NOTE ON POINTWISE CONVERGENCE ON THE CHOQUET BOUNDARY

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In [6] J. Rainwater obtained the following theorem.

THEOREM. Let N be a normed linear space, $\{x_n\}$ a bounded sequence of elements in N and $x \in N$. If $\lim_n f(x_n) = f(x)$ for each extreme point f of the unit ball of N^*, then $\{x_n\}$ converges weakly to x.

Now let X be a compact Hausdorff space and H a linear subspace of $C(X)$ (all real-valued continuous functions on X) which separates the points of X and contains the constant functions. If $x \in X$, then $M_x(H)$ denotes the set of positive linear functionals μ on $C(X)$ such that $\mu(h) = h(x)$ for all h in H. $\nabla_H X$, the Choquet boundary of X relative to H, is the set of x in X for which $M_x(H)$ contains only the evaluation functional on $C(X)$ at x (i.e., the Dirac measure at x). Note that $\nabla_H X = \nabla_{\bar{H}} X$ where $\bar{H}$ is the sup norm closure of H. The closure of $\nabla_H X$ coincides with the Šilov boundary $\partial_H X$ of X relative to H, that is, the smallest closed subset E of X such that $\sup h(E) = \sup h(X)$ for all h in H (see [1] for details). $\hat{H}$ is the set of f in $C(X)$ such that $\mu(f) = f(x)$ for all $x \in X$ and $\mu \in M_x(H)$.

An application of the Choquet-Bishop-de Leeuw theorem and the Lebesgue bounded convergence theorem gives:

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THEOREM 1. Let X be a compact Hausdorff space and H a separating linear subspace of $C(X)$ which contains the constant functions. If $\{h_n\} \subset H$ is a uniformly bounded sequence which converges pointwise to $h_o \in H$ on $\nabla_H X$, then $\{h_n\}$ converges pointwise to h_o on X. Equivalently, $\{h_n\}$ converges weakly to h_o.

Proof. Let $x \in X$ and μ_x be a measure on the σ-ring generated by $\nabla_H X$ and the Baire sets such that $\mu_x(\nabla_H X) = 1$ and $\int h d\mu_x = h(x)$ for all $h \in H$ (see [2, Theorem 5.5] or [4]). Since $\{h_n\}$ converges to h_o almost everywhere μ_x, by the Lebesgue bounded convergence theorem,

$$\int h_n d\mu_x \quad \text{converges to} \quad \int h_o d\mu_x \text{, i.e.,}$$

$$h_n(x) \quad \text{converges to} \quad h_o(x) \quad .$$

The purpose of this note is to show that the above theorem is equivalent to Rainwater's theorem.

U_H will denote the positive face of the unit sphere of H^* (the conjugate space of H where H has the sup norm). Equip H^* with the w*-topology and let $L: X \to H^*$ be defined by $L(x) = \phi_x^H$ where $\phi_x^H(h) = h(x)$ for all h in H. Then L maps X homeomorphically onto $L(X)$ and $L(\nabla_H X) = \mathcal{E}(U_H)$ where $\mathcal{E}$ denotes the set of extreme points (see [1, Hilfssatz 8] or [2, Lemma 4.3]).

The following statements are equivalent:

I. Theorem 1

II. Let K be a compact convex subset of a locally convex Hausdorff space E and $\mathfrak{A}$ the family of continuous affine functions on K. If $\{A_n\} \subset \mathfrak{A}$ is a uniformly bounded sequence which converges pointwise to $A \in \mathfrak{A}$ on $\mathcal{E}(K)$, then $\{A_n\}$ converges pointwise to A on K.

III. Let X and H be as in Theorem 1. If $\{h_n\} \subset H$ is

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uniformly bounded and converges pointwise to 0 on $\nabla_H X$, then there is a sequence of convex combinations of the h_n which converges uniformly to 0 on $\nabla_H X$.

IV. Let K and $\mathfrak{A}$ be as in II. If $\{A_n\} \subset \mathfrak{A}$ is uniformly bounded and converges pointwise to 0 on $\mathcal{E}(K)$, then there is a sequence of convex combinations of the A_n which converges uniformly to 0 on $\mathcal{E}(K)$.

V. Rainwater's theorem.

Proof. Let $\mathcal{L}$ be the class of continuous affine functions on E (i.e., functions of the form $\ell + \alpha$ where ℓ is a continuous linear functional on E and α is a real scalar) restricted to K. By a result of Bauer [1, Korollar, p.119], $\mathcal{L}$ is a separating linear subspace of C(K) which contains the constant functions and $\nabla_{\mathcal{L}} K = \mathcal{E}(K)$. Bauer has also shown [1, Korollar, p.117] that $\hat{\mathcal{L}} = \mathfrak{A}$. Consequently, $\nabla_{\mathfrak{A}} K = \nabla_{\hat{\mathcal{L}}} K = \nabla_{\mathcal{L}} K = \mathcal{E}(K)$. (Alternatively, $\nabla_{\mathfrak{A}} K = \nabla_{\mathcal{L}} K$ since $\mathcal{L}$ is uniformly dense in $\mathfrak{A}$ [4, p.31].) It follows that Theorem 1 implies II and III implies IV.

We note that under our general setting if we set $H \circ L^{-1} = \{h \circ L^{-1} : h \in H\}$, then $H \circ L^{-1}$ is exactly the restriction of the continuous affine functions on H^* to L(X). (cf. [3, Theorem 3, p.18]). Since $L(\nabla_H X) = \mathcal{E}(U_H)$, II implies I and IV implies III.

If $\{h_n\} \subset H$ is uniformly bounded and converges pointwise to 0 on $\nabla_H X$, then by I, $\{h_n\}$ converges pointwise to 0 on X. It is well-known that then the zero function on X can be uniformly approximated on X by convex combinations of the h_n (see [5, 2.1] for a non-measure-theoretic proof). Thus, I implies III.

If III holds, then by a result of Pták [5, Theorem 5.3] $h_n | \partial_H X$ converges weakly to $h | \partial_H X$ where $\partial_H X$ is the Šilov boundary of X relative to H. Since H is isometrically isomorphic to $H | \partial_H X$, h_n converges weakly to h.

Statement II provides exactly the crucial step in Rainwater's proof (see [6]) so that II implies V.

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Now suppose V holds and let $\{h_n\}$ be a uniformly bounded sequence in H which converges pointwise to $h \in H$ on $\nabla_H X$. By hypothesis, $\phi_x^H(h_n) \to \phi_x^H(h)$ for all $x \in \nabla_H X$. Since $L(\nabla_H X) = \mathcal{E}(U_H)$, $\mu(h_n) \to \mu(h)$ for all $\mu \in \mathcal{E}(U_H) \bigcup \mathcal{E}(-U_H)$. If $\mu \in H^*$ is such that $\|\mu\| = 1$ and μ is neither positive nor negative, then μ is not an extreme point of the unit ball of H^*. For let ν be a Hahn-Banach extension of μ to C(X). Since $C(X)^*$ is an (AL)-space, $\nu = \|\nu^+\| \left(\frac{\nu^+}{\|\nu^+\|} \right) + \|\nu^-\| \left(\frac{-\nu^-}{\|\nu^-\|} \right)$ where ν^+, ν^- denote the positive and negative parts of ν. Consequently, $\nu | H = \mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ where $0 < \lambda < 1$, $\mu_1 \in U_H$ and $\mu_2 \in -U_H$. Thus, the set of extreme points of the unit ball of H^* is contained in $\mathcal{E}(U_H) \bigcup \mathcal{E}(-U_H)$. It follows from V that $h_n \to h$ weakly in H so that V implies I,.

Problem. Find an elementary non-measure-theoretic proof of Theorem 1 or of Rainwater's theorem.

A reason for hoping that a solution might exist is Pták's Theorem 2.1 which proves III, without using measure theory, when $\nabla_H X$ is closed. Theorem 5.3 in Pták's paper may be useful here.

We state next two corollaries of Theorem 1 which follow immediately from known characterizations of $\nabla_H X$.

COROLLARY 1. Let Ω be a bounded open set in E^n and $\{h_n\}$ a uniformly bounded sequence of real-valued functions continuous on $\bar{\Omega}$ and harmonic in Ω. If $\{h_n\}$ converges pointwise on the set of regular points of the boundary of Ω to h continuous on $\bar{\Omega}$ and harmonic in Ω, then $\{h_n\}$ converges pointwise to h on $\bar{\Omega}$.

Proof. If H is the set of functions in $C(\bar{\Omega})$ which are harmonic in Ω, then $\nabla_H \bar{\Omega}$ coincides with the set of regular points of the boundary of Ω (see [1, Satz 16]).

COROLLARY 2. Let A be a separating, uniformly closed

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subalgebra of the algebra of all complex-valued continuous functions on the compact metric space X with $1 \in A$. If $\{f_n\}$ is a uniformly bounded sequence of functions in A which converges pointwise to $f \in A$ on the set of peak points for A, then $\{f_n\}$ converges pointwise to f on X.

Proof. If $H = \{\text{Re } f : f \in A\}$, then $\nabla_H X$ coincides with the set of peak points for A (see [2, Theorem 6.5]).

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