so that $m_U(\,\cdot\,) = \operatorname{Tr}(K_U \,\cdot\,)$, where the density matrix $K_U \in B(\mathcal{H}_U)$ is given by

$$K_U = \frac{1}{d_U}(\varphi \odot \iota)(U(\sigma \otimes \iota)U^*),$$

σ denoting the automorphism of $\mathcal{A}$ given by $\sigma(a) = f_1 * a * f_1,\ a \in \mathcal{A}$. We remark that if φ is tracial, then $f_1 = \varepsilon$, hence $K_U = \frac{1}{d_U} I$, that is, the left-invariant mean for U is just τ.

Now assume that U is irreducible. We write $U = \sum_{i,j} u_{ij} \otimes m_{ji}$, where $u_{ij} \in \mathcal{A}$ and the m_{ij}'s form a system of matrix units for $B(\mathcal{H}_U)$ such that $m_{lk} m_{sr} = \delta_{lr} m_{sk}$ and $m_{lk}^* = m_{kl}$. Using the orthogonality relation

$$\varphi((u_{km})^* u_{ln}) = (1/M_U)\delta_{mn} f_{-1}(u_{lk}),$$

where M_U denotes the quantum dimension of U, we get

$$\begin{aligned}
\sum_{i,j,k,l} \varphi((u_{ij})^* u_{kl}) m_{lk} m_{ji}^* &= \sum_{i,j,k,l} (1/M_U)\delta_{jl} f_{-1}(u_{ki})\delta_{lj} m_{ik} \\
&= \sum_{i,j,k} (1/M_U) f_{-1}(u_{ki}) m_{ik} = (d_U/M_U) \sum_{i,k} f_{-1}(u_{ki}) m_{ik} \\
&= (d_U/M_U)(f_{-1} \odot \iota)\Big(\sum_{i,k} u_{ki} \otimes m_{ik}\Big) \\
&= (d_U/M_U)(f_{-1} \odot \iota)(U).
\end{aligned}$$

Hence, in this case, we get $K_U = (1/M_U)(f_{-1} \odot \iota)(U)$. We summarize what we have shown.

Proposition 6.5 *Assume that $(\mathcal{A}, \Delta)$ is of compact type and let U be a finite-dimensional unitary representation of $(\mathcal{A}_r, \Delta_r)$. Let d_U (resp., M_U) denote the usual (resp., quantum) dimension of U, and let σ be the automorphism of $\mathcal{A}$ given by $\sigma(a) = f_1 * a * f_1,\ a \in \mathcal{A}$. Then a left-invariant mean m_U for U is given by $m_U(\,\cdot\,) = \mathrm{Tr}(K_U\,\cdot\,)$, with density matrix K_U given by*

$$K_U = \frac{1}{d_U}(\varphi \odot \iota)(U(\sigma \otimes \iota)U^*).$$

If U is irreducible, then $K_U = \frac{1}{M_U}(f_{-1} \odot \iota)(U)$.

7 On Weak Containment

We discuss in this section the notion of weak containment for representations and corepresentations of algebraic quantum groups. We begin by discussing the stronger (and easier) notion of containment.

7.1 Strong Containment

We recall the following definition.

Definition 7.1 Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and let π_1, π_2 be two non-degenerate $*$-representations of $\mathcal{A}_u$. We say that π_1 is *contained* in π_2, and write $\pi_1 < \pi_2$, if there exists an isometry $T \in \mathrm{Mor}(\pi_1, \pi_2)$.

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Observe that $K = T(\mathcal{H}_{\pi_1})$ is then a closed invariant subspace for π_2. Therefore, if π_2 is irreducible, then any non-degenerate $*$-representation π_1 of $\mathcal{A}_u$ contained in π_2 is unitarily equivalent to π_2.

The interesting case when $\pi_1 = \varepsilon_u$ may be characterized as follows.

Proposition 7.2 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and consider a non-degenerate $*$-representation π of A_u. Write $\pi = \pi_U$ for a unique unitary corepresentation U of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$. The following conditions are equivalent:*

(1) *$\varepsilon_u < \pi_U$;*
(2) *there exists a unit vector $\xi \in \mathcal{H}_U$ such that $(\iota\bar{\otimes}\omega_\xi)U = I \in M(\hat{\mathcal{A}}_r)$;*
(3) *there exists a unit vector ξ in H_U such that $U(v \otimes \xi) = v \otimes \xi$, $\forall\, v \in \mathcal{H}$.*

Proof We first show (1) implies (2). Assume that (1) holds. Then there exists a linear map $T\colon \mathbb{C} \to \mathcal{H}_U$ such that $T^*T = 1$ and $T\varepsilon_u(a) = \pi_U(a)T$ for all $a \in \mathcal{A}_u$. Consider the unit vector $\xi = T(1)$ of $\mathcal{H}_U$. Then the adjoint $T^*\colon \mathcal{H}_U \to \mathbb{C}$ is given by $T^*(\eta) = (\eta, \xi)$ for all $\eta \in \mathcal{H}_U$. Now, for all $a \in \mathcal{A} \subset \mathcal{A}_u$, we have

$$\begin{aligned} Q(a)(I) &= (Q(a)\bar{\otimes}\iota)(I \otimes 1) = \varepsilon(a) = (\varepsilon_u(a)1, 1)_{\mathbb{C}} \\ &= (T^*\pi_U(a)T(1), 1)_{\mathbb{C}} = (\pi_U(a)T(1), T(1))_{\mathcal{H}_U} \\ &= (\pi_U(a)\xi, \xi)_{\mathcal{H}_U} = ((Q(a)\bar{\otimes}\iota)U\xi, \xi)_{\mathcal{H}_U} = Q(a)((\iota\bar{\otimes}\omega_\xi)U)\,. \end{aligned}$$

Since $Q(\mathcal{A})$ is dense in $\hat{\mathcal{M}}_*$ it follows that $I = (\iota\bar{\otimes}\omega_\xi)U$.

Next, we show that (2) implies (1). Given a unit vector ξ satisfying (2), we define the linear isometry $T\colon \mathbb{C} \to \mathcal{H}_U$ by $T(1) = \xi$. By reversing the above calculations, we see that $\varepsilon_u(a) = T^*\pi_U(a)T$ holds for all $a \in \mathcal{A}_u$. It is easily checked that $T \in \mathrm{Mor}(\varepsilon_u, \pi_U)$.

Finally, to prove the equivalence between (2) and (3), observe first that

$$(\iota\bar{\otimes}\omega_\xi)U = I$$

if and only if $(\omega_v\bar{\otimes}\omega_\xi)U = 1$ for all unit vectors $v \in \mathcal{H}$. Now, for a unit vector $v \in \mathcal{H}$, one easily checks that

$$(U(v \otimes \xi), v \otimes \xi) = 1 \Leftrightarrow \|U(v \otimes \xi) - v \otimes \xi\|_2 = 0.$$

Hence, this equivalence is clear, and the proof is finished. ■

Let U, V be unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$. We say that U is (strongly) *contained* in V, and write $U < V$ if there is an isometry $T \in \mathrm{Mor}(U, V)$. It is an easy exercise to check that $U < V$ if and only if $\hat{\pi}_U < \hat{\pi}_V$. One may then clearly obtain a result similar to Proposition 7.2.

Example Let $(\mathcal{A}, \Delta)$ be of compact type and U be an irreducible unitary representation of $(\mathcal{A}_r, \Delta_r)$. Let $\{e_i\}$ denote an orthonormal basis for $\mathcal{H}_U$ and $\tilde{J}$ be defined as in Proposition 4.2. Then the isometry R from $\mathbb{C}$ into $\overline{\mathcal{H}}_U \otimes \mathcal{H}_U$ determined by

$$R(1) = \sum_i \tilde{J}^{*-1}(e_i) \otimes e_i$$

satisfies $R \in \operatorname{Mor}(I \otimes 1, \overline{U} \times U)$, as shown in [32]. Hence, $I \otimes 1 < \overline{U} \times U$. It follows that $\xi = \sum_i \tilde{J}^{*-1}(e_i) \otimes e_i \in \overline{\mathcal{H}}_U \otimes \mathcal{H}_U$ satisfies $(\overline{U} \times U)(\eta \otimes \xi) = \eta \otimes \xi$ for all $\eta \in \mathcal{H}$. Since $(\mathcal{A}, \Delta)$ is compact by assumption, we know that all unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are left- (and right-) amenable, as pointed out in the previous section. Indeed, we have seen in Proposition 6.5 that a left-invariant mean m_U for U is given by

$$m_U(x) = \operatorname{Tr}(K_U x), \quad x \in B(\mathcal{H}_U),$$

where $K_U = \frac{1}{M_U}(f_{-1} \odot \iota)U$.

Now, let

$$\widetilde{\mathcal{V}(\xi)} = \frac{1}{(M_U)^{1/2}} \mathcal{V}(\xi),$$

where $\mathcal{V}\colon \overline{\mathcal{H}}_U \otimes \mathcal{H}_U \to HS(\mathcal{H}_U)$ is defined as in Section 4. It is then not difficult to check by direct computation that we also have

$$m_U(\,\cdot\,) = \operatorname{Tr}(\widetilde{\mathcal{V}(\xi)}^{\,*} \cdot \widetilde{\mathcal{V}(\xi)}),$$

that is, we have $\widetilde{\mathcal{V}(\xi)}\widetilde{\mathcal{V}(\xi)}^{\,*} = K_U$.

7.2 Weak Containment

Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and let π_1, π_2 be non-degenerate $*$-representations of $\mathcal{A}_u$. As usual for representations of C^*-algebras, we say that π_1 is *weakly contained* in π_2, and write $\pi_1 \prec \pi_2$, if $\operatorname{Ker} \pi_2 \subset \operatorname{Ker} \pi_1$. This relation is obviously transitive and reflexive, and, of course, $\pi_1 < \pi_2$ implies $\pi_1 \prec \pi_2$.

Proposition 7.3 *With notation as above, we have $\pi_1 \prec \pi_2$ if and only if there exists a unique surjective $*$-homomorphism $\theta\colon \pi_2(\mathcal{A}_u) \to \pi_1(\mathcal{A}_u)$ such that $\theta\pi_2(a) = \pi_1(a)$, $\forall\, a \in \mathcal{A}$.*

Proof This proof is easy and left to the reader. ∎

An almost immediate consequence of this proposition is the following.

Corollary 7.4 $\varepsilon_u \prec \pi_r$ *if and only if $(\mathcal{A}, \Delta)$ is co-amenable.*

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Remark Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. Note that $(\mathcal{A}, \Delta)$ is co-amenable if and only if $\pi \prec \pi_r$ for every non-degenerate $*$-representation π of $\mathcal{A}_u$. Indeed, if $(\mathcal{A}, \Delta)$ is co-amenable, then $\mathcal{A}_u = \mathcal{A}_r$, that is, $\pi_r \colon \mathcal{A}_u \to \mathcal{A}_r$ is injective (see [6]). Therefore, $\{0\} = \operatorname{Ker} \pi_r \subset \operatorname{Ker} \pi$. On the other hand, if $(\mathcal{A}, \Delta)$ is not co-amenable, then ε_u is not weakly contained in π_r.

We also remark that the condition $\varepsilon_u \prec \pi$ and the condition $\pi \prec \pi_r$ are generally independent of each other. In fact, if $\pi = \varepsilon_u$, then the first is trivially satisfied, while the second holds if and only if $(\mathcal{A}, \Delta)$ is co-amenable. On the other hand, if $\pi = \pi_r$, then second is trivially satisfied, while the first holds if and only if if $(\mathcal{A}, \Delta)$ is co-amenable.

Definition 7.5 Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. A non-degenerate $*$-representation π of $\mathcal{A}_u$ is said to have the weak containment property (WCP) if $\varepsilon_u \prec \pi$, that is, if $\operatorname{Ker} \pi \subset \operatorname{Ker} \varepsilon_u$.

Thus π has the WCP if and only if there exists a $*$-homomorphism $\theta \colon \pi(\mathcal{A}_u) \to \mathbb{C}$ such that $\theta\pi(a) = \varepsilon(a)$ for all $a \in \mathcal{A} \subset \mathcal{A}_u$.

Definition 7.6 Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group, U, V be unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ and let π_U, π_V be the associated $*$-representations of $(\mathcal{A}_u, \Delta_u)$.

We say that U is weakly contained in V if π_U is weakly contained in π_V. Moreover, we say that U has the weak containment property (WCP) if the trivial corepresentation $I \otimes 1$ is weakly contained in U, that is, if π_U has the WCP.

Corollary 7.7 *An algebraic quantum group $(\mathcal{A}, \Delta)$ is co-amenable if and only if $\hat{W}$, as a unitary corepresentation of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$, has the WCP.*

Proof As $\pi_{\hat{W}} = \pi_r$, this is just a reformulation of Corollary 7.4 (see Theorem 3.3). ■

The weak containment property for unitary corepresentations may be characterized as follows.

Theorem 7.8 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group, U be a unitary corepresentation of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ and let π_U be the associated $*$-representation of $\mathcal{A}_u$. The following conditions are equivalent:*

(1) *$I \otimes 1 \prec U$, that is, U has the WCP;*
(2) *there exists $\psi \in S(B(\mathcal{H}_U))$ such that $\psi((\omega \bar{\otimes} \iota)U) = \omega(I)$, $\forall\, \omega \in \hat{\mathcal{M}}_*$;*
(3) *there exists a net (ξ_i) of unit vectors in H_U such that*

$$\lim_i \|U(v \otimes \xi_i) - v \otimes \xi_i\|_2 = 0 \quad \forall\, v \in \mathcal{H};$$

(4) *there exists a net (ξ_i) of unit vectors in H_U such that*

$$\lim_i (U(v \otimes \xi_i), v \otimes \xi_i) = 1, \quad \forall\, v \in \mathcal{H}, \|v\|_2 = 1.$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Further, any of these conditions implies that U is left-amenable, right-amenable and hypertracial.

Proof The equivalence between (3) and (4) is elementary.

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4): Assume that $I \otimes 1 \prec U$. By the remark following Definition 7.5, there exists a $*$-homomorphism $\theta\colon \pi_U(\mathcal{A}_u) \to \mathbb{C}$ such that $\theta(\pi_U(x)) = \varepsilon_u(x)$ for all $x \in \mathcal{A}_u$. We extend the state θ to a state ψ on $B(\mathcal{H}_U)$. Then, for all $a \in \mathcal{A} \subset \mathcal{A}_u$, we have

$$\begin{aligned}\psi((Q(a)\bar{\otimes}\iota)U) &= \psi((Q_r(a) \otimes \iota)U) = \psi(\pi_U(a)) \\ &= \varepsilon_u(a) = (Q_r(a) \otimes \iota)(I \otimes 1) = Q_r(a)(I) = Q(a)(I).\end{aligned}$$

Since $Q(\mathcal{A})$ is dense in $\hat{\mathcal{M}}_*$, we get by continuity, $\psi((\omega\bar{\otimes}\iota)U) = \omega(I)$, $\forall\, \omega \in \hat{\mathcal{M}}_*$, which shows (2).

Further, as ε_u is a $*$-homomorphism on $\mathcal{A}_u$, it is a pure state on $\mathcal{A}_u$. From [12, Proposition 3.4.2, ii)], we can then conclude that there exists a net of unit vectors $(\xi_i) \in \mathcal{H}_U$ such that $\varepsilon_u(x) = \lim_i(\pi_U(x)\xi_i, \xi_i)$ for all $x \in \mathcal{A}_u$. Since $\varepsilon_u = \psi \circ \pi_U$ as above, this means that $\psi(y) = \lim_i \omega_{\xi_i}(y)$ for all $y \in \pi_U(\mathcal{A}_u)$. As $(\omega\bar{\otimes}\iota)U \in \pi_U(\mathcal{A}_u)$ (see [7, Theorem 3.3]) and $\psi((\omega\bar{\otimes}\iota)U) = \omega(I)$, for all $\omega \in \hat{\mathcal{M}}_*$, we get $\lim_i \omega_{\xi_i}((\omega_\eta\bar{\otimes}\iota)U) = 1$ for all unit vectors η in $\mathcal{H}$. This just says that $\lim_i(U(\eta \otimes \xi_i), \eta \otimes \xi_i) = 1$ for all unit vectors η in $\mathcal{H}$, hence that (4) holds.

(2) $\Rightarrow$ (1): Assume that (2) holds, and let ψ be as in (2). Let $x \in \operatorname{Ker} \pi_U$. Choose a sequence (a_n) in $\mathcal{A}$ converging to $x \in \mathcal{A}_u$ with respect to the norm $\|\cdot\|_u$. Then, by continuity of π_U,we get $(Q_r(a_n) \otimes \iota)U = \pi_U(a_n) \to \pi_U(x) = 0$. Using the assumption, we have $\psi((Q(a_n)\bar{\otimes}\iota)U) = Q(a_n)(I)$ for all n. By continuity of ψ we therefore get

$$\begin{aligned}\varepsilon_u(a_n) &= Q_r(a_n)(I) = Q(a_n)(I) \\ &= \psi((Q(a_n)\bar{\otimes}\iota)U) = \psi((Q_r(a_n) \otimes \iota)U) \to 0.\end{aligned}$$

Thus, by continuity of ε_u, we get $\varepsilon_u(x) = \lim_n \varepsilon_u(a_n) = 0$, so $x \in \operatorname{Ker} \varepsilon_u$. Hence, (1) holds.

(4) $\Rightarrow$ (2): Let $(\xi_i)_i$ be a net satisfying condition (4). Using Alaoglu's theorem, and passing to a subnet if necessary, there exists a $\psi \in S(B(H_U))$ such that $\psi(x) = \lim_i \omega_{\xi_i}(x)$, for all $x \in B(H_U)$. Since $\hat{\mathcal{M}}$ is in standard form on H, any normal state ω on $\hat{\mathcal{M}}$ is of the form ω_v for some unit vector $v \in \mathcal{H}$. Then $\psi((\omega_v\bar{\otimes}\iota)U) = \lim_i \omega_{\xi_i}((\omega_v\bar{\otimes}\iota)U) = \lim_i (U(v\otimes\xi_i), v\otimes\xi_i) = 1 = \omega_v(I)$, so ψ satisfies condition (2).

Hence, we have established the equivalence between conditions (1)–(4).

Finally, assume that (2) holds and set $m_U = \psi$. Let $\omega \in \hat{\mathcal{M}}^+_{*,1}$. Then

$$m_U((\omega\bar{\otimes}\iota)U) = \omega(I) = 1.$$

As U is a unitary in $\hat{\mathcal{M}}\bar{\otimes}B(\mathcal{H}_U)$, it follows from the Cauchy-Schwarz inequality that the state $m_U((\omega\bar{\otimes}\iota)(\,\cdot\,))$ on $\hat{\mathcal{M}}\bar{\otimes}B(\mathcal{H}_U)$ is multiplicative at U and at U^*. Hence,

$$\begin{aligned}m_U((\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)) &= m_U((\omega\bar{\otimes}\iota)U^*)m_U((\omega\bar{\otimes}\iota)(I \otimes x))m_U((\omega\bar{\otimes}\iota)U) \\ &= m_U(x) = \omega(I)m_U(x)\end{aligned}$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

for all $x \in B(\mathcal{H}_U)$ and $\omega \in \hat{\mathcal{M}}^+_{*,1}$. It easily follows that m_U is a left-invariant mean for U. Similarly, m_U is a right-invariant mean for U, and it also serves to show that U is hypertracial. This finishes the proof. ■

Weak containment and WCP for unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are defined in an analogous way, via weak containment and WCP for the associated representations of $\hat{\mathcal{A}}_u$. From a conceptual point of view, it is better to work in this setting, and we will often do this in the sequel. All statements concerning WCP for unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ have an analogous statement concerning WCP for unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$. For example, we have the following counterpart to Theorem 7.8.

Theorem 7.9 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. The following conditions are equivalent:*

(1) *$I \otimes 1 \prec U$, that is, U has the WCP;*
(2) *there exists $\psi \in S(B(\mathcal{H}_U))$ such that $\psi((\omega \bar{\otimes} \iota)U) = \omega(I)$, $\forall\, \omega \in \mathcal{M}_*$;*
(3) *there exists a net (ξ_i) of unit vectors in $\mathcal{H}_U$ such that*

$$\lim_i \|U(v \otimes v_i) - v \otimes v_i\|_2 = 0, \quad \forall\, v \in \mathcal{H};$$

(4) *there exists a net (ξ_i) of unit vectors in $\mathcal{H}_U$ such that*

$$\lim_i (U(v \otimes \xi_i), v \otimes \xi_i) = 1, \quad \forall\, v \in \mathcal{H},\ \|v\|_2 = 1;$$

Further, any of these conditions implies that U is left-amenable, right-amenable and hypertracial.

We will illustrate in the next section that amenability of U does not in general imply that U has the WCP. We now collect some elementary facts about the WCP.

Proposition 7.10 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and let U, V be unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$.*

(1) *If U has the WCP, then $\overline{U}$ also has the WCP.*
(2) *If U and V have the WCP, then $U \times V$ also has the WCP.*
(3) *If U has the WCP, then U_{HS} also has the WCP.*
(4) *If $U \times V$ has the WCP, then U is left-amenable and V is right-amenable.*
(5) *If U_{HS} has the WCP, then $\overline{U}$ is left-amenable and U is right-amenable.*

Proof The proof of assertion (1) is an easy exercise, left to the reader. Assertion (2) is a straightforward application of condition (2) in Theorem 7.9. Assertion (3) follows from (1) and (2). To prove assertion (4), assume that $U \times V$ has the WCP. The last assertion in Theorem 7.9 tells us then that $U \times V$ is left- and right-amenable. If now $M \in S(B(\mathcal{H}_U \otimes \mathcal{H}_V))$ is a left-invariant (resp., right-invariant) mean for $U \times V$, then one checks without difficulty that $m_U(x) = M(x \otimes I_{\mathcal{H}_V})$ (resp., $m'_V(y) = M(I_{\mathcal{H}_U} \otimes y)$), $x \in B(\mathcal{H}_U)$ (resp., $y \in B(\mathcal{H}_V)$), is a left-invariant (resp., right-invariant) mean for U (resp., V). This shows (4). Finally, assertion (5) follows clearly from (4). ■

Remark Let u denote a unitary representation of a discrete group Γ. One of the main results of Bekka in [8] is that u is amenable if and only if its associated Hilbert-Schmidt representation weakly contains the trivial representation. An interesting question is whether some quantum group version of this result is true, that is, whether the converse of assertion (5) in Proposition 7.10 holds, at least in some cases. We will show that this is true for algebraic quantum groups of discrete type in Section 9 (*cf.* Theorem 9.5).

Corollary 7.11 *Let U be a finite-dimensional unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$. Assume that $(R \odot \iota)U = U^*$. (This is known to hold in the Kac algebra case,* cf. [14, Proposition 1.5.1] *).*

Then $I \otimes 1 < U_{HS}$, and U is both left- and right-amenable.

Proof Write $U = \sum_i a_i \otimes b_i$, where $a_i \in \mathcal{M}, b_i \in B(\mathcal{H}_U), i = 1, \ldots, n$. We use the notation introduced in Section 4. Using Proposition 4.4 we may write $U_{HS} = (\iota \odot l)U(R \odot r)U$. As $(R \odot r)U = (\iota \odot r)U^*$, we have $(R \odot r)U = \sum_j a_j^* \otimes r(b_j^*)$. Let $\xi = I_{\mathcal{H}_U} \in HS(\mathcal{H}_U)$. For any $\eta \in \mathcal{H}$ we get

$$\begin{aligned}
U_{HS}(\eta \otimes \xi) &= \big((\iota \odot l)U(R \odot r)U\big)(\eta \otimes \xi)\\
&= \Big(\sum_i a_i \otimes l(b_i)\Big)\Big(\sum_j a_j^* \otimes r(b_j^*)\Big)(\eta \otimes \xi)\\
&= \Big(\sum_{ij} a_i a_j^* \otimes l(b_i) r(b_j^*)\Big)(\eta \otimes \xi) = \sum_{ij} a_i a_j^* \eta \otimes b_i b_j^*\\
&= \Big(\sum_{ij} a_i a_j^* \otimes b_i b_j^*\Big)(\eta \otimes \xi) = \eta \otimes \xi
\end{aligned}$$

where, in the last equality, we have used the fact that $\sum_{ij} a_i a_j^* \otimes b_i b_j^* = UU^* = I_{\mathcal{H}} \otimes I_{\mathcal{H}_U}$. Thus, appealing to Proposition 7.2, we have shown that $I \otimes 1 < U_{HS}$. We may then apply Proposition 7.10 (5) and conclude that U is right-amenable. Finally, as $\overline{U}$ is also finite-dimensional, we then easily deduce that $\overline{U}$ is right-amenable, hence that U is left-amenable. ■

Proposition 7.12 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and consider its multiplicative unitary W as a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$.*

Then W has the WCP if and only if W_{HS} has the WCP.

Proof Using assertion (3) of Proposition 7.10, it suffices to show that W has the WCP whenever W_{HS} has it. So assume that $I \otimes 1 \prec W$. Using the absorbing property of W (*cf.* our remark after Proposition 3.4), we get that $W_{HS} = \overline{W} \times W$ is unitarily equivalent to $I_{\overline{W}} \times W = W_{13}(I_{\mathcal{H}} \otimes I_{\overline{\mathcal{H}}})_{12} = W_{13}$. According to Theorem 7.9, there exists $\psi \in S(B(\overline{\mathcal{H}} \otimes \mathcal{H}))$ such that $\psi((\omega \bar{\otimes} \iota \bar{\otimes} \iota)W_{13}) = \omega(I)$, $\omega \in \hat{\mathcal{M}}_*$. Define a state ψ' on $B(\mathcal{H})$ by $\psi'(x) = \psi(I_{\overline{\mathcal{H}}} \otimes x)$, $x \in B(\mathcal{H})$. Then

$$\psi'((\omega \bar{\otimes} \iota)W) = \psi(I_{\overline{\mathcal{H}}} \otimes (\omega \bar{\otimes} \iota)W) = \psi((\omega \bar{\otimes} \iota \bar{\otimes} \iota)W_{13}) = \omega(I).$$

Using Theorem 7.9 again, we deduce that W has the WCP, as desired. ∎

Corollary 7.13 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group and consider the multiplicative unitary $\hat{W}$ as a unitary corepresentation of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$. Then $(\mathcal{A}, \Delta)$ is co-amenable if and only if $\hat{W}_{HS}$ has the WCP.*

Proof We just have to combine the dual version of Proposition 7.12 with Corollary 7.7. ∎

Our interest in such a result is that it is presumably easier to establish that $\hat{W}_{HS}$ has the WCP than to establish that $\hat{W}$ has the WCP if one wants to show that $(\mathcal{A}, \Delta)$ is co-amenable.

We conclude this subsection with another proposition involving containment and amenability.

Proposition 7.14 *Let $(\mathcal{A}, \Delta)$ denote an algebraic quantum group and U, V be unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$.*

If U is left- (resp., right-) amenable and $U < V$, then V is left- (resp., right-) amenable.

Proof Let $T \in \mathrm{Mor}(U, V)$ be such that $T^*T = I$ and assume that U is left-amenable. Define $\tilde{T}\colon B(\mathcal{H}_V) \to B(\mathcal{H}_U)$ by $\tilde{T}(x) = T^*xT$, and note that $\tilde{T}$ is a normal $*$-preserving completely positive unital linear map. Then note that since $T \in \mathrm{Mor}(U, V)$ and $T^*T = I$, we have

$$\begin{aligned}(\omega\bar{\otimes}\iota)U &= T^*(\omega\bar{\otimes}\iota)VT\\ &= \tilde{T}(\omega\bar{\otimes}\iota)V\\ &= (\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\tilde{T})V\end{aligned}$$

for all $\omega \in \mathcal{M}_*$ (the last equality can be checked for elementary V first and then use continuity). As $\{(\omega\bar{\otimes}\iota) \mid \omega \in \mathcal{M}_*\}$ separates the elements of $\mathcal{M}\bar{\otimes}B(\mathcal{H}_U)$, we get $U = (\iota\bar{\otimes}\tilde{T})V$.

Now, let $m_U \in S(B(\mathcal{H}_U))$ be a left-invariant mean for U, so that

$$m_U\Big((\omega\bar{\otimes}\iota)(U^*(I \otimes y)U)\Big) = \omega(I)m_U(y)$$

for all $y \in B(\mathcal{H}_U)$ and $\omega \in \mathcal{M}_*$.

Define $m_V \in S(B(\mathcal{H}_V))$ by $m_V = m_U\tilde{T}$. Then, for $x \in B(\mathcal{H}_V)$ and $\omega \in \mathcal{M}_*$, we get

$$m_V(x) = m_U\tilde{T}(x) = m_U\Big((\omega\bar{\otimes}\iota)(U^*(I \otimes \tilde{T}(x))U)\Big).$$

Since $(\iota\bar{\otimes}\tilde{T})V = U$ is unitary and $\iota\bar{\otimes}\tilde{T}$ is completely positive, it follows from a well known result of M. D. Choi (see *e.g.*, [34, 9.2]) that $\iota\bar{\otimes}\tilde{T}$ is multiplicative at V and V^*. Hence, we get

$$(\iota\bar{\otimes}\tilde{T})(V^*(I \otimes x)V) = (\iota\bar{\otimes}\tilde{T})V^*(I \otimes \tilde{T}(x))(\iota\bar{\otimes}\tilde{T})V.$$

Thus

$$\begin{aligned} m_V(x) &= m_U\left((\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\tilde{T})(V^*(I\otimes x)V)\right) \\ &= m_U\left(\tilde{T}(\omega\bar{\otimes}\iota)(V^*(I\otimes x)V)\right) \\ &= m_V\left((\omega\bar{\otimes}\iota)(V^*(I\otimes x)V)\right). \end{aligned}$$

So m_V is a left-invariant mean for V and V is left-amenable. The proof of the resp. part of the statement is similar. ■

It would be interesting to know whether this result still holds if one replaces strong containment with weak containment. Bekka has shown [8, Corollary 5.3] that this is true in the classical case and we will also show this for discrete quantum groups in Section 9.

7.3 On Property (T)

We introduce a version of Kazhdan's property (T) [19] for algebraic quantum groups. Then, as in the classical case, we show that every compact quantum group has property (T). This implies that none of the non-trivial irreducible unitary corepresentations of a compact quantum group has the WCP. Furthermore, we show that compactness may be characterized by having property (T) together with co-amenability of the dual quantum group.

Definition 7.15 Let $(\mathcal{A},\Delta)$ be an algebraic quantum group. We say that $(\mathcal{A},\Delta)$ has property (T) if $I\otimes 1 \prec U \Rightarrow I\otimes 1 < U$ for all unitary corepresentations U of $(\mathcal{A}_r,\Delta_r)$, in other words, if $\hat{\varepsilon}_u \prec \hat{\pi} \Rightarrow \hat{\varepsilon}_u < \hat{\pi}$ for all non-degenerate $*$-representations $\hat{\pi}$ of $\hat{\mathcal{A}}_u$.

Theorem 7.16 *Let $(\mathcal{A},\Delta)$ be an algebraic quantum group of compact type. Then $(\mathcal{A},\Delta)$ has property (T).*

Proof Let U be a unitary corepresentation of $(\mathcal{A}_r,\Delta_r)$ and assume that U has the WCP. To show the theorem, we have to show that $\hat{\varepsilon}_u < \hat{\pi}_U$.

Since $(\mathcal{A},\Delta)$ is of compact type, $(\mathcal{A}_r,\Delta_r)$ is a compact quantum group in the sense of Woronowicz. Its Haar state φ_r is then left- and right-invariant, and it has a unique extension to a normal state on $\mathcal{M}$ which we also denote by φ_r.

Now, let $\xi\in\mathcal{H}_U$ and set $\eta = ((\varphi_r\otimes\iota)U)\xi\in\mathcal{H}_U$. Then, for all $v\in\mathcal{H}$, we have

$$U(v\otimes\eta) = U(I\otimes(\varphi_r\otimes\iota)U)(v\otimes\xi),$$

while

$$v\otimes\eta = (I\otimes(\varphi_r\otimes\iota)U)(v\otimes\xi).$$

But

$$
\begin{aligned}
I \otimes ((\varphi_r \otimes \iota)U) &= (\varphi_r(\cdot)I \otimes \iota)U \\
&= (\iota \otimes \varphi_r \otimes \iota)(\Delta_r \otimes \iota)U \quad \text{(using invariance of } \varphi_r) \\
&= (\iota \otimes \varphi_r \otimes \iota)(U_{13}\, U_{23}) = U(I \otimes (\varphi_r \otimes \iota)U).
\end{aligned}
$$

Hence, we get $U(v \otimes \eta) = v \otimes \eta$ for all $v \in \mathcal{H}$.

Using the dual version of Proposition 7.2, we will then have shown that $\hat{\varepsilon}_u < \hat{\pi}_U$ if we can show that the vector η may be chosen to be non-zero. This may be seen as follows. Since U has the WCP, we know from Theorem 7.9 that there exists a state ψ on $B(\mathcal{H}_U)$ such that $\psi((\varphi_r \bar{\otimes} \iota)U) = 1$. This implies that $(\varphi_r \bar{\otimes} \iota)U \neq 0$. Hence, there exists at least one $\xi \in \mathcal{H}_U$ such that

$$0 \neq ((\varphi_r \bar{\otimes} \iota)U)\xi = ((\varphi_r \otimes \iota)U)\xi = \eta,$$

as desired. ■

Theorem 7.17 *Let $(\mathcal{A}, \Delta)$ be an algebraic quantum group. Then $(\mathcal{A}, \Delta)$ is of compact type if and only if $(\mathcal{A}, \Delta)$ has property (T) and $(\hat{\mathcal{A}}, \hat{\Delta})$ is co-amenable.*

Proof If $(\mathcal{A}, \Delta)$ is of compact type, then we know from Theorem 7.16 that $(\mathcal{A}, \Delta)$ has property (T). Further, $(\hat{\mathcal{A}}, \hat{\Delta})$ is then of discrete type and therefore co-amenable [6].

Conversely, assume that $(\mathcal{A}, \Delta)$ has property (T) and $(\hat{\mathcal{A}}, \hat{\Delta})$ is co-amenable. Then, using the dual version of Corollary 7.4, we get $\hat{\varepsilon}_u \prec \hat{\pi}_W$, hence $\hat{\varepsilon}_u < \hat{\pi}_W$. This means that there exists a $T\colon \mathbb{C} \to \mathcal{H}$ such that $T^*T = 1$ and $\hat{\varepsilon}_u(y) = T^*\hat{\pi}_W(y)T$ for all $y \in \hat{\mathcal{A}}_u$. Thus we have $\hat{\varepsilon}_u(y) = (\hat{\pi}_W(y)\eta, \eta)$ for all $y \in \hat{\mathcal{A}}_u$, where $\eta = T(1)$ is a unit vector in $\mathcal{H}$.

Let ψ denote the vector state ω_η on $B(\mathcal{H})$. Then, proceeding as in the proof of Theorem 7.8, (1) implies (2), we get $\psi(\omega \bar{\otimes} \iota)W = \omega(I_{\mathcal{H}})$ for all $\omega \in \mathcal{M}_*$. As ψ is normal, this gives $\omega(\iota \bar{\otimes} \psi)W = \omega(I_{\mathcal{H}})$ for all $\omega \in \mathcal{M}_*$, hence $(\iota \bar{\otimes} \psi)W = I_{\mathcal{H}}$. Since $\mathcal{A}_r$ is the norm closure of $\{(\iota \bar{\otimes} \phi)W \mid \phi \in B(\mathcal{H})_*\}$, we get $I_H \in \mathcal{A}_r$, that is, $(\mathcal{A}_r, \Delta_r)$ is compact, as desired. ■

It is clear that a more detailed study of property (T) for algebraic quantum groups would be an interesting task (see [30] for the case of Kac algebras), but this would take us too far away from the main theme of this paper.

8 Amenability vs. Co-amenability vs. WCP

Let $(\mathcal{A}, \Delta)$ denote an algebraic quantum group and U be a unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$.

We show that some of the notions introduced in the previous sections concerning U are different from each other by producing counterexamples to the various possible implications. We consider here only left-amenability, as we may obtain similar statements for right-amenability by considering the conjugate of U.

(1) U co-amenable does not imply that U has the WCP. In fact, pick $(\mathcal{A}, \Delta)$ of discrete type and such that $(\hat{\mathcal{A}}, \hat{\Delta})$ is not co-amenable, *e.g.*, $\mathcal{A} = C_c(\mathbb{F}_2)$. Then $(\mathcal{A}, \Delta)$ is co-amenable since it is of discrete type, *cf.* [6, Theorem 4.1]. Hence, every corepresentation of it is co-amenable, by Theorem 5.2. In particular U is co-amenable. But U has not the WCP since $(\hat{\mathcal{A}}, \hat{\Delta})$ is not co-amenable, by the dual version of Corollary 7.7.

(2) U has the WCP does not imply that U is co-amenable. Indeed, pick $(\mathcal{A}, \Delta)$ non co-amenable and of compact type, *e.g.*, $\mathcal{A} = \mathbb{C}[\mathbb{F}_2])$. Again pick $U = W$. Now, $(\hat{\mathcal{A}}, \hat{\Delta}_{\mathrm{op}})$ is co-amenable (being of discrete type). Hence, U has the WCP, using the dual version of Corollary 7.7. On the other hand, U is not co-amenable, according to Theorem 5.2.

(3) U left-amenable does not imply that U is co-amenable. Again, pick $(\mathcal{A}, \Delta)$ non co-amenable and of compact type and let $U = W$. Since any compact quantum group is amenable (see the paragraph preceding Theorem 4.7 in [6]), U is left-amenable according to Theorem 6.3. On the other hand, according to Theorem 5.2, U is not co-amenable.

(4) U co-amenable does not imply that U is left-amenable. Let $(\mathcal{A}, \Delta)$ be non-amenable and of discrete type. Being co-amenable, all its unitary corepresentations are then co-amenable. However, they cannot all be amenable.

(5) U left-amenable does not imply that U has the WCP. Indeed, let Γ be any non-trivial finite group and let $\mathcal{A} = C(\Gamma)$. Let Δ be the usual co-product on $\mathcal{A}$. Then pick a non-trivial irreducible unitary representation u of Γ and let U be the unitary corepresentation of $(\mathcal{A}_r, \Delta_r)$ associated with u. Now, it is clear that $(\mathcal{A}_r, \Delta_r)$ is amenable and has property (T) (since it is compact). Then U is amenable (by Theorem 6.3), but U has not the WCP (as remarked at the beginning of subsection 7.3).

Remark Let $(\mathcal{A}, \Delta)$ be of compact type. As used several times by now, $(\mathcal{A}, \Delta)$ is then amenable and all the unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$ are therefore amenable. If $(\mathcal{A}, \Delta)$ is also co-amenable (*e.g.*, we may take the compact matrix pseudogroup $\mathcal{A} = \mathrm{SU}_q(2)$, *cf.* [3, 5]), all these corepresentations are then also co-amenable. Further, as $(\mathcal{A}, \Delta)$ has property (T), we get that none of the non-trivial irreducible corepresentations of $(\mathcal{A}_r, \Delta_r)$ satisfies the WCP.

On the other hand, $(\hat{\mathcal{A}}, \hat{\Delta})$ is always co-amenable since it is of discrete type. Hence, all the unitary corepresentations of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ are co-amenable. If $(\mathcal{A}, \Delta)$ is also co-amenable, then we know that $(\hat{\mathcal{A}}, \hat{\Delta})$ is amenable, hence all these corepresentations are then also amenable.

9 Amenability and Discrete Quantum Groups

As we pointed out in connection with Proposition 7.10, it would be interesting to know whether the converse of Proposition 7.10 (5) holds. This amounts to asking whether right-amenability of U always implies that U_{HS} has the WCP. It does in the classical case and this is one of the most interesting results in [8]. The problem of passing from the right-amenability of U to the WCP for U_{HS} seems more delicate in the general case. We present here a proof of this fact for the case of a discrete quantum group.

For notational reasons, we let $(\mathcal{A}, \Delta)$ be an algebraic quantum group of compact type and consider its dual $(\hat{\mathcal{A}}, \hat{\Delta})$ which is then of discrete type. We use the description of $(\hat{\mathcal{A}}, \hat{\Delta})$ given in Proposition 2.2 and the notation introduced there. We denote by $\tilde{S} = \hat{S}_{\mathrm{op}}$ the antipode of $(\hat{\mathcal{A}}, \hat{\Delta}_{\mathrm{op}})$, and by $\tilde{R}$ the anti-unitary antipode of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$ (which is defined on $\hat{\mathcal{M}}$). For each $\alpha \in A$, we denote the central minimal projection of $\hat{\mathcal{M}}$ which is given by $\hat{\pi}_r(p_\alpha)$ with the same symbol p_α. Further, we identify $\hat{\pi}_r(\hat{\mathcal{A}}_\alpha) = p_\alpha \hat{\pi}_r(\hat{\mathcal{A}})$ with $\hat{\mathcal{A}}_\alpha = M_{d_\alpha}(\mathbb{C})$ and let Tr_α denote its canonical trace satisfying $\mathrm{Tr}_\alpha(p_\alpha) = d_\alpha$. Finally, we denote the canonical injection from $\hat{\mathcal{A}}$ into $\mathcal{H}$ by $\hat{\Lambda}$.

Now let U be a unitary corepresentation of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$. We remark that using the above identifications and the properties of p_α, one easily deduces that

$$(p_\alpha \otimes I)U = U(p_\alpha \otimes I) \in \hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U), \;\; U(p_\alpha \otimes y)U^* \in \hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$$

for all $\alpha \in A$ and $y \in HS(\mathcal{H}_U)$. We denote by T_α the trace on $\hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U)$ given by $T_\alpha = \mathrm{Tr}_\alpha \odot \mathrm{Tr}$, where Tr denotes the canonical trace on $B(\mathcal{H}_U)$. Further, we denote by $\|\cdot\|_{1,\alpha}$ and $\|\cdot\|_{2,\alpha}$ the associated norms on $\hat{\mathcal{A}}_\alpha \odot TC(\mathcal{H}_U)$ and $\hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$, respectively.

We establish a series of lemmas.

Lemma 9.1 *For each $\alpha \in A$, set*

$$b_\alpha = M_\alpha \sum_{i,j=1}^{d_\alpha} f_1(u_{ji}^\alpha) u_{ij}^\alpha,$$

so that we have $\hat{b}_\alpha = p_\alpha$. The following conditions are equivalent:

(1) *U has the WCP.*

(2) *There exists a net (ξ_i) of unit vectors in $\mathcal{H}_U$ such that*

$$\lim_i \|U(\hat{\Lambda}(p_\alpha) \otimes \xi_i) - \hat{\Lambda}(p_\alpha) \otimes \xi_i\|_2 = 0 \quad \forall\, \alpha \in A.$$

(3) *There exists a state ϕ on $B(\mathcal{H}_U)$ such that*

$$\phi((\omega_{\hat{\Lambda}(p_\alpha)} \bar{\otimes} \iota)U) = \omega_{\hat{\Lambda}(p_\alpha)}(I), \quad \forall\, \alpha \in A.$$

(4) *There exists a state ϕ on $B(\mathcal{H}_U)$ such that*

$$\phi(\pi_U(b_\alpha)) = M_\alpha^2, \quad \forall\, \alpha \in A.$$

(5) *There exists a state ϕ on $B(\mathcal{H}_U)$ such that $\phi\pi_U = \varepsilon_u$.*

Proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow as in the proof of Theorem 7.8.

(3) $\Rightarrow$ (4): Assume that (3) holds and let ϕ be as in (3). We will show that ϕ satisfies (4). Fix $\alpha \in A$.

We first observe that $S(b_\alpha^*) = b_\alpha$. Indeed,

$$\begin{aligned}
S(b_\alpha^*) &= M_\alpha \sum_{i,j} \overline{f_1(u_{ji}^\alpha)} S((u_{ij}^\alpha)^*) \\
&= M_\alpha \sum_{i,j} \overline{f_1(u_{ji}^\alpha)} \sum_{k,l} f_1(u_{jk}^\alpha) f_{-1}(u_{li}^\alpha) u_{kl}^\alpha \\
&= M_\alpha \sum_{i,j,k,l} f_{-1}((u_{ji}^\alpha)^*) f_{-1}(u_{li}^\alpha) f_1(u_{jk}^\alpha) u_{kl}^\alpha \\
&= M_\alpha \sum_{j,k,l} f_{-1}\Big(\sum_i (u_{ji}^\alpha)^* u_{li}^\alpha\Big) f_1(u_{jk}^\alpha) u_{kl}^\alpha \\
&= M_\alpha \sum_{j,k,l} f_{-1}(\delta_{jl} I) f_1(u_{jk}^\alpha) u_{kl}^\alpha = M_\alpha \sum_{k,l} f_1(u_{lk}^\alpha) u_{kl}^\alpha = b_\alpha.
\end{aligned}$$

Thus, we have

$$\hat{b}_\alpha (S(b_\alpha^*))^\wedge = p_\alpha p_\alpha = p_\alpha = (S(b_\alpha^*))^\wedge.$$

Therefore, using the result from [7] recalled at the beginning of subsection 3.1, we have

$$Q(b_\alpha) = \omega_{\Lambda(b_\alpha),\Lambda(b_\alpha)} = \omega_{\hat{\Lambda}(p_\alpha)}.$$

Using [7, Theorem 3.2] and the assumption that ϕ satisfies (3), we get

$$\begin{aligned}
\phi(\pi_U(b_\alpha)) &= \phi((Q(b_\alpha) \bar{\otimes} \iota) U) \\
&= \phi((\omega_{\hat{\Lambda}(p_\alpha)} \bar{\otimes} \iota) U) = \omega_{\hat{\Lambda}(p_\alpha)}(I) = \hat{\psi}(p_\alpha^* p_\alpha) \\
&= \hat{\psi}(p_\alpha) = \hat{\psi}(\hat{b}_\alpha) = \varepsilon(b_\alpha) \\
&= M_\alpha \sum_{i,j} f_1(u_{ji}^\alpha) \varepsilon(u_{ij}^\alpha) = M_\alpha \sum_i f_1(u_{ii}^\alpha) = M_\alpha^2,
\end{aligned}$$

and (4) is proved.

(4) $\Rightarrow$ (5): Assume (4) holds and let ϕ be as in (4). Let η_u be the state on $\mathcal{A}_u$ given by $\eta_u = \phi \pi_U$ and let η denote the restriction of η_u to $\mathcal{A}$. To show that ϕ satisfies (5), that is $\eta_u = \varepsilon_u$, it suffices to show that $\eta = \varepsilon$.

Fix $\alpha \in A$. As $\sum_{i,j} f_1(u_{ji}^\alpha) u_{ij}^\alpha = \sum_i f_1 * u_{ii}^\alpha$, we have

$$(\star) \qquad \phi\Big(\sum_i f_1 * u_{ii}^\alpha\Big) = M_\alpha = f_1\Big(\sum_i u_{ii}^\alpha\Big).$$

Set $d = \sum_i u_{ii}^\alpha$ and observe that we may write $(\star)$ as $\eta f_1(d) = f_1(d)$.

Further, set $X_{ij} = f_{1/2} * u^{\alpha}_{ij} - f_{1/2}(u^{\alpha}_{ij})I \in \mathcal{A}$. Then

$$\begin{aligned}
\eta\Big(\sum_{i,j} X^*_{ij}X_{ij}\Big) &= \sum_{i,j} \eta\big((f_{1/2} * u^{\alpha}_{ij})^*(f_{1/2} * u^{\alpha}_{ij})\big) - 2Re\big(\overline{f_{1/2}(u^{\alpha}_{ij})}\eta(f_{1/2} * u^{\alpha}_{ij})\big) \\
&\quad + |f_{1/2}(u^{\alpha}_{ij})|^2 \\
&= \sum_{i,j} \eta\Big(\sum_{k,l}(u^{\alpha}_{ik})^* \overline{f_{1/2}(u^{\alpha}_{kj})} u^{\alpha}_{il} f_{1/2}(u^{\alpha}_{lj}) \\
&\quad - 2Re(\overline{f_{1/2}(u^{\alpha}_{ij})} \sum_{k} f_{1/2}(u^{\alpha}_{kj})\eta(u_{ik})\Big) + |f_{1/2}(u^{\alpha}_{ij})|^2 \\
&= \sum_{j,k,l}\big(\delta_{kl}\eta(I)\overline{f_{1/2}(u^{\alpha}_{kj})} f_{1/2}(u_{lj})\big) \\
&\quad - 2Re\Big(\sum_{i,j,k}\eta(u_{ik}) f_{1/2}(u^{\alpha}_{ji}) f_{1/2}(u^{\alpha}_{kj})\Big) + \sum_{i} f_1(u^{\alpha}_{ii}) \\
&= \sum_{j,k} f_{1/2}(u^{\alpha}_{jk}) f_{1/2}(u^{\alpha}_{kj}) - 2Re(\eta f_1(u^{\alpha}_{ii})) + \sum_{i} f_1(u^{\alpha}_{ii}) \\
&= 2\sum_{i} f_1(u^{\alpha}_{ii}) - 2Re(\eta f_1(u^{\alpha}_{ii})) = 2\big(f_1(d) - Re(\eta f_1(d))\big).
\end{aligned}$$

Now, as $\eta f_1(d) = f_1(d) = M_{\alpha}$ is real, we get

$$\eta\Big(\sum_{i,j} X^*_{ij}X_{ij}\Big) = 0.$$

Since η is a positive linear functional, this implies that $\eta(X^*_{ij}X_{ij}) = 0$ for all i, j. Using the Cauchy-Schwarz inequality, we obtain $\eta(X_{ij}) = 0$ for all i, j, that is,

$$\eta(f_{1/2} * u^{\alpha}_{ij}) = f_{1/2}(u^{\alpha}_{ij}), \quad \forall\, i, j,$$

hence

$$\eta(f_{1/2} * a) = f_{1/2}(a), \quad \forall\, a \in \mathcal{A},$$

by linearity.

Now, given $b \in \mathcal{A}$, let $a = f_{-1/2} * b$ and apply the above: this gives

$$\eta(f_{1/2} * f_{-1/2} * b) = f_{1/2}(f_{-1/2} * b),$$

that is, $\eta(b) = \varepsilon(b)$. Thus, we have shown that $\eta = \varepsilon$, as desired.

(5) $\Rightarrow$ (1): Assume (5) holds. Then we clearly have $\operatorname{Ker} \pi_U \subset \operatorname{Ker} \varepsilon_u$, that is, U has the WCP. ■

Lemma 9.2 *Let $\alpha \in A$ and set $p_{\beta} = \tilde{S}(p_{\alpha})$. Then we have*

$$(\tilde{S} \odot \iota)((p_{\alpha} \otimes I)U) = U^*(p_{\beta} \otimes I).$$

Proof From the proof of [24, Proposition 3.4], we know that

$$\tilde{S}((\iota\bar{\otimes}\omega)U) = (\iota \otimes \omega)(U^*), \quad \omega \in B(\mathcal{H}_U)_*.$$

Hence, we get

$$\begin{aligned}(\iota\bar{\otimes}\omega)(\tilde{S} \odot \iota)((p_\alpha \otimes I)U) &= \tilde{S}(p_\alpha(\iota\bar{\otimes}\omega)U)) \\ &= \tilde{S}((\iota\bar{\otimes}\omega)U)p_\beta = (\iota \otimes \omega)(U^*)p_\beta \\ &= (\iota\bar{\otimes}\omega)(U^*(p_\beta \otimes I))\end{aligned}$$

for all $\omega \in B(\mathcal{H}_U)_*$. ■

Lemma 9.3 *Assume that U is right-amenable. Then there exists a net (y_i) in $\{y \in HS(\mathcal{H}_U) \mid y \geq 0, \|y\|_2 = 1\}$ such that*

$$\lim_i \|U(p_\alpha f_{-1/2} \otimes y_i)U^* - p_\alpha f_{-1/2} \otimes y_i\|_{2,\alpha} = 0, \quad \forall\, \alpha \in A.$$

Proof We begin as in the proof of Proposition 6.4. Let m'_U be a right-invariant mean for U, so

$$m'_U((\omega\bar{\otimes}\iota)U(I \otimes x)U^*) = \omega(I)m'_U(x)$$

for all $x \in B(\mathcal{H}_U), \omega \in \hat{\mathcal{M}}_*$. As the normal states are weak*-dense in $S(B(\mathcal{H}_U))$, we may pick a net $(s_i) \subset TC(\mathcal{H}_U)_1^+$ such that m'_U is a weak*-limit point of the net $(\mathrm{Tr}(s_i\,\cdot)) \subset S(B(\mathcal{H}_U))$.

Now, we define a net (y_i) in $\{y \in HS(\mathcal{H}_U) \mid y \geq 0, \|y\|_2 = 1\}$ by setting $y_i = s_i^{1/2}$ for all i.

Let $\alpha \in A$. Hereafter, we write $\hat{a}_\alpha$ to denote $\hat{a}p_\alpha \in \hat{\mathcal{A}}_\alpha$ whenever $a \in \mathcal{A}$.

Let $b, b' \in \mathcal{A}$. Set $c_\alpha = \hat{\rho}^{-1}(\hat{b'}_\alpha)\hat{b}^*_\alpha$. Then we have

$$\begin{aligned}\lim_i \omega_{\hat{\Lambda}(\hat{b'}_\alpha),\hat{\Lambda}(\hat{b}_\alpha)}(I)\,\mathrm{Tr}(xs_i) &= \lim_i \mathrm{Tr}\big((\omega_{\hat{\Lambda}(\hat{b'}_\alpha),\hat{\Lambda}(\hat{b}_\alpha)}\bar{\otimes}\iota)(U(I \otimes x)U^*)s_i\big) \\ &= \lim_i (\hat{\psi} \odot \mathrm{Tr})\big((\hat{b}^*_\alpha \otimes y_i)U(I \otimes x)U^*(\hat{b'}_\alpha \otimes y_i)\big).\end{aligned}$$

Thus

$$\lim_i(\hat{\psi} \odot \mathrm{Tr})\big(\hat{b}^*_\alpha\hat{b'}_\alpha \otimes xy_i^2 - (\hat{b}^*_\alpha \otimes y_i)U(I \otimes x)U^*(\hat{b'}_\alpha \otimes y_i)\big) = 0,$$

which gives

$$\lim_i(\hat{\psi} \odot \mathrm{Tr})\big(c_\alpha \otimes xy_i^2 - (c_\alpha \otimes y_i^2)U(I \otimes x)U^*\big) = 0.$$

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Write $(p_\alpha \otimes I)U = \sum_r a_r \otimes x_r \in \hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U)$. Then, using that Tr is a trace and Proposition 2.2, we get that

$$\begin{aligned}
(\hat{\psi} \odot \mathrm{Tr})(c_\alpha \otimes xy_i^2 - (c_\alpha \otimes y_i^2)U(I \otimes x)U^*) &= (\hat{\psi} \odot \mathrm{Tr})(c_\alpha \otimes xy_i^2) \\
&\quad - \sum_{r,s} (\hat{\psi} \odot \mathrm{Tr})(c_\alpha a_r a_s^* \otimes y_i^2 x_r x x_s^*) \\
&= (\hat{\psi} \odot \mathrm{Tr})(c_\alpha \otimes xy_i^2) - \sum_{r,s} (\hat{\psi} \odot \mathrm{Tr})(c_\alpha a_r a_s^* \otimes x x_s^* y_i^2 x_r) \\
&= (\hat{\psi} \odot \mathrm{Tr})\big((c_\alpha \otimes x)(p_\alpha \otimes y_i^2 - \sum_{r,s}(a_r a_s^* \otimes x_s^* y_i^2 x_r)\big) \\
&= (\hat{\psi} \odot \mathrm{Tr})\big((c_\alpha \otimes x)(p_\alpha \otimes y_i^2 - \sum_{r,s} a_r p_\alpha a_s^* \otimes x_s^* y_i^2 x_r)\big) \\
&= M_\alpha (\mathrm{Tr}_\alpha \odot \mathrm{Tr})\Big((c_\alpha \otimes x)\big(p_\alpha f_{-1} \otimes y_i^2 \\
&\qquad - \sum_{r,s} a_r p_\alpha a_s^* f_{-1} \otimes x_s^* y_i^2 x_r\big)\Big)
\end{aligned}$$

converges to zero. Note that if we let b and b' vary in $\mathcal{A}$, then c_α will give all elements in $\hat{\mathcal{A}}_\alpha$ (using that $\hat{\mathcal{A}}^2 = \hat{\mathcal{A}}$ and Proposition 2.2). Hence, adapting Namioka's argument [18, Proof of Theorem 2.4.2] by considering the locally convex product space $\prod\{\hat{\mathcal{A}}_\alpha \otimes TC(\mathcal{H}_U), \alpha \in A\}$ with the product of the $\|\cdot\|_{1,\alpha}$-norm topologies, we may assume that

$$\lim_i \Big\| p_\alpha f_{-1} \otimes y_i^2 - \sum_{r,s} a_r p_\alpha a_s^* f_{-1} \otimes x_s^* y_i^2 x_r \Big\|_{1,\alpha} = 0.$$

Now, as $\hat{\mathcal{A}}_\alpha$ is a matrix algebra, we can apply the linear map $\tilde{S}(\cdot f_2)$ on the first tensor factor and still keep convergence in norm. Thus we get

$$(\star) \qquad \lim_i \Big\| p_\beta f_{-1} \otimes y_i^2 - \sum_{r,s} f_{-1} \tilde{S}(a_s^*) p_\beta \tilde{S}(a_r) \otimes x_s^* y_i^2 x_r \Big\|_{1,\alpha} = 0.$$

Lemma 9.2 says that $(\tilde{S} \odot \iota)((p_\alpha \otimes I)U) = U^*(p_\beta \otimes I)$. Since $\tilde{S}^2(\hat{a}) = f_1 \hat{a} f_{-1}$ for all $\hat{a} \in \hat{A}$, we also get

$$(\tilde{S} \odot \iota)(U^*(p_\alpha \otimes I)) = (f_1 \otimes I)U(p_\beta f_{-1} \otimes I).$$

Combining these two facts, equation $(\star)$ tells us that

$$\lim_i \big\| p_\beta f_{-1} \otimes y_i^2 - U(p_\beta f_{-1} \otimes y_i^2)U^* \big\|_{1,\alpha} = 0.$$

Using the Powers-Størmer inequality (see [8, Lemma 4.2]), we then get

$$\lim_i \big\| p_\beta f_{-1/2} \otimes y_i - U(p_\beta f_{-1/2} \otimes y_i)U^* \big\|_{2,\alpha} = 0.$$

As $\alpha \to \beta$ is a bijection of A, we are done. ∎

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press

Lemma 9.4 *Assume that U is right-amenable. Then there exists a net (y_i) in $\{y \in HS(\mathcal{H}_U) \mid y \geq 0,\ \|y\|_2 = 1\}$ such that*

$$\lim_i \|U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y_i) - \hat{\Lambda}(p_\alpha) \otimes y_i\|_2 = 0 \quad \forall\, \alpha \in A.$$

Proof Let $\alpha \in A$, $y \in HS(\mathcal{H}_U)$ and set $p_\beta = \tilde{S}(p_\alpha)$ $(= \tilde{R}(p_\alpha))$. Recalling from Proposition 4.4 that

$$U_{HS} = (\iota \bar{\otimes} l)U(\tilde{R} \bar{\otimes} r)U$$

we have

$$U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y) = (\iota \bar{\otimes} l)U(\tilde{R} \bar{\otimes} r)(U)(p_\alpha \otimes I)(\hat{\Lambda}(p_\alpha) \otimes y).$$

Now, using Lemma 9.2 and the formula $\tilde{R}(\,\cdot\,) = f_{-1/2}\tilde{S}(\,\cdot\,)f_{1/2}$, which is easily checked, we get

$$\begin{aligned}(\tilde{R} \bar{\otimes} r)(U)(p_\alpha \otimes I) &= (\tilde{R} \bar{\otimes} r)((p_\beta \otimes I)U)\\ &= (\iota \odot r)(\tilde{R} \odot \iota)((p_\beta \otimes I)U)\\ &= (\iota \odot r)((f_{-1/2} \otimes I)U^*(p_\alpha f_{1/2} \otimes I)).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y) &= (\iota \bar{\otimes} l)U(\iota \odot r)\big((f_{-1/2} \otimes I)U^*(p_\alpha f_{1/2} \otimes I)\big)(\hat{\Lambda}(p_\alpha) \otimes y)\\ &= (\iota \odot l)(U(p_\alpha \otimes I))(\iota \odot r)\big((f_{-1/2} \otimes I)U^*(p_\alpha f_{1/2} \otimes I)\big)\\ &\qquad \times (\hat{\Lambda}(p_\alpha) \otimes y)\\ &= (\hat{\Lambda} \odot \iota)\big(U(p_\alpha f_{-1/2} \otimes y)U^*(f_{1/2} \otimes I)\big).\end{aligned}$$

Using Proposition 2.2, one sees that

$$\|(\hat{\Lambda} \odot \iota)(B)\|_2 = M_\alpha^{1/2}\|B(f_{-1/2} \otimes I)\|_{2,\alpha}$$

for all $B \in \hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$.

Therefore, choosing the net (y_i) as the one provided by Lemma 9.3, we get

$$\begin{aligned}&\lim_i \|U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y_i) - \hat{\Lambda}(p_\alpha) \otimes y_i\|_2\\ &\qquad = \lim_i \big\| (\hat{\Lambda} \odot \iota)(U(p_\alpha f_{-1/2} \otimes y_i)U^*(f_{1/2} \otimes I)) - (\hat{\Lambda} \odot \iota)(p_\alpha \otimes y_i)\big\|_2\\ &\qquad = \lim_i M_\alpha^{1/2}\big\| U(p_\alpha f_{-1/2} \otimes y_i)U^* - p_\alpha f_{-1/2} \otimes y_i\big\|_{2,\alpha} = 0,\end{aligned}$$

as desired. ■

We are now in position to derive the following analog of [8, Theorem 5.1].

Theorem 9.5 *Assume that $(\mathcal{A}, \Delta)$ is of compact type. Let U be a unitary corepresentation of $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\mathrm{op}})$. Then U is right-amenable if and only if U_{HS} has the WCP.*

Proof Assume that U is right-amenable. Combining Lemma 9.4 with Lemma 9.1, we deduce that U_{HS} has the WCP. The converse implication is shown in Proposition 7.10 (5). ■

Remark Let $(\mathcal{A}, \Delta)$ and U be as in Theorem 9.5. Recall from 4.2 that we can associate with U another Hilbert-Schmidt corepresentation $U_{HS'} \simeq (\overline{U})_{HS}$. As U is left-amenable if and only if $\overline{U}$ is right-amenable, we deduce from Theorem 9.5 that U is left-amenable if and only if $U_{HS'}$ has the WCP.

As a consequence of Theorem 9.5, we present a new proof of the fact that amenability of a discrete quantum group is equivalent to co-amenability of its dual. This result is due to Z.-J. Ruan [33, Theorem 4.5] in the case when the dual (compact) quantum group is assumed to have a tracial Haar state (see also [7]), and has recently been shown independently by E. Blanchard and S. Vaes [10] and by R. Tomatsu [35] for any discrete quantum group.

Corollary 9.6 *Assume that $(\mathcal{A}, \Delta)$ is of compact type. Then $(\mathcal{A}, \Delta)$ is co-amenable if and only if $(\hat{\mathcal{A}}, \hat{\Delta})$ is amenable.*

Proof We know from [6, Theorem 4.7] that $(\hat{\mathcal{A}}, \hat{\Delta})$ is amenable whenever $(\mathcal{A}, \Delta)$ is co-amenable. Assume now that $(\hat{\mathcal{A}}, \hat{\Delta})$ is amenable. From Theorem 6.3, we deduce that $\hat{W}$ is right-amenable. Using Theorem 9.5, we obtain that $\hat{W}_{HS}$ has the WCP. It follows from Corollary 7.13 that $(\mathcal{A}, \Delta)$ is co-amenable. ■

Finally, we improve Proposition 7.14 in the discrete case.

Corollary 9.7 *Let $(\mathcal{A}, \Delta)$ denote an algebraic quantum group of discrete type and U, V be unitary corepresentations of $(\mathcal{A}_r, \Delta_r)$.*

If U is left- (resp., right-) amenable and $U \prec V$, then V is left- (resp., right-) amenable.

Proof We leave to the reader to verify that $U_{HS} \prec V_{HS}$ whenever $U \prec V$. The right-version of the assertion follows then by applying twice Theorem 9.5 (to the compact quantum group $(\hat{\mathcal{A}}, \hat{\Delta}_{\mathrm{op}})$). The left-version is shown similarly. ■

In view of Theorem 6.3 we can now conclude that a discrete quantum group is amenable if and only if all its irreducible unitary corepresentations are left- (resp., right-) amenable.

Acknowledgements Part of this work was done while R. Conti visited the Department of Mathematics at the University of Oslo, May 2000 and June 2001, and the Centre for Advanced Studies in Oslo, October 2001. He thanks these institutions and the Operator Algebra team in Oslo for their friendly hospitality.

References

[1] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*- algèbres.* Ann. Sci. École. Norm. Sup. **26**(1993), 425–488.
[2] T. Banica, *Representations of compact quantum groups and subfactors.* J. Reine Angew. Math. **509**(1999), 167–198.
[3] ———, *Fusion rules for representations of compact quantum groups.* Exposition. Math. **17**(1999), 313–338.
[4] E. Bédos, *Notes on hypertraces and C^*-algebras.* J. Operator Theory **34**(1995), 285–306.
[5] E. Bédos, G. J. Murphy and L. Tuset, *Co-amenability of compact quantum groups.* J. Geom. Phys. **40**(2001), 130–153.
[6] ———, *Amenability and co-amenability for algebraic quantum groups.* Int. J. Math. Math. Sci. **31**(2002), 577–601.
[7] ———, *Amenability and co-amenability for algebraic quantum groups, II.* J. Funct. Anal. **201**(2003), 303–340.
[8] M. E. B. Bekka, *Amenable unitary representation of locally compact groups.* Invent. Math. **100**(1990), 383–401 .
[9] E. Blanchard, *Déformations de C^*- Algèbres de Hopf.* Bull. Soc. Math. France **124**(1996), 141–215.
[10] E. Blanchard and S. Vaes, *A remark on amenability of discrete quantum groups.* Preliminary version (2002).
[11] P. Desmedt, J. Quaegebeur and S. Vaes, *Amenability and the bicrossed product construction.* Illinois J. Math. **46**(2003), 1259–1273.
[12] J. Dixmier, *Les C^*-algèbres et leurs représentations.* Gauthiers-Villars, Paris, 1969.
[13] E. G. Effros and Z.-J. Ruan, *Discrete quantum groups. I. The Haar measure.* Internat. J. Math. **5**(1994), 681–723.
[14] M. Enock and J. M. Schwartz, *Kac algebras and duality of locally compact groups.* Springer-Verlag, Berlin, 1992.
[15] ———, *Algèbres de Kac moyennables.* Pacific. J. Math. **125**(1986), 363–379.
[16] J. M. G. Fell, *Weak containment and induced representations of groups.* Canad. J. Math. **14**(1962), 237–268.
[17] L. R. Ghez and J. E. Roberts, *W^*-categories.* Pacific. J. Math. **120**(1985), 79–109.
[18] F. P. Greenleaf, *Invariant means on topological groups.* Van Nostrand, New York, 1969.
[19] P. de la Harpe, and A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts.* Asterisque 175, Soc. Math. de France, 1989.
[20] A. Jacobs, and A. Van Daele, *The quantum $E(2)$ as a locally compact quantum group.* Preprint, K.U. Leuven (2003) (in preparation).
[21] J. Kustermans, *Examining the dual of an algebraic quantum group.* Preprint Odense Universitet (1997). (arXiv:funct-an/9704006).
[22] ———, *Universal C^*- algebraic quantum groups arising from algebraic quantum groups.* Preprint Odense Universitet (1997). (arXiv:funct-an/9704004).
[23] J. Kustermans and A. Van Daele, *C^*-algebraic quantum groups arising from algebraic quantum groups.* Int. J. Math. **8**(1997), 1067–1139 .
[24] J. Kustermans, and S. Vaes, *Locally compact quantum qroups.* Ann. Sci. École Norm. Sup. **33**(2000), 837–934.
[25] R. Longo, and J. E. Roberts, *A theory of dimension.* *K*–Theory **11**(1997), 103–159.
[26] C.-K. Ng, *Amenability of Hopf C^*-algebras.* In: Operator theoretical works. Theta Found., Bucharest, 2000, pp. 269–284.
[27] ———, *An Example of amenable Kac algebras.* Proc. Amer. Math. Soc. **130**(2002), 2995–2998.
[28] ———, *Amenable representations and Reiter's property for Kac algebras.* J. Funct. Anal. **187**(2001), 163–182.
[29] A. L. Paterson, *Amenability.* Math. Surveys and Monographs 29, American Mathematical Society, Providence, RI, 1988.
[30] S. Petrescu and M. Joita, *Property (T) for Kac algebras.* Rev. Roum. Math. Pures Appl. **37**(1992), 163–178.
[31] P. Podles and S. L. Woronowicz, *Quantum deformation of Lorentz group.* Comm. Math. Phys. **130**(1990), 381–431.
[32] J. E. Roberts, and L. Tuset, *On the equality of q-dimension and intrinsic dimension.* J. Pure Appl. Algebra **156**(2001), 329–343 .
[33] Z.-J. Ruan, *Amenability of Hopf von Neumann algebras and Kac algebras.* J. Funct. Anal. **139**(1996), 466–499.
[34] S. Stratila, *Modular theory in operator algebras.* Abacus Press, Tunbridge Wells, Kent, 1981.

[35] R. Tomatsu, *Amenable discrete quantum groups.* Preprint, University of Tokyo (2003).
[36] A. Van Daele, *Multiplier Hopf algebras.* Trans. Amer. Math. Soc. **342**(1994), 917–932.
[37] ———, *An algebraic framework for group duality.* Adv. Math. **140**(1998), 323–366.
[38] D. Voiculescu, *Amenability and Katz Algebras.* Algèbres d'Opérateurs et leurs Applications en Physique Mathématique, 274, Colloq. Internation. C.N.R.S., 1977, pp. 451–457.
[39] S. L. Woronowicz, *Compact matrix pseudogroups.* Comm. Math. Phys. **111**(1987), 613–665.
[40] ———, *Compact quantum groups.* In: Symétries Quantiques, North Holland, Amsterdam, 1998, pp. 845–884.

Institute of Mathematics
University of Oslo
P.B. 1053 Blindern
0316 Oslo
Norway
e-mail: bedos@math.uio.no

Mathematisches Institut
Friedrich-Alexander Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
D-91054 Erlangen
Germany
e-mail: conti@mi.uni-erlangen.de

Faculty of Engineering
Oslo University College
Cort Adelers Gate 30
0254 Oslo
Norway
e- mail: Lars.Tuset@iu.hio.no

https://doi.org/10.4153/CJM-2005-002-8 Published online by Cambridge University Press