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ON SINGULAR SETS OF FLAT HOLOMORPHIC MAPPINGS WITH ISOLATED SINGULARITIES

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Introduction

In [4] B. Iversen studied critical points of algebraic mappings, using algebraic-geometry methods. In particular when algebraic maps have only isolated singularities, he shows the following relation; Let V and S be compact connected non-singular algebraic varieties of $\dim_C V = n$, and $\dim_C S = 1$, respectivelly. Suppose f is an algebraic map of V onto S with isolated singularities. Then it follows that

$$\chi(V) = (-1)^n \sum_p \mu_f(p) + \chi(S)\chi(F) , \tag{0.1}$$

where χ denotes the Euler number, $\mu_f(p)$ is *the Milnor number* of f at the singular point p, and F is the general fiber of $f: V \to S$.

The purpose of this paper is to generalize the above relation (0.1) as follows; Let V and W be connected compact complex manifolds of $\dim_C = n$, and k respectively. And let f be a flat holomorphic map of V into W with isolated singularities ([Def. 1.2]). Moreover we assume that rank $f \geqq k-1$. Then for generic points p on singular set $\Sigma(f)$ of f ([Def. 1.1]), we can define obstruction numbers $\mu_f(p) \in \boldsymbol{Z}$ ([Def. 1.3]) associated with f and p which are Milnor numbers of f at p in the case $k=1$. However these numbers $\mu_f(p)$ are constant on irreducible component containing p of $\Sigma_\ell(f)$. Therefore we put, with respect to, the irreducible decomposition $\Sigma(f) = \bigcup_{j=1} \Sigma^{(j)}$.

$$\mu_f(\Sigma^j) = \mu_f(p) \qquad \text{for any generic } p \in \Sigma^{(j)}$$

Now our main theorem is to show the next relation; For the general fiber F or $f: V \to W$,

$$\chi(V) = (-1)^n \sum_{j=1}^{\ell} C(\mathscr{B}_j)\mu_f(\Sigma^j) + \chi(W)\chi(F) \tag{0.2}$$

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where $C(\mathscr{B}_j)$ are the Chern numbers of line bundles $\mathscr{B}_j$ over Σ^j induced from f (rank $f \geqq k-1$).

In §1 we state some properties of singular set $\Sigma(f)$ ([3]) and give the definition of $\mu_f(p)$. In §2 we review differential geometrical definitions of connections, curvature and boundary forms ([2] and [6]). In §3 we show the duality formula of boundary forms which play an important role in proving the main theorem. The proof of the main theorem is done in §4.

§1. Singularities of holomorphic maps

1.1. Let V be a complex manifold of complex dimension n and T^*V be the holomorphic cotangent bundle of V. Let f be a holomorphic mapping of V into a complex k-dimensional complex manifold W ($n \geqq k$).

DEFINITION 1.1. A point p of V is called a *singular point* of f, if $\operatorname{rank}_p f < k$, where $\operatorname{rank}_p f$ denotes the rank of the linear map $f_p^*: T_{f(p)}^* W \to T_p^* V$. Moreover we denote by $\Sigma(f)$ the set of all singular points of f, called the *singular set* of f.

DEFINITION 1.2. $p \in \Sigma(f)$ is *generic* if the following conditions are satisfied;

i) $\operatorname{rank}_p f = k-1$,

ii) there exists a neighborhood U of p in V such that $U \cap \Sigma(f)$ is a $(k-1)$-dimensional complex submanifold of V.

Let $p \in \Sigma(f)$ be generic. In order to define a topological number concerned with p, we take holomorphic coordinates $\{z^i\}_{i=1}^n$ on an open set $U_p \ni p$ in V with $z^i(p) = 0, i = 1, \cdots, n$ and also local coordinates $\{w^\alpha\}_{\alpha=1}^k$ of $f(p)$ in W such that $w^\alpha(f(p)) = 0, \alpha = 1, \cdots, k$. Set $g^\alpha = w^\alpha \circ f$ and $g = (g^1, \cdots, g^k)$. Further let $\partial g/\partial z$ be the Jacobian matrix of g, that is,

$$\frac{\partial g}{\partial z} = \left\| \begin{matrix} \dfrac{\partial g^1}{\partial z^1}, \cdots, \dfrac{\partial g^k}{\partial z^1} \\ \dfrac{\partial g^1}{\partial z^n}, \cdots, \dfrac{\partial g^k}{\partial z^n} \end{matrix} \right\| .$$

If we write $V(n, k; \boldsymbol{C})$ for the *Stiefel manifold* consisting of all k-frames of $\boldsymbol{C}^n$, then we have the holomorphic map

$$\frac{\partial g}{\partial z}: U_p - \Sigma(g) \to V(n, k; \boldsymbol{C}) .$$

As p is generic, we can take a complex submanifold $\Sigma^{\perp}(p)$ as follows;

(α) $\Sigma^{\perp}(p) \cap \Sigma(g) = \{p\}$.

(β) $\Sigma^{\perp}(p)$ intersects transversally to $\Sigma(g)$ at p.

(γ) the boundary $\partial\Sigma^{\perp}(p)$ of $\Sigma^{\perp}(p)$ is a smooth manifold which is diffeomorphic to the $2(n-k)+1$-dimensional unit sphere $S^{2(n-k)+1}$ in $\boldsymbol{C}^{2(n-k+1)}$.

We call the above submanifold $\Sigma^{\perp}(p)$ a complemental submanifold to $\Sigma(g)$ at p. Finally we choose $2(n-k)+1$-form $\omega_{n,k}$ on $V(n,k;\boldsymbol{C})$ whose cohomology class $\hat{\omega}_{n,k}$ is the generator of $2(n-k)+1$ dimensional cohomology group of $V(n,k;\boldsymbol{C})$, $H^{2(n-k)+1}(V(n,k;\boldsymbol{C});Z) \cong Z$. Here put

$$\tilde{\mu}_g(p) = \int_{\partial\Sigma^{\perp}(p)} \left(\frac{\partial g}{\partial z}\right)^* \omega_{n,k} . \tag{1.1}$$

One notes that $\tilde{\mu}_g(p)$ is an integer and that $\tilde{\mu}_g(p)$ is independent of choosing local coordinates $\{z^i\}_{i=1}^n$, $\{w^\alpha\}_{\alpha=1}^k$ and complemental submanifolds $\Sigma^{\perp}(p)$ to $\Sigma(g)$ at p. Therefore the following definition is well-defined.

DEFINITION 1.3. Let $p \in \Sigma(f)$ be generic. Then *the obstruction number $\mu_f(p)$ at p of f* is defined by

$$\mu_f(p) = \tilde{\mu}_g(p) . \tag{1.1$'$}$$

1.2. Isolated singularities

We shall restrict our discussion to a holomorphic map $f: V \to W$ such that

i) f has only isolated singularities, i.e., for any point $q \in W$, $f^{-1}(q) \cap \Sigma(f)$ is an isolated points set in V.

ii) f is *flat*.

For simplicity we call f satisfying the above conditions i) and ii) *an (IF)-holomorphic map*. The following proposition is well-known.

PROPOSITION 1.4. *Let $f: V \to W$ be (IF)-holomorphic. Then the singular set $\Sigma(f)$ of f is an analytic set of V such that*

$$\dim_C \Sigma(f) = \dim_C W - 1 .$$

From now on we assume that V and W are connected compact complex manifolds of $\dim_C V = n$, and $\dim_C W = k$ and that $f: V \to W$ is (IF)-holomorphic such that rank $f \geqq k-1$.

LEMMA 1.5. *Let $f: V \to W$ be as above and let $\Sigma(f) = \bigcup_{i=1}^{\ell} \Sigma^{(i)}$ be an irreducible decomposition of $\Sigma(f)$. Then for each $\Sigma^{(i)}$ the obstruction number $\mu_f(p_i)$ of f at any generic point $p_i \in \Sigma^{(i)}$ is defined and constant.*

Proof. It is clear from Proposition 1.4 and rank $f \geqq k-1$ that generic points of $\Sigma(f)$ become regular points of the analytic set $\Sigma(f)$ in V, and conversely. Since the regular set of $\Sigma^{(i)}$ is connected, this lemma is trivial from the definition of the obstruction number. Q.E.D.

By the above lemma we can put $\mu_f(\Sigma^{(i)}) = \mu_f(p_i)$ for any generic point $p_i \in \Sigma^{(i)}$.

Next les us consider the following sequence concerning with $f: V \to W$;

$$\text{(I)} \qquad 0 \longrightarrow \operatorname{Ker} T(f) \longrightarrow f^{\#}(T^*W) \xrightarrow{T(f)} T^*V ,$$

where $f^{\#}(T^*W)$ is the induced bundle of T^*W by f and $T(f)$ is the linear map defined by

$$T(f)(p, v) = f_p^* v \qquad \text{for any } (p, v) \in f^{\#}(T^*W) .$$

As rank $f \geqq k-1$ ($\dim_C W = k$), we have

$$(\operatorname{Ker} T(f))_p = \{0\} \qquad \text{if } p \notin \Sigma(f)$$

and

$$\dim_C (\operatorname{Ker} T(f))_p = 1 \qquad \text{for } p \in \Sigma(f) .$$

Thus the restricted bundle $\operatorname{Ker} T(f)|\Sigma(f)$ becomes a topological one-dimensional complex vector bundle. Let us denote by $Q(\Sigma(f))$ the quotient bundle $f^{\#}(T^*W)/\operatorname{Ker} T(f)|\Sigma(f)$, and let $\sim : f^{\#}(T^*W)|\Sigma(f) \to Q(\Sigma(f))$ be the natural projection. Now let $p \in \Sigma(f)$ be generic. Suppose that ω is a type (1,0)-differential form defined on an open set $U \ni f(p)$ such that

$$\text{(1.2)} \qquad \textit{zeros of } f^*(\omega) \cap f^{-1}(U) = p .$$

Here let $f^{\#}(\omega)$ be the cross-section of $f^{\#}(T^*W)$ defined by

$$f^{\#}(\omega)(p) = (p, \omega_{f(p)}) \qquad \text{for any } p \in U .$$

Then $\widetilde{f^{\#}(\omega)}$ is the continuous section of $Q(\Sigma(f))$ on $f^{-1}(U) \cap \Sigma(f)$, and from (1.2) we get

$$\text{zeros of } \widetilde{f^{\#}(\omega)} = \{p\} ,$$

that is, $f^{\#}(\omega)(p) \in \operatorname{Ker} T(f)_p$. Since p is generic, there exists a neighborhood U_p of p in $\Sigma(f)$, which is a complex submanifold of V of $\dim_C (k-1)$, included in $f^{-1}(U)$. Therefore $Q(\Sigma(f))\,|\,U_p$ is a $(k-1)$-dimensional holomorphic vector bundle. Here we can define canonically the *order* of *zeros* $I_p(\widetilde{f^{\#}(\omega)})$ of $\widetilde{f^{\#}(\omega)}$ at p.

DEFINITION 1.6. Let $p \in \Sigma(f)$ be generic and let ω be a (1,0)-type differential form on an open set $U \ni f(p)$ satisfying (1.2). Then the restricted index $\tilde{I}_p(\omega)$ of ω at p is the *order* of *zeros* of $f^{\#}(\omega)$ at p, i.e.

$$\tilde{I}_p(\omega) = I_p(\widetilde{f^{\#}(\omega)}) . \tag{1.3}$$

Before we state our theorem, we need

DEFINITION 1.7. Let $f: V \to W$ be as before, we call *a* (1,0)-*type differential form* ω on W, an f-form, when the following conditions are satisfied;

i) The *zeros* of ω is a finite points set such that

$$f(\Sigma(f)) \cap \textit{zeros} \text{ of } \omega = \phi$$

ii) $\Sigma(f) \cap$ *zeros* of $f^*\omega$ is also a finite points set whose points are all generic.

Let ω be an f-form. Then from the above condition ii) we can define the restricted index $\tilde{I}_p(\omega)$ for each $p \in \Sigma(f) \cap$ zeros of $f^*\omega$. The following existence proposition is proved by using the transversality theorem in [5].

PROPOSITION 1.8. *Let V and W be compact complex manifolds and let $f: V \to W$ be an (IF)-holomorphic map with* $\operatorname{rank} f \geqq k-1$, $(\dim_C W = k)$. *Then there exists an f-form ω.*

We shall prove this proposition in Appendix. Now we are in a position to state the following

THEOREM 1.9. *Let V and W be connected compact complex manifolds of* $\dim_C V = n$ *and* $\dim_C W = k$. *Suppose that $f: V \to W$ is an (IF)-holomorphic mapping such that* $\operatorname{rank} f \geqq k-1$. *Then if F denotes a* *general fiber of* $f: V \to W$, *we have*

$$\chi(V) = \sum_{j=1}^{\ell} (-1)^n \tilde{I}_{p_j}(\omega)\mu_f(p_j) + \chi(W)\chi(F)\,, \tag{1.4}$$

where χ *represents the Euler number and* ω *is an* f*-form such that* $\Sigma(f) \cap$ *zeros of* $f^*\omega = \{p_1, \cdots, p_\ell\}$, (*cf. Def.* 1.7).

Moreover we claim that (1.4) is independent of f-forms ω.

MAIN THEOREM. *Under assumptions in Theorem* 1.9, *let* $\Sigma(f) = \bigcup_{i=1}^{r} \Sigma^{(i)}$ *be the irreducible decomposition of the singular set* $\Sigma(f)$ *of* f. *Let* $Q(f) = f^{\#}(T^*W)/\mathrm{Ker}\, T(f) \,|\, \Sigma(f)$ *as in* 1.1. *Here put* $\mathscr{B}_i = Q(f) \,|\, \Sigma^{(i)}$, $(i = 1, \cdots, r)$. *Then it follows that*

$$\chi(V) = (-1)^n \sum_{i=1}^{r} C(\mathscr{B}_i)\mu_f(\Sigma^{(i)}) + \chi(W)\chi(F)\,, \tag{1.5}$$

where $C(\mathscr{B}_i)$ *denotes the topological* $(k-1)$*-th Chern number of the complex* $(k-1)$*-dimensional vector bundle* $\mathscr{B}_i$ *over* $\Sigma^{(i)}$, *in the sense of Steenrod* [7].

Proof. Without loss of generality we can assume that $\Sigma(f)$ is irreducible, because any point of $\Sigma^{(i)} \cap \Sigma^{(j)}$ $(i \neq j)$ is not generic. Then we have from Lemma 1.5 and (1.4)

$$\chi(V) = (-1)^n \sum_{j=1}^{\ell} \tilde{I}_{p_j}(\omega)\mu_f(\Sigma(f)) + \chi(W)\chi(F)\,. \tag{1.6}$$

Since $\Sigma(f)$ is a compact analytic set of V, $\Sigma(f)$ becomes a compact CW-complex. Thus one can define the $(k-1)$-th chern number $C(Q(f))$ of $Q(f)$ in virtue of Steenrod. On the other hand, $\widetilde{f^{\#}(\omega)}$ is a continuous cross-section of $Q(f)$ and we see zero $(\widetilde{f^{\#}(\omega)}) = \{p_1, \cdots, p_\ell\}$, here $\tilde{}$ is the natural projection of $f^{\#}(T^*W) \,|\, \Sigma(f)$ onto $Q(f)$. The Steenrod's theorem shows that

$$C(Q(f)) = \sum_{j=1}^{\ell} I_{p_j}(\widetilde{f^{\#}(\omega)})\,,$$

that is

$$C(Q(f)) = \sum_{j=1}^{\ell} \tilde{I}_{p_j}(\omega)\,.$$

Combining this fact with (1.6), we can prove (1.5). Q.E.D.

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In particular in case of $k=1$, that is, W is a compact Riemann surface, $\Sigma(f)$ is a finite points set, say $\{p_1, \cdots, p_r\}$ and $\mu_f(p_j)$ is the Milnor number of f at the isolated singular point p_j, $j=1, \cdots, r$. Here from the main theorem we have

COROLLARY 1.10 [4]. *Let W be a connected compact Riemann surface and f a holomorphic map of a connected compact complex manifold V into W with isolated singularities, say $\Sigma(f)=\{p_1, \cdots, p_\ell\}$. Then we get*

$$\chi(V) = (-1)^n \sum_{j=1}^{\ell} \mu_f(p_j) + \chi(W)\chi(F) ,$$

where F is a generic fibre of $f: V \to W$ and $n = \dim_C V$.

Proof. We know that f is flat, because $\Sigma(f)$ is a finite points set and $\dim_C W = 1$. Here the proof is trivial. Q.E.D.

§ 2. Connections and boundary forms ([2], [6])

In this section we review several geometrical definitions in [2] and [6] to be used in the next section.

2.1. Let V be a complex manifold and $A^k(V)$ the set of all k-forms on V, $k=1, \cdots, 2m$ ($\dim_R V = 2m$). Let E be a holomorphic vector bundle of fibre dimension n over V and N be a hermitian norm on E. Here we denote by $\langle\, , \rangle$ the inner product induced by N. Then we can define *a canonical connection* $D(N)$ on E as in [2], as follows; Let U be an open set of V such that there exists a holomorphic frame $s=(s_1, \cdots, s_n)$, where s_i is a holomorphic section of $E|U$. Put $N(s) = \|\langle s_i, s_j\rangle\|_{1\le i,j\le n}$, and

(2.1) $$\theta = \theta(s,N) = d'N(s)\cdot N(s)^{-1} , \quad \text{i.e.,} \quad \theta_{ij} = \sum_{k=1}^{n} d'N(s)_{ik}\cdot (N(s)^{-1})_{kj} ,$$

where d' is the type $(1,0)$-derivation. For a section $\xi = \sum_{k=1}^n \xi^k s_k$ of $E|U$, we define the covariant differential $D\xi = \sum_{k=1}^n d\xi^k\cdot s_k + \sum \xi^k Ds_k$, where $Ds_k = \sum_j \theta_{kj} s_j$. Then $D\xi$ is an E-valued 1-form and $d\langle \xi, \eta\rangle = \langle D\xi, \eta\rangle + \langle \xi, D\eta\rangle$ for sections ξ, η of E.

We call the above connection $D = D(N)$ *the N-connection* of E. Moreover *the curvature form* $K(s,D) = \|K_{ij}\|$ is given by

(2.2) $$K(s, D(N)) = d\theta(s,N) - \theta(s,N) \wedge \theta(s,N) .$$

$K(s, D(N))$ is called the *curvature matrix* of $D(N)$ relative to the frame s. Take another frame $s' = (s'_1, s'_n)$ of $E|U$, and put $s'_i = \sum_j A_{ij} \cdot s_j$. Then we get

$$AK(s, D)A^{-1} = K(s', D) . \tag{2.3}$$

Now we shall define the k-th chern form $C_k(E)$ of E associated with the norm N. Let M_n be the $n \times n$-complex matrices. Let $b^n_k : \overbrace{M_n \times \cdots \times M_n}^{k} \to \boldsymbol{C}$ be a k-linear map defined by

$$b^n_k(A_1, \cdots, A_k) = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k}\bigg|_{\lambda_1 = \cdots = \lambda_k = 0} \det (1_n + \lambda_1 A_1 + \cdots + \lambda_k A_k)$$

for $A_i \in M_n$, where 1_n denotes the unit matrix of degree n. For simplicity we set $b^n_k(A, \cdots, A) = b^n_k((A))$. Then it follows from (2.3) and the definition of b^n_k that $b^n_k\left(\left(\frac{\sqrt{-1}}{2\pi} K(s, D)\right)\right)$ is independent of the frame s.

DEFINITION 2.1. Let $D = D(N)$ be an N-connection on E. Then the *k-th Chern form* $C_k(E, D)$ induced by N is a type (k, k)-real form on V defined as follows; for any frame s of $E|U$ (s may be smooth),

$$C_k(E, D)|_U = b^n_k\left(\left(\frac{\sqrt{-1}}{2\pi} K(s, D)\right)\right) .$$

The next proposition is directly proved from the above definition ([1]).

PROPOSITION 2.2. *$C_k(E, D)$ is closed, i.e., $dC_k(E, D) = 0$.*

2.2. Duality formula and boundary form.

Suppose now that the following sequence (Σ) of holomophic vector bundles over a complex manifold V is exact,

$$0 \to E_{\mathrm{I}} \to E \to E_{\mathrm{II}} \to 0 . \tag{Σ}$$

If E has a hermitian norm N, then in virtue of the sequence (Σ) norms N_{I} and N_{II} are induced from N, on E_{I} and E_{II}, respectively. Let $p_{\mathrm{I}}: E \to E_{\mathrm{I}}$ and $p_{\mathrm{II}}: E \to E_{\mathrm{II}}$ be orthogonal projections and $D = D(N)$ be the N-connection on E. It follows then that $P_i D P_i = D_i$ becomes the N_i-connection on E_i for $i = \mathrm{I}, \mathrm{II}$. Moreover put $D_t = D + (e^t - 1)P_{\mathrm{II}} D P_{\mathrm{I}}$ $(t \in \boldsymbol{R})$. D_t is also a connection on E, called $\boldsymbol{R}$-family of D. The following proposition is used to define the boundary form of E.

Duality formula [2]. Let all notations be as above. Snppose that $\dim E = n$ and $\dim E_{\mathrm{I}} = k$. Then it follows that

$$
\begin{aligned}
C_n(E, D) &= C_k(E_{\mathrm{I}}, D_{\mathrm{I}}) C_{n-k}(E_{\mathrm{II}}, D_{\mathrm{II}}) \\
&\quad - \frac{1}{4\pi} d\Big\{\lim_{t\to-\infty} d^c \int_t^0 b_n^n\Big(\Big(\frac{\sqrt{-1}}{2\pi} K[E, D_t]\,;\, \frac{\sqrt{-1}}{2\pi} P_{\mathrm{I}}\Big)\Big) dt\Big\}
\end{aligned}
\tag{2.4}
$$

where $d^c = \sqrt{-1}(d'' - d')$, $b_n^n((A\,;\,b)) = b_n^n(B, \overbrace{A, \cdots}^{n-1}, A) + \cdots + b(A, \cdots, A, B)$ and $K[E, D_t]$ is the curvature element of D_t, defined as follows; for any frame s of E,

$$
K[E, D_t](s) = K(s, D_t)\,.
$$

Now let E be a holomorphic vector bundle with a norm N over V of fiber dimension n and let π be the projection of E onto V. Set $E_0 = \{v \in E\,;\, v \neq 0\}$ and $\pi | E_0 = \pi_0$. And let $\pi_0^\sharp(E)$ be the induced bundle of E by π_0. Then one defines naturally the induced norm $\pi_0^\sharp(N)$ by π_0 and N. Let $L(E) = \bigcup_{e \in E_0} \{(e, Ce)\}$ be the line bundle over E_0 and $Q(E)$ the quotient bundle of $\pi_0^\sharp(E)$ by $L(E)$. Then clearly we get the following exact sequence;

$$
\Sigma(E)\,, \qquad 0 \to L(E) \to \pi_0^\sharp(E) \to Q(E) \to 0\,,
$$

At first we have

PROPOSITION 2.3. [6].

$$
C_n(\pi_0^\sharp(E), D(\pi_0^\sharp N)) = \pi_0^* C_n(E, D(N))\,. \tag{2.5}
$$

Let $\Delta : E_0 \to L(E)$ be the global holomorphic section defined by $\Delta(e) = (e, e)$ for $e \in E_0$. Then we know

$$
C_1(L(E), D(N_{\mathrm{I}})) = \frac{-1}{4\pi} dd^c \log(\pi_0^\sharp N)(\Delta)\,, \tag{2.6}
$$

where N_{I} is the norm on $L(E)$ defined by the sequence $\Sigma(E)$ and the norm $\pi_0^\sharp(N)$ on $\pi_0^\sharp(E)$ as before. Applying the duality formula (2.4) to the sequence $\Sigma(E)$, we obtain from (2.5) and (2.6)

$$
\begin{aligned}
\pi_0^* C_n(E, D(N)) &= \frac{-1}{4\pi} d\Big\{d^c \log \pi_0^\sharp N(\Delta) \cdot C_{n-1}(Q(E), D(N_{\mathrm{II}})) \\
&\quad + \lim_{t\to-\infty} d^c \int_t^0 b_n^n((\kappa K[\pi_0^\sharp E, D_t(\pi_0^\sharp N)]\,;\, \kappa P_{\mathrm{I}})) dt\Big\}\,,
\end{aligned}
$$

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where $\kappa = \dfrac{\sqrt{-1}}{2\pi}$.

Here let us put

$$\tag{2.7} \begin{aligned} \eta_n(E, N, \Delta) = \frac{-1}{4\pi}\Big\{ & d^c \log \pi_0^\sharp N(\Delta)\cdot C_{n-1}(Q(E)) \\ & + \lim_{t\to-\infty} d^c \int_t^0 b_n^n((\kappa K[\pi_0^\sharp E, D_t(\pi_0^\sharp N)]\,;\,\kappa P_{\mathrm{I}}))\Big\}\,. \end{aligned}$$

It is trivial

$$\tag{2.8} d\eta_n(E, N, \Delta) = \pi_0^* C_n(E, D(N))\,.$$

We resume the above discussions as the following

DEFINITION 2.4 [6]. Let E be a holomorphic n-dimensional vector bundle over a complex manifold V with a norm N. Then the $(2n-1)$-form $\eta_n(E, N, \Delta)$ on E_0 defined by (2.7) is called the *boundary form* of E with the norm N.

For simplicity we abbreviate $\eta_n(E, N, \Delta)$ to $\eta_n(E)$ or $\eta_n(E, N)$.

§3. Duality formula of boundary forms

Let V be a complex manifold and $A^k(V)$ the k-forms on V. Assume there exists an exact sequence (Σ) of holomorphic vector bundles over V

$$\tag{Σ} 0 \to E_{\mathrm{I}} \to E \to E_{\mathrm{II}} \to 0\,.$$

Put $(E_{\mathrm{I}})_0 = \{v \in E_{\mathrm{I}}\,;\, v \neq 0\}$ and let $\prod_{\mathrm{I}}$ be the projection of $(E_{\mathrm{I}})_0$ onto V. Moreover let $\iota: (E_{\mathrm{I}})_0 \to E$ be the inclusion linear map in (Σ). The purpose of this section is to prove the following

MAIN LEMMA. All notations are as in §2. Let $\dim E_{\mathrm{I}} = k$ and $\dim E = n$, and let N be a norm on E. Then it follows that, on $(E_{\mathrm{I}})_0$

$$\tag{3.1} \iota^*\eta_n(E) = \eta_k(E_{\mathrm{I}}) \textstyle\prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}}) + \prod_{\mathrm{I}}^* \xi + d\Phi\,,$$

where $\xi \in A^{2n-1}(V)$ and $\Phi \in A^{2n-2}((E_{\mathrm{I}})_0)$.

3.1. In order to show (3.1) we need several lemmas. At first we have the following

LEMMA 3.1. *There exists an element $\xi' \in A^{2n-1}(V)$ satisfying the following condition; Let us put*

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(3.2) $$\psi = \iota^*\eta_n(E) - \eta_k(E_{\mathrm{I}}) \prod\nolimits_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}}) + \prod\nolimits_{\mathrm{I}}^* \xi' .$$

Then ψ is closed, that is, $d\psi = 0$.

Proof. Let $\prod_0 : E_0 \to V$ be the natural projection as before ($E_0 \subset E$). It is clear by (2.8) that

$$\begin{aligned} d(\iota^*\eta_n(E) &- \eta_k(E_{\mathrm{I}}) \textstyle\prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}})) \\ &= \iota^* \textstyle\prod_0^* C_n(E) - \prod_{\mathrm{I}}^* C_k(E_{\mathrm{I}}) \prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}}) \\ &= \textstyle\prod_{\mathrm{I}}^*\{C_n(E) - C_k(E_{\mathrm{I}})C_{n-k}(E_{\mathrm{II}})\} . \end{aligned}$$

Here we get $\xi'' \in A^{2n-1}(V)$ by the duality formula in § 2 such that

$$C_n(E) - C_k(E_{\mathrm{I}})C_{n-k}(E_{\mathrm{II}}) = d\xi'' .$$

Thus putting $\xi' = -\xi''$, we can prove (3.2). Q.E.D.

Next let $S(E_{\mathrm{I}})$ be the *sphere bundle* of E_{I}, that is, $S(E_{\mathrm{I}}) = \{v \in E_{\mathrm{I}};\ N_{\mathrm{I}}(v) = 1\}$, and let $P: S(E_{\mathrm{I}}) \to V$ be the projection. Clearly each fibre of $S(E_{\mathrm{I}})$ has the canonical orientation. One can here define the fibre integral $P_*: A^{\ell}(S(E_{\mathrm{I}})) \to A^{\ell-(2k-1)}(V)$ as follows; for any $\theta \in A^{\ell}(S(E_{\mathrm{I}}))$,

$$P_*(\theta)_x = \int_{P^{-1}(x)} \theta \qquad (x \in V) .$$

The following lemma is proved in Hirzebruch-Borel [1].

LEMMA 3.2 [4]. *There is $\omega \in A^{2k-1}(S(E_{\mathrm{I}}))$ such that $d\omega = 0$ and $P_*(\omega) = 1$. Moreover for any closed form $\theta \in A^{\ell}(S(E_{\mathrm{I}}))$ we write $\hat\theta$ the cohomology class of θ. Then it follows that*

$$\hat\theta = \overline{P^*(P_*(\theta)) \wedge \omega} + P^*(\xi_1)$$

where $\xi_1 \in H^{\ell}(V; \boldsymbol{R})$.

PROPOSITION 3.3. *Let $\iota_s: S(E_{\mathrm{I}}) \to (E_{\mathrm{I}})_0$ be the inclusion map and let ψ be the $(2n-1)$-form on $(E_{\mathrm{I}})_0$ as in Lemma 3.1. Then we obtain*

(3.3) $$P_*\widehat{(\iota_s^*\psi)} = 0 .$$

If (3.3) is proved, we get our main lemma as Corollary. Indeed it follows from Lemma 3.2 that

$$\widehat{\iota_s^*(\psi)} = \overline{P^*(P_*(\iota_s^*\psi)) \wedge \omega} + P^*(\hat\xi_1)$$

where ξ_1 is a $(2n-1)$-closed form on V. Therefore we have from (3.3)

$$\widehat{\iota_s^*(\psi)} - \widehat{P^*(\xi_1)} = 0 .$$

Set here $\varphi = \psi - \prod_{\mathrm{I}}^*(\xi_{\mathrm{I}})$ ($\prod_{\mathrm{I}}^*$ *is the projection of* $(E_{\mathrm{I}})_0$ *onto* V). *Noticing* $P = \prod_{\mathrm{I}} \circ \iota_s$, *it follows*

$$\widehat{\iota_s^* \varphi} = \widehat{\iota_s^* \psi} - \widehat{P^*(\xi_1)} = 0 .$$

However since $H^{2n-1}((E_{\mathrm{I}})_0; \boldsymbol{R})$ *is isomorphic to* $H^{2n-1}(S(E_{\mathrm{I}}); \boldsymbol{R})$ *by* ι_s^*, *we have* $\hat{\varphi} = 0$ *in* $H^{2n-1}((E_{\mathrm{I}})_0; \boldsymbol{R})$. *Thus it follows* $\varphi = \psi - \prod_{\mathrm{I}}^*(\xi_1) = d\Phi$ *for some* $\Phi \in A^{2n-2}((E_{\mathrm{I}})_0)$ *and so* (3.1) *in main lemma is proved.*

3.2. The proof of Proposition 3.3.

All notations in § 2 are used in this paragraph.

Let N be the norm on E and let N_i be the norm on E_i induced by N ($i = \mathrm{I}, \mathrm{II}$) in terms of the sequence $(\Sigma): 0 \to E_{\mathrm{I}} \to E \to E_{\mathrm{II}} \to 0$. The inclusion map $\iota: (E_{\mathrm{I}})_0 \to E_0 (\subset E)$ induces the canonical linear map of $\prod_{\mathrm{I}}^{\#}(E)$ into $\prod_0^{\#}(E)$, which we denote by ι without confusion. Note $\iota^{\#}(\prod_0^{\#}(E)) = \prod_{\mathrm{I}}^{\#}(E)$ and $\iota^{\#}(L(E)) = L(E_{\mathrm{I}})$. Then we have the following diagram

$$\begin{array}{ccccccccccl}
(\Sigma_1): 0 & \longrightarrow & L(E_{\mathrm{I}}) & \longrightarrow & \prod_{\mathrm{I}}^{\#}(E) & \longrightarrow & \iota^{\#}Q(E) & \longrightarrow & 0 & & \text{on } (E_{\mathrm{I}})_0 \\
 & & \downarrow \iota & \circlearrowright & \downarrow \iota & \circlearrowright & \downarrow & & & & \\
(\Sigma_2): 0 & \longrightarrow & L(E) & \longrightarrow & \prod_0^{\#}(E) & \longrightarrow & Q(E) & \longrightarrow & 0 & & \text{on } E_0 .
\end{array}$$

Remark. (Σ_1) is the restriction of (Σ_2) to $(E_{\mathrm{I}})_0$ and Σ_i is exact for $i = 1, 2$.

First of all, using Proposition 3.5 in [6], it follows from the exact sequence (Σ_2) that

$$\begin{aligned}
\iota^*\eta_n(E, N) = -\frac{1}{4\pi}\Big\{ & d^c \log \textstyle\prod_{\mathrm{I}}^{\#}(N_{\mathrm{I}})(\Delta \circ \iota) \cdot \iota^* C_{n-1}(Q(E)) \\
& + \lim_{t \to -\infty} d^c \int_t^0 b_n^n (C\kappa K[\textstyle\prod_{\mathrm{I}}^{\#}(E), D_t(\prod_{\mathrm{I}}^{\#} N)]; \kappa P_{\mathrm{I}})) \Big\}
\end{aligned} \tag{3.4}$$

where $\Delta: E_0 \to L(E)$ is the section defined by

$$\Delta(e) = (e, e) \qquad \text{for } e \in E_0 .$$

Let us consider the form $\iota^* C_{n-1}(Q(E))$ in the left hand side of (3.4). For this purpose take the following commutative diagram over $(E_{\mathrm{I}})_0$;

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & (E_{\mathrm{I}}) & \rightrightarrows & L(E_{\mathrm{I}}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & \prod{}^{\#}_{\mathrm{I}}(E_{\mathrm{I}}) & \longrightarrow & \prod{}^{\#}_{\mathrm{I}}(E) & \longrightarrow & \prod{}^{\#}_{\mathrm{I}}(E_{\mathrm{II}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\Sigma_3): & 0 \longrightarrow & Q(E_{\mathrm{I}}) & \longrightarrow & \iota^{\#}Q(E) & \longrightarrow & \prod{}^{\#}_{\mathrm{I}}(E_{\mathrm{II}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where each sequence is exact.

It is well-known (cf. (3.23) in [6] that

$$\iota^* C_{n-1}(Q(E)) = C_{n-1}(\iota^{\#}Q(E)) \, .$$

Therefore using the exact sequence (Σ_3) in the above diagram, we find from duality formula,

$$
\begin{aligned}
\iota^* C_{n-1}(Q(E)) &= C_{k-1}(Q(E_{\mathrm{I}})) \prod{}^{*}_{\mathrm{I}} C_{n-k}(E_{\mathrm{II}}) \\
&\quad - \frac{\pi}{4} d \lim_{t \to -\infty} d^c \int_t^0 b^{n-1}_{n-1}((\kappa K[\iota^{\#}Q(E), D^Q_t]\, ; \kappa P^Q_{\mathrm{I}}))dt
\end{aligned} \tag{3.5}
$$

where $P^Q_{\mathrm{I}} : \iota^{\#}Q(E) \to Q(E_{\mathrm{I}})$ is the orthogonal projection associated with (Σ_3) and the norm $\iota^{\#}(N^Q)$ on $\iota^{\#}Q(E)$ induced by N, and D^Q_t is $\boldsymbol{R}$-family of $D(\iota^{\#}N^Q)$ as in §2.

Furthermore, noting that $\Delta \circ \iota : (E_{\mathrm{I}})_0 \to L(E_{\mathrm{I}})$ is the canonical section for the boundary form $\eta_k(E_{\mathrm{I}})$, we have from the first vertical exact sequence in the above diagram,

$$
\begin{aligned}
\eta_k(E_{\mathrm{I}}) = -\frac{1}{4\pi}\Big\{ & d^c \log \prod{}^{\#}_{\mathrm{I}} N_{\mathrm{I}}(\Delta \circ \iota) C_{k-1}(Q(E_{\mathrm{I}})) \\
&+ \lim_{t \to -\infty} d^c \int_t^0 b^k_k((\kappa K[\prod{}^{\#}_{\mathrm{I}} E, D_t(\prod{}^{\#}_{\mathrm{I}} N_{\mathrm{I}})]\, ; \kappa P^{E_{\mathrm{I}}}_{\mathrm{I}}))dt \Big\} \, ,
\end{aligned} \tag{3.6}
$$

where $P^{E_{\mathrm{I}}}_{\mathrm{I}} : \prod{}^{\#}_{\mathrm{I}}(E_{\mathrm{I}}) \to L(E_{\mathrm{I}})$ is the orthogonal projection.

Here it follows from (3.4), (3.5) and (3.6) that

$$\iota^*\eta_n(E) - \eta_k(E_{\mathrm{I}}) \textstyle\prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}})$$

$$\begin{aligned} &= \text{const. } d^c \log \textstyle\prod_{\mathrm{I}}^{\#} N_{\mathrm{I}}(\Delta \circ \iota) \cdot d \lim_{t \to -\infty} d^c \int_t^0 b_{n-1}^{n-1}((K[\iota^{\#} Q(E), D_t^Q]; P_{\mathrm{I}}^Q))dt \\ &\quad + \text{const. } \textstyle\prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}}) \\ &\quad\quad \cdot \textstyle\prod_{\mathrm{I}}^* C_{n-k}(E_{\mathrm{II}}) \lim_{t \to -\infty} d^c \int_t^\infty b_k^k((K[\textstyle\prod_{\mathrm{I}}^{\#} E_{\mathrm{I}}, D_t(\textstyle\prod_{\mathrm{I}}^{\#} N)]; P_{\mathrm{I}}^{E_{\mathrm{I}}}))dt \\ &\quad + \text{const. } \lim_{t \to -\infty} d^c \int_t^0 b_n^n((K[\textstyle\prod_{\mathrm{I}}^{\#} E, D_t(\textstyle\prod_{\mathrm{I}}^{\#} N)]; P_{\mathrm{I}}))dt , \end{aligned} \tag{3.7}$$

(note P_{I} is the orthogonal projection of E onto E_{I} in the sequence Σ).

We shall show next that the first term of the right hand side of (3.7) has zero fibre integral, that is, if P_* denotes the fibre integral of the sphere bundle $S(E_{\mathrm{I}})$ of E_{I}, then

$$P_*(d^c \log \textstyle\prod_{\mathrm{I}}^{\#} N_{\mathrm{I}}(\Delta \circ \iota) \cdot d \lim_{t \to -\infty} d^c \int_t^0 b_{n-1}^{n-1}((K[\iota^{\#} Q(E), D_t^Q]; P_{\mathrm{I}}^Q))dt = 0 , \tag{3.7$'$}$$

where $\iota_s : (E_{\mathrm{I}})_0 \to S(E_{\mathrm{I}})$ is the inclusion.

For this aim let $1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq k$, and $1 \leq A, B \leq n - k$. In the sequence $(\Sigma) : 0 \to E_{\mathrm{I}} \to E \to E_{\mathrm{II}} \to 0$, n and k are dimensions of E and E_{I}, respectively, and N is the norm on E and N_i denotes the induced norm on E_i defined by (Σ) and $N, i = \mathrm{I}, \mathrm{II}$.

Now let $\{\tilde{e}_i\}_{i=1}^n$ be an orthonormal frame of E over an open set $\tilde{U} \subset V$ such that

$$\{\tilde{e}_\alpha\}_{\alpha=1}^k \text{ is the orthonormal frame of } E \mid \tilde{U} .$$

If $D = D(N)$ is the N-connection of E, one finds

$$D\tilde{e}_i = \Sigma_j \tilde{\theta}_{ij} \tilde{e}_j , \qquad \tilde{\theta}_{ij} \in A^1(\tilde{U}) .$$

Let $\prod_{\mathrm{I}} : (E_{\mathrm{I}})_0 \to V$ be the natural projection and put $\prod_{\mathrm{I}}^{-1}(\tilde{U}) = U$. Since $\{\tilde{e}_\alpha\}$ is the frame of $E_{\mathrm{I}} | \tilde{U}$, we find $U \cong \tilde{U} \times C^k - \{0\}$ (diffeomorphic). Let us denote the canonical coordinates of C^k by $\{z^\alpha\}_{\alpha=1}^k$.

Clearly $\{\tilde{e}_i \circ \prod_{\mathrm{I}}\}_{i=1}^n$ becomes the orthonormal frame of $\prod_{\mathrm{I}}^{\#}(E) | U$, with respect to the induced norm $\prod_{\mathrm{I}}^{\#}(N)$. Putting $\tilde{e}_i \circ \prod_i = e_i$ $(i = 1, \cdots, n)$ we get

$$D(\textstyle\prod_{\mathrm{I}}^{\#} N)e_i = \Sigma_j \theta_{ij} e_j , \qquad \theta_{ij} \in A^1(U) \tag{3.8}$$

where $\theta_{ij} = \prod_{\mathrm{I}}^* (\tilde{\theta}_{ij})$.

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On the other hand for $\Delta\circ\iota$, we find

$$\Delta\circ\iota(x, z^1, \cdots, z^k) = \Sigma_\alpha z^\alpha e_\alpha(x) \qquad \text{for } (x, z^1, \cdots, z^k)\in \tilde{U}\times \boldsymbol{C}^k - \{0\} .$$

Let here u_1 be the smooth-section of $L(E_{\mathrm{I}})$ defined by

$$\begin{aligned} u_1(x, z^1, \cdots, z^k) &= \left(\frac{\Delta\circ\iota}{\prod_{\mathrm{I}}^{\#} N_{\mathrm{I}}}\right)(x, z^1, \cdots, z^k) \\ &= \Sigma_{\alpha=1}^k \frac{z^k}{\sqrt{z^1\bar{z}^1 + \cdots + z^k\bar{z}^k}} e_\alpha(x) . \end{aligned}$$

Set $U_1 = \{(x, z^1, \cdots, z^k); z^1 \neq 0\}\subset U$. Then we can choose another sections $u_2, \cdots, u_k$ of $\prod_{\mathrm{I}}^{\#}(E_{\mathrm{I}})\,|\,U_1$ such that $\{u_\alpha\}_{\alpha=1}^k$ is the orthonormal frame. Thus we have

$$\text{(3.9)} \qquad \begin{cases} u_\alpha = \Sigma_\beta a_{\alpha\beta} e_\beta & a_{\alpha\beta}\in A^0(U_1) \\ e_\beta = \Sigma_\alpha b_{\beta\alpha} u_\alpha & b_{\alpha\beta}\in A^0(U_1) \end{cases}$$

Then $a = \|a_{\alpha\beta}\|$, and $b = \|b_{\alpha\beta}\|$, are elements of the group of $k\times k$-unitary matrices $U(k)$ and $a = b^{-1}$. Here let us put

$$\omega_{\alpha\beta} = \Sigma_\gamma da_{\alpha\gamma}\cdot b_{\gamma\beta} .$$

For simplicity we consider $U(k)$ as the subspace of $n\times n$-matrices M_n in the following way; $a\in U(k)$ corresponds to $\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right)\in M_n$.

Now let $D^{\#} = D(\prod_{\mathrm{I}}^{\#} N)$ be the $\prod_{\mathrm{I}}^{\#} N$-connection of $\prod_{\mathrm{I}}^{\#} E$. Then it follows directly from (3.8) and (3.9) that with respect to the orthonormal frame $\{u_1, \cdots, u_k, e_{k+1}, \cdots, e_n\}$,

$$\text{(3.10)} \qquad \begin{cases} D^{\#} u_\alpha = \sum_{\beta=1}^{k} \{\omega_{\alpha\beta} + (a\cdot\theta\cdot b)_{\alpha\beta}\} u_\beta + \sum_{A=1}^{n-k} (a\cdot\theta)_{\alpha,k+A} e_{k+A} \\ D^{\#} e_{k+A} = \sum_{\alpha} (\theta\cdot b)_{k+A,}\, u_\alpha + \sum_{B} \theta_{k+A,k+B} e_{k+B} \end{cases}$$

where $\theta = \|\theta_{ij}\|_{1\le i,j\le n}$.

Next let $P_{\mathrm{II}}\colon \prod_{\mathrm{I}}^{\#}(E)\to \iota^{\#}Q(E)$ be the orthogonal projection associated with the exact sequence: $0\to L(E_{\mathrm{I}})\to \prod_{\mathrm{I}}^{\#}(E)\to \iota^{\#}Q(E)\to 0$. Remember that if we denote by D^Q the canonical connection of $\iota^{\#}Q(E)$, then

$$D^Q = P_{\mathrm{II}} D^{\#} P_{\mathrm{II}}$$

and that $\{u_2, \cdots, u_k, e_{k+1}, \cdots, e_n\}$ is the orthonormal frame of $\iota^{\#}Q(E)$. Here let $2\le \tilde{\alpha}, \tilde{\beta}\le k$. Then it follows from (3.10) that

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$$(3.11)\qquad \begin{cases} D^Q u_{\tilde{\alpha}} = \sum_{\tilde{\beta}} \{\omega_{\tilde{\alpha}\tilde{\beta}} + (a\cdot\theta\cdot b)_{\tilde{\alpha}\tilde{\beta}}\} u_{\tilde{\beta}} + \sum_A (a\cdot\theta)_{\tilde{\alpha},k+A} e_{k+A} \\ D^Q e_{k+A} = \sum_{\tilde{\alpha}} (\theta\cdot b)_{k+A,\tilde{\alpha}} u_{\tilde{\alpha}} + \sum_B \theta_{k+A,k+B} e_{k+B}\,. \end{cases}$$

For calculations of $b^{n-1}_{n-1}((K[i^{\#}Q(E), D^Q_t]; P^Q_{\mathrm{I}}))$ recall that $D^Q_t = D^Q + (e^t - 1)P^Q_{\mathrm{II}} D^Q P^Q_{\mathrm{I}}$, where $P^Q_{\mathrm{I}}: \iota^{\#}Q(E) \to Q(E_{\mathrm{I}})$ and $P^Q_{\mathrm{II}}: \iota^{\#}Q(E) \to \prod^{\#}_{\mathrm{I}}(E_{\mathrm{II}})$ are the orthogonal projections induced from the exact sequence

$$0 \longrightarrow Q(E_{\mathrm{I}}) \longrightarrow \iota^{\#}Q(E) \longrightarrow \textstyle\prod^{\#}_{\mathrm{I}}(E_{\mathrm{II}}) \longrightarrow 0\,.$$

Let $K(D^Q_t)$ be the curvature matrix of D^Q_t with respect to the above frame $\{u_2, \cdots, u_k, e_{k+1}, \cdots, e_n\}$ of $\iota^{\#}Q(E)$ and put

$$K(D^Q_t) = \begin{matrix} k-1\{ \\ n-k\{ \end{matrix}\left\| \begin{array}{c|c} \overbrace{K^t_{\mathrm{II}}}^{k-1} & \overbrace{K^t_{\mathrm{I\,II}}}^{n-k} \\ \hline K^t_{\mathrm{II\,I}} & K^t_{\mathrm{II\,II}} \end{array} \right\|\,.$$

Then we get (cf. Lemma 4.8 [1])

$$(3.12)\qquad \begin{cases} K^t_{\mathrm{I\,I}} = K[Q(E_{\mathrm{I}}), P^Q_{\mathrm{I}} D^Q P^Q_{\mathrm{I}}] + e^t \,\square_{\mathrm{I}} \\ K^t_{\mathrm{II\,II}} = K[\prod^{\#}_{\mathrm{I}}(E^{\mathrm{II}}), P^Q_{\mathrm{II}} D^Q P^Q_{\mathrm{II}}] + e^t \,\square_{\mathrm{II}} \\ K^t_{\mathrm{I\,II}} = e^t P_{\mathrm{II}}(D^Q)^2 P_{\mathrm{I}},\, K^t{}_{\mathrm{II\,I}} = e^t P_{\mathrm{I}}(D^Q)^2 P_{\mathrm{II}} \end{cases}$$

where

$$\square_{\mathrm{I}} = P^Q_{\mathrm{I}} D^Q P^Q_{\mathrm{II}} D^Q P^Q_{\mathrm{I}}$$

and

$$\square_{\mathrm{II}} = P^Q_{\mathrm{II}} D^Q P^Q_{\mathrm{I}} D^Q P^Q_{\mathrm{II}}\,.$$

Remark. It is clear from the choice of the frame $\{u_2, \cdots, u_k, e_{k+1}, \cdots, e_n\}$ that

$$\begin{cases} P^Q_{\mathrm{I}} u_{\tilde{\alpha}} = u_{\tilde{\alpha}}\,, \qquad P^Q_{\mathrm{I}} e_{k+A} = 0 \\ P^Q_{\mathrm{II}} u_{\tilde{\alpha}} = 0 \quad \text{and} \quad P^Q_{\mathrm{II}} e_{k+A} = e_{k+A}\,. \end{cases}$$

Using this remark and (3.11), we shall compute each term in (3.12). At first we introduce the following notations. In general let P be a differential fibre bundle over a differential manifold M, and $\prod$ be the projection. Let ω be any differential form on P. We say that ω is *at most of k-fibre degree*, denoted by $F(\omega) \leq k$, when the following condition is satisfied; Let y be any point of P and F_y be the fibre space passing through y. Then for any $k+1$ vectors $X_1, \cdots, X_{k+1} \in T_y(F_y)$ the

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inner derivative $X_1 \wedge \cdots \wedge X_{k+1} \lrcorner\, \omega$ of ω with respect to $X_1, \cdots, X_{k+1}$ is zero, i.e., $X_1 \wedge \cdots \wedge X_{k+1} \lrcorner\, \omega = 0$.

Under this notation each term in the right hand side of (3.12) is calculated as below.

a) $K[Q(E_{\mathrm{I}}), P_{\mathrm{I}}^Q D^Q P_{\mathrm{I}}^Q] = \|\omega_{\bar{\alpha}1} \wedge \omega_{1\bar{\beta}}\|_{2\le\bar{\alpha},\bar{\beta}\le k} + \Phi_1$, where $F(\Phi_1) \le 1$.

Proof. First of all find from (3.11)

$$P_{\mathrm{I}}^Q D^Q u_{\bar{\alpha}} = \sum_{\bar{\beta}} \{\omega_{\bar{\alpha}\bar{\beta}} + (a\cdot\theta\cdot b)_{\bar{\alpha}\bar{\beta}}\} u_{\bar{\beta}} \,.$$

Thus

$$K[Q(E_{\mathrm{I}}), P_{\mathrm{I}}^Q D^Q P_{\mathrm{I}}^Q] = \left\| d\omega_{\bar{\alpha}\bar{\beta}} - \sum_{\bar{\gamma}} \omega_{\bar{\alpha}\bar{\gamma}} \wedge \omega_{\bar{\gamma}\bar{\beta}} \right\| + \Phi_1$$

where

$$\Phi_1 = \left\| d(a\cdot\theta\cdot b)_{\bar{\alpha}\bar{\beta}} - \sum_{\bar{\gamma}} (a\cdot\theta\cdot b)_{\bar{\alpha}\bar{\gamma}} \wedge (\omega_{\bar{\gamma}\bar{\beta}} + (a\cdot\theta\cdot b)_{\bar{\gamma}\bar{\beta}}) \right.$$
$$\left. - \sum_{\bar{\gamma}} (\omega_{\bar{\alpha}\bar{\gamma}} + (a\cdot\theta\cdot b)_{\bar{\alpha}\bar{\gamma}}) \wedge (a\cdot\theta\cdot b)_{\bar{\beta}\bar{\gamma}} \right\| .$$

However as $d\omega_{\alpha\beta} - \sum_{\gamma=1}^{k} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = 0$, we get

$$K[Q(E_{\mathrm{I}}), P_{\mathrm{I}}^Q D^Q P_{\mathrm{I}}^Q] = \|\omega_{\bar{\alpha}1} \wedge \omega_{1\bar{\beta}}\| + \Phi_1 \,.$$

The fact that Φ_1 is at most fibre degree one is shown from $\theta = \prod_{\mathrm{I}}^*(\tilde{\theta})$ (note $\prod_{\mathrm{I}} : (E_{\mathrm{I}})_0 \to V$). Q.E.*D*.

b) $K[\prod_{\mathrm{I}}^{\#} E_{\mathrm{II}}, P_{\mathrm{II}}^Q D^Q P_{\mathrm{II}}^Q] = \|d\theta_{k+A,k+B} - \sum_C \theta_{k+A,k+C} \wedge \theta_{k+C,k+B}\|_{A,B}$.

This is trivial.

c) $F(\Box_{\mathrm{I}}) = 0$.

Proof. As $P_{\mathrm{II}}^Q D^Q u_{\bar{\alpha}} = \sum_A (a\cdot\theta)_{\bar{\alpha},k+A} e_{k+A}$, we have

$$P_{\mathrm{I}}^Q D^Q P_{\mathrm{II}} D^Q u_{\bar{\alpha}} = \sum_A (a\cdot\theta)_{\bar{\alpha},k+A} P_{\mathrm{I}}^Q D^Q e_{k+A}$$
$$= \sum_{A,\bar{\beta}} (a\cdot\theta)_{\bar{\alpha},k+A} (\theta\cdot b)_{k+A,\bar{\beta}} u_{\bar{\beta}} \,.$$

This fact shows that $F(\Box_{\mathrm{I}}) = 0$.

d) $\Box_{\mathrm{II}} = -\|\sum_\beta \theta_{k+A,\beta}\theta_{\beta,k+\beta}\| + \|\sum_{\beta,\gamma} \theta_{k+A,\beta} a_{1\gamma} \bar{a}_{1\beta} \theta_{\gamma,k+B}\|$.

Proof. Direct calculations show that

$$\square_{\rm II} = \left\| -\sum_{\bar\alpha} (\theta\cdot b)_{k+A,\bar\alpha} \wedge (a\cdot\theta)_{\bar\alpha,k+B} \right\|_{A,B}$$
$$= -\left\| \sum_{\beta\bar\alpha\gamma} \theta_{k+A,\beta} b_{\beta\bar\alpha} a_{\bar\alpha\gamma} \theta_{\gamma,k+B} \right\|$$
$$= -\left\| \sum_{\beta\gamma} \theta_{k+A}(\delta_{\beta\gamma} - b_{\beta 1} a_{1\gamma}) \theta_{\gamma,k+B} \right\|$$
$$= -\left\| \sum_{\beta} \theta_{k+A,\beta} \wedge \theta_{\beta,k+B} \right\| + \left\| \sum_{\beta\gamma} \theta_{k+A,\gamma} a_{1\beta} \bar a_{1\beta} \theta_{\gamma,k+B} \right\| .$$

here we used ${}^t\bar a = a^{-1} = b$ in the third equality. Q.E.D.

e) $P^Q_{\rm II}(D^Q)^2 P^Q_{\rm I} = \|\sum_{\gamma} \omega_{\bar\alpha 1} a_{1\gamma} \theta_{\gamma,k+A}\|_{\bar\alpha,A} + \Phi_0$,

where $F(\Phi_0) = 0$.

f) $P^Q_{\rm I}(D^Q)^2 P_{\rm II} = -{}^t\|\bar\omega_{\bar\alpha 1} \bar a_{1\gamma} \bar\theta_{\gamma,k+A}\| + \Phi'_0$

where ${}^t\|\cdot\|$ denotes the transpose of the matrix $\|\cdot\|$ and $F(\Phi'_0) = 0$.

Since we can obtain e) and f) by the same computation before, we omit these calculations.

Hence it follows from a) ~ f) that, with respect to the frame $\{u_2, \cdots, u_k, e_{k+1}, \cdots, e_n\}$ of $i^*Q(E)$,

(3.13) $K[i^*Q(E), D^Q_t]$

$$= \begin{matrix} k-1\{ \\ \\ n-k\{ \end{matrix} \left\| \begin{array}{c|c} \overbrace{\|\omega_{\bar\alpha 1} \wedge \omega_{1\bar\beta}\|_{\bar\alpha,\bar\beta} + \Phi_1}^{k-1} & \overbrace{e\ {}^t\|\omega_{\bar\alpha 1} \wedge a_{1\gamma}\theta_{\gamma,k+A}\|_{\bar\alpha,A} + \Phi'_0}^{n-k} \\ \hline -{}^t\|\bar\omega_{\bar\alpha 1} \wedge \bar a_{1\gamma}\bar\theta_{\gamma,k+A}\| + \Phi'_0 & \begin{array}{l} \left\| d\theta_{k+A,k+B} - \sum_C \theta_{k+A,k+C} \wedge \theta_{k+C,k+B} \right\|_{A,B} \\ \quad - e^t \left\| \sum_\beta \theta_{k+A,\beta} \wedge \theta_{\beta,k+B} \right\| \\ \quad + e^t \left\| \sum_{\beta-1} \theta_{k+A,\beta} \bar a_{1\beta} a_{1\gamma} \theta_{\gamma,k+B} \right\| \end{array} \end{array} \right\| ,$$

where $F(\Phi_1) \leq 1$ and $F(\Phi_0) = F(\Phi'_0) = 0$.

On the other hand for $P^Q_{\rm I}$ we get

$$P^Q_{\rm I} = \left\| \begin{array}{c|c} 1_{k+1} & 0 \\ \hline 0 & 0 \end{array} \right\| , \tag{3.14}$$

with respect to $\{u_2, \cdots, u_k, e_{k+1}, \cdots, e_n\}$.
finally we need the following elementary

LEMMA 3.4. *Let $A \in M_n$ and let $\Delta_{ij}(A)$ be the (i, j)-cofactor of A. Then one has*

$$b^n_n\left(\left(A ; \left(\begin{array}{c|c} 1_k & 0 \\ \hline 0 & 0 \end{array}\right)\right)\right) = \sum_{j=1}^{k} \Delta_{jj}(A) .$$

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Proof. From the definition of $b_n^n((A;B))$ in §2 this lemma is trivial. Q.E.D.

Now applying Lemma 3.4 to (3.13) and (3.14), we have

$$b_{n-1}^{n-1}((K[\iota^\# Q(E), D_t^Q]; P_1^Q)) = \sum_{i=1}^{k-1} \Delta_{ii}(K[\iota^\# Q(E), D_t^Q]) .$$

Thus from (3.13) it follows that any term in $\Delta_{ii}(K[\iota^\# Q(E); D_t^Q])$ $(i = 1, \cdots, k-1)$ is of type

$$(3.15)\quad \Phi\Big(\sum_{\tilde\alpha=2}^{k} \omega_{21} \wedge \cdots \overset{\overset{\tilde\alpha}{\vee}}{\wedge} \cdots \wedge \omega_{k1} \wedge \bar\omega_{21} \wedge \cdots \overset{\overset{\tilde\alpha}{\vee}}{\wedge} \cdots \wedge \bar\omega_{k1}\Big) \cdot (a_{1\tilde\gamma_1} \cdots a_{1\tilde\gamma_t} \bar a_{1\tilde\beta_1} \cdots \bar a_{1\tilde\beta_t}) + \Phi_{2k-3}$$

where $\Phi \in A^0(\tilde U), F(\Phi_{2k-3}) \leq 2k-3$ and $2 \leq \tilde\gamma_j, \tilde\beta_j \leq k$, $(j = 1, \cdots, t)$.

Let us here represent $a_{1\alpha}$ using the coordinates $\{z^1, \cdots, z^k\}$ of C^k. From (3.9), $u_1 = \sum_{\alpha=1}^k a_{1\alpha} e_\alpha$. But as $u_1 = \sum \dfrac{z^\alpha}{\alpha\sqrt{z_1\bar z_1 + \cdots + z_k \bar z_k}} e_\alpha$ it follows that

$$a_{1\alpha} = \frac{z^\alpha}{|z|} , \qquad (|z| = \sqrt{z^1\bar z^1 + \cdots + z^k \bar z^k}) .$$

Put $\Omega = dd^c \log |z|^2$ on $C^k - \{0\}$. Then we have the following

LEMMA 3.5. *Let $\omega_{\alpha\beta}$ be as before $(1 \leq \alpha, \beta \leq k)$. Then it follows that*

$$\sum_{\tilde\alpha=2}^{k} \omega_{21} \wedge \cdots \overset{\overset{\tilde\alpha}{\vee}}{\wedge} \cdots \wedge \omega_{k1} \wedge \bar\omega_{21} \wedge \cdots \overset{\overset{\tilde\alpha}{\vee}}{\wedge} \cdots \wedge \bar\omega_{k1} = \text{const.}\ \Omega^{k-2} ,$$

where $\Omega^{k-2} = \Omega \overbrace{\wedge \cdots \wedge}^{k-2 \text{ times}} \Omega$.

Proof. Noting that $\omega_{\alpha\beta} = \sum_\gamma da_{\alpha\gamma} b_{\gamma\beta}$, we get

$$\sum_{\tilde\alpha} \omega_{\tilde\alpha 1} \wedge \bar\omega_{\tilde\alpha 1} = \sum_\beta da_{1\beta} \wedge d\bar a_{1\beta} .$$

But from the representations of $a_{1\beta}$ with respect to $\{z^1, \cdots, z^k\}$, we see that

$$\sum_{\bar{\alpha}} \omega_{\bar{\alpha}1} \wedge \bar{\omega}_{\bar{\alpha}1} = \sum_{\beta} d\Big(\frac{z^\beta}{|z|}\Big) \wedge d\Big(\frac{\bar{z}^\beta}{|z|}\Big)$$
$$= -\sum_{\beta} d\Big(d\Big(\frac{z^\beta}{|z|}\Big)\cdot\frac{\bar{z}^\beta}{z}\Big)$$
$$= \text{const. } \Omega \ .$$

From this fact lemma is proved. Q.E.D.

Moreover we have the next lemma concerned with the above one.

LEMMA 3.6. *Let $S(C^k)$ be the unit sphere of C^k and $j_S: S(C^k) \to C^k - \{0\}$ be the inclusion. Furthermore let $\alpha = (\alpha_1, \cdots, \alpha_t)$ and $\beta = (\beta_1, \cdots, \beta_t)$ be t-multiple indices for any positive integer t such that $1 \leq \alpha_i, \beta_i \leq k$ $(i = 1, \cdots, t)$. Then we find*

$$j_S^*(\Omega^{k-1}) \wedge d^c\Big(\frac{z^\alpha \bar{z}^\beta}{|z|^{2t}}\Big) = 0 \ , \tag{3.16}$$

where $z^\alpha = z^{\alpha_1}\cdots z^{\alpha_t}$ and $\bar{z}^\beta = \bar{z}^{\beta_1}\cdots\bar{z}^{\beta_t}$.

Proof. It suffices to prove (3.16) in case of $t = 1$, that is $1 \leq \alpha$, $\beta \leq k$. Clearly we get

$$j_S^*(\Omega) = \sum_{\alpha,\beta=1}^{k} (\delta_{\alpha\beta} - \bar{z}^\alpha z^\beta) dz^\alpha \wedge d\bar{z}^\beta \ ,$$

where $\delta_{\alpha\beta}$ is Kronecker index.

Here let us put

$$B = \begin{Vmatrix} 1 - \bar{z}^1 z^1, & -\bar{z}^1 z^2, & \cdots, & -\bar{z}^1 z^k \\ -\bar{z}^2 z^1, & 1 - \bar{z}^2 z^2 & & \\ & & \ddots & \\ -\bar{z}^k z^1, & \cdots, & & 1 - \bar{z}^k z^k \end{Vmatrix} .$$

Then

$$j_S^*(\Omega^{k-1}) = \text{const.} \sum_{\alpha,\beta} \Delta_{\alpha\beta}(B) dz^1 \wedge \cdots \overset{\overset{\alpha}{\vee}}{\wedge} \cdots \wedge dz^k \wedge d\bar{z}^1 \wedge \cdots \overset{\overset{\beta}{\vee}}{\wedge} \cdots \wedge d\bar{z}^k \ ,$$

where $\Delta_{\alpha\beta}(B)$ denotes the (α, β)-cofactor of B. Now we find

$$d'\Big(\frac{z^\alpha \bar{z}^\beta}{|z|^2}\Big) = \frac{dz^\alpha \bar{z}^\beta}{|z|^2} - \frac{z^\alpha \bar{z}^\beta}{|z|^4}\Big(\sum_{\gamma} \bar{z}^\gamma dz^\gamma\Big) .$$

And so it follows that

$$j_S^*(\Omega^{k-1} \wedge d'\Big(\frac{z^\alpha \bar{z}^\beta}{|z|^2}\Big)$$

$$= \text{const.}\Big\{\bar{z}^\beta \sum_\gamma \Big((-1)^{\alpha-1}\Delta_{\alpha\gamma}(B) - \sum_\delta (-1)^{\delta-1} z^\alpha \bar{z}^\delta \Delta_{\delta_\gamma}(B)\Big) \overset{\gamma}{\vee} \times dz^1 \wedge \cdots \wedge dz^k \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^k\Big\} .$$

Fixing γ in the right hand side of this equation, we obtain

$$(-1)^{\alpha-1}\Big(\Delta_{\alpha\gamma}(B) - \sum_\delta (-1)^{\delta+\alpha} z^\alpha \bar{z}^\delta \Delta_{\delta_\gamma}(B)\Big)$$

$$= (-1)^{\alpha-1}\Big\{\sum_\delta (-1)^{\alpha+\delta}(\delta_{\alpha,\delta} - z^\alpha \bar{z}^\delta)\Delta_{\delta_\gamma}(B)\Big\} \tag{3.17}$$

$$= \begin{cases} \det(B) = 1 - |z|^2, & \text{if } \alpha = \gamma \\ 0, & \text{otherwise.} \end{cases}$$

But as $|z| = 1$ on $S(\boldsymbol{C}^k)$, the right hand side of (3.17) equals to zero for any γ, so that

$$j_S^*\Big(d'\Big(\frac{z^\alpha \bar{z}^\beta}{|z|^2}\Big) \wedge \Omega^{k-1}\Big) = 0 .$$

Similarly we can prove

$$j_S^*\Big(d''\Big(\frac{z^\alpha \bar{z}^\beta}{|z|^2}\Big) \wedge \Omega^{k-1}\Big) = 0 .$$

Thus from $d^c = i(d'' - d')$, (3.16) follows. Q.E.D.

Now we are in a position to prove (3.7)′ For simplicity, set

$$\mathrm{I} = d^c \log\big(\textstyle\prod_{\mathrm{I}}^{\sharp} N_{\mathrm{I}}\big)(\Delta \circ \iota) \wedge \lim_{s\to-\infty} d^c \int_s^0 b_{n-1}^{n-1}((K[\iota_Q^\sharp(E), D_t^Q]\,; P_{\mathrm{I}}^Q))ds .$$

By the definition of the canonical section Δ (see the above of (3.9)),

$$\log\big(\textstyle\prod_{\mathrm{I}}^{\sharp} N_{\mathrm{I}}\big)\Delta \circ \iota(x, z^1, \cdots, z^k) = \log|z|^2 ,$$

and so we find from (3.15) and Lemma 3.5 that

$$\mathrm{I} = \sum_{\substack{t \\ \alpha=(\alpha_1,\cdots,\alpha_t) \\ \beta=(\beta_1,\cdots,\beta_t)}} \psi_{\alpha\beta,t} d^c \log|z|^2 \wedge dd^c\Big(\Omega^{k-1}\cdot\frac{z^\alpha \bar{z}^\beta}{|z|^{2t}}\Big) + \Phi_{2k-2} , \tag{3.18}$$

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where the $\Phi_{\alpha\beta,t}$ are functions on the base space $\tilde{U}$ $(\subset V)$, and $F(\Phi_{2k-2}) \leq 2k-2$. Further using $d'\Omega = d''\Omega = 0$, it follows from (3.18) that

$$\mathrm{I} = \sum_{\substack{t \\ \alpha,\beta}} \psi_{\alpha\beta,t}\left\{-d\left(d^c \log|z|^2 \wedge \Omega^{k-2} \wedge d^c\left(\frac{z^\alpha \bar{z}^\beta}{|z|^{2t}}\right)\right)\right.$$
$$\left. + \Omega^{k-1} \wedge d^c\left(\frac{z^\alpha \bar{z}^\beta}{|z|^{2t}}\right)\right\} + \Phi_{2k-2}\,.$$

As before let ι_S be the inclusion of $S(E_{\mathrm{I}})$ into $(E_{\mathrm{I}})_0$. Then by (3.16),

$$\iota_S^*\mathrm{I} = \sum_{\substack{t \\ \alpha,\beta}} \iota_S^*(-\psi_{\alpha\beta,t}) d\iota_S^*\left(d^c \log|z|^2 \wedge \Omega^{k-2} \wedge d^c\left(\frac{z^\alpha \bar{z}^\beta}{|z|^{2t}}\right)\right) + \iota_S^*\Phi_{2k-2}\,.$$

And noting that $\psi_{\alpha\beta,t}$ is constant on each fibre of $\Pi_{\mathrm{I}}\colon (E_{\mathrm{I}})_0 \to V$, we see that the fibre integral $P_*(\iota_S^*\mathrm{I})$ of $\iota_S^*\mathrm{I}$ is zero;

$$P_*(\iota_S^*\mathrm{I}) = 0\,.$$

Thus the fibre integral of the first term in the right hand side of (3.7) is equal to zero and we can prove similarly that other terms are so.

The above facts show Proposition 3.3, and so our main lemma is proved as stated after this Proposition.

§ 4. Proof of Theorem 1.9

As in the statement of Theorem 1.9 let V and W be compact connected complex manifolds of $\dim_C V = n$ and $\dim_C W = k$. Moreover let f be an (IF)-holomorphic map with $\operatorname{rank} f \geqq k-1$, and let us denote by Σ the singular set of f (cf. Definition 1.1).

Then in terms of Proposition 1.8 we take an f-form ω on W (Definition 1.7) such that

i) ω is a $(1,0)$-type differential form on W

ii) zeros of ω is isolated, say, $\{q_1, \cdots, q_m\}$ and $f(\Sigma) \cap \text{zeros}\,(\omega) = \phi$.

iii) $\Sigma \cap (\text{zeros of } f^*\omega) = \{P_1, \cdots, P_\iota\}$, where the P_j are generic points of Σ.

Let N be a hermitian norm on the holomorphic cotangent bundle T^*V of V and let $C_n(T^*V)$ be the n-th Chern form defined by N as in § 2. First it is well-known that

$$\chi(V) = (-1)^n \int_V C_n(T^*V)\,, \tag{4.1}$$

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where $\chi(V)$ is the Euler number of V.

On the other hand, as rank $f \geqq k-1$, it is easy to see, from the above conditions i), ii) and iii), that $f^*(\omega)$ becomes the smooth-section of $(T^*V)_0 = \{vd^*V; v \neq 0\}$, over $V - \{P_1, \cdots, P_\ell\} \cup \bigcup_{j=1}^m f^{-k}(q_j)$. Moreover since q_j is the regular values of f by ii), $(j = 1, \cdots, m)$, we can choose the ε-balls $U_\varepsilon(p_j)$ with center q_j in W such that for each j

$$f : f^{-1}(U_\varepsilon(q_j)) \to U_\varepsilon(q_j) \qquad \text{is a fibre bundle,}$$

and

$$U_\varepsilon(q_i) \cap U_\varepsilon(q_j) = \phi \qquad \text{for } i \neq j .$$

Take also the ε-ball $V_\varepsilon(p_j)$ for each p_j such that

$$V_\varepsilon(p_i) \cap V_\varepsilon(p_j) = \phi \qquad \text{for } i \neq j$$

and

$$V_\varepsilon(p_j) \cap F^{-1}(U_\varepsilon(q_i)) = \phi \qquad \text{for } 1 \leq j \leq \ell,\ 1 \leq i \leq m .$$

Here put

$$V_\varepsilon = V - \bigcup_{j=1}^{\ell} V_\varepsilon(p_j) - \bigcup_{j=1}^{m} f^{-1}(U_\varepsilon(q_j)) .$$

One remarks that f is onto, because f is flat and both of V and W are connected compact manifold. Then if we write $\eta_n(T^*V)$ the boundary form of T^*V with the norm N (Definition 2.4), we have from (2.8) and Stokes' formula,

$$\begin{aligned} \int_V C_n(T^*V) &= \lim_{\varepsilon\to 0} \int_{V_\varepsilon} C_n(T^*V) \\ &= -\sum_{j=1}^{m} \lim_{\varepsilon\to 0} \int_{\partial f^{-1}(U_\varepsilon(q_j))} (f^*\omega)\eta_n(T^*V) \\ &\quad + \sum_{j=1}^{\ell} \lim_{\varepsilon\to 0} \int_{\partial U_\varepsilon(p_j)} (f^*\omega)^*\eta_n(T^*V) . \end{aligned} \tag{4.2}$$

We shall actually compute in two parts (A) and (B) each term in the last right hand side of (4.2).

(A) Calculation of $\lim_{\varepsilon\to 0} \int_{\partial f^{-1}(U_\varepsilon(q_j))} (f^*\omega)^*\eta_n(T^*V)$.

For simplicity set $q = q_j$, and take a sufficiently small ε_0 such that $f : f^{-1}(U_{\varepsilon_0}(q)) \to U_{\varepsilon_0}(q)$ is a fibre bundle. Let ε_0 be fixed. Here we put

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$$V_0 = f^{-1}(U_{\varepsilon_0}(q)) \quad \text{and} \quad W_0 = U_{\varepsilon_0}(q) ,$$

and so $f|_{V_0}: V_0 \to W_0$ is the fibre bundle. Recall that $T(f): f^\sharp(T^*W) \to T^*V$ is the bundle map defined by

$$T(f)(p, v) = f_p^* v \qquad \text{for } (p, v) \in f^\sharp(T^*W) .$$

Then we clearly obtain the following exact sequence of holomorphic vector bundles over V_0;

$$0 \longrightarrow f^\sharp(T^*W_0) \xrightarrow{T(f)} T^*V_0 \longrightarrow T^*V_0/T(f)(f^\sharp(T^*W_0)) \longrightarrow 0 .$$

Put

$$Q(V_0, W_0) = T^*V_0/T(f)(f^\sharp(T^*W_0)) .$$

Since T^*V_0 has naturally the norm N, we can apply the above exact sequence to Main lemma in §3, so that on $(f^\sharp(T^*W_0))_0$,

$$T(f)^*\eta_n(T^*V_0) = \eta_k(f^\sharp(T^*W_0))\pi_{\mathrm{I}}^* C_{n-k}(Q(V_0, W_0)) + \pi_{\mathrm{I}}^*(\xi) + d\eta , \tag{4.3}$$

where π_{I} is the natural projection of $(f^\sharp(T^*W_0))_0$ onto V_0. Let $f^\sharp(\omega)$ be the cross-section of $f^\sharp(T^*W)$ defined by

$$f^\sharp(\omega)(p) = (p, \omega_{f(p)}) \qquad \text{for } p \in V .$$

Hence noting that $T(f)f^\sharp(\omega) = f^*\omega$, we find from (4.3),

$$\begin{aligned}(f^*\omega)^*\eta_n(T^*V_0) = (f^\sharp\omega)^*\eta_k(f^\sharp(T^*W_0)\pi_{\mathrm{I}}^* C_{n-k}(Q(V_0, W_0)) \\ + \pi_{\mathrm{I}}^*(\xi) + d(f^\sharp(\omega))^*\eta .\end{aligned} \tag{4.4}$$

Now put

$$\mathrm{I} = \lim_{\varepsilon \to 0} \int_{\partial f^{-1}(U_\varepsilon(q))} (f^*\omega)^*\eta_n(T^*V) .$$

First we remark that for any $\varepsilon < \varepsilon_0$.

$$\int_{\partial f^{-1}(U_\varepsilon(q))} (f^*\omega)^*\eta_n(T^*V) = \int_{\partial f^{-1}(U_\varepsilon(q))} (f^*\omega)^*\eta_n(T^*V_0) , \tag{4.5}$$

because of $(f^*\omega)(f^{-1}(U_\varepsilon(q))) \subset T^*V_0$. Then it follows from (4.4) and (4.5) that

$$\mathrm{I} = \lim_{\varepsilon \to 0} \int_{\partial f^{-1}(U_\varepsilon(q))} (f^\sharp(\omega))^*\{\eta_k(f^\sharp(T^*W_0))\pi_{\mathrm{I}}^* C_{n-k}(Q(V_0, W_0)) + \pi_{\mathrm{I}}^*\xi\} . \tag{4.6}$$

First of all we shall show

$$\lim_{\varepsilon \to 0} \int_{\partial f^{-1}(U_\varepsilon(q))} (f^\#(\omega))^* \pi_{\mathrm{I}}^* \xi = 0 . \tag{4.7}$$

In fact as $f^{-1}(U_\varepsilon(q))$ is diffeomorphic to $U_\varepsilon(q) \times f^{-1}(q)$ and $f^{-1}(q)$ is compact, we see

$$\partial f^{-1}(U_\varepsilon(q)) \cong \partial U_\varepsilon(q) \times f^{-1}(q) \qquad \text{(diffeo.)} .$$

Here noting that $f^{-1}(q)$ is the compact manifold of real dimension $2(n-k)$ $(k \geq 1)$ and that ξ is the $(2n-1)$-form on V_0, we can prove (4.7) by virtue of $(f^\#(\omega))^* \cdot \pi_{\mathrm{I}}^* = \text{identity}$.

Next for the purpose of computations of

$$\lim_{\varepsilon \to 0} \int_{\partial f^{-1}(U_\varepsilon(q))} (f^\# \omega)^* \eta_k(f^\#(T^* W_0)) \wedge C_{n-k}(Q(V_0, W_0)) ,$$

let N_W be a hermitian norm on T^*W and let $f^\#(N_W)$ be the induced norm on $f^\#(T^*W)$ by f and N_W. We denote by $\eta_k(f^\#(T^*W), f^\#(N_W))$ the boundary form of $f^\#(T^*W)$ associated with the norm $f^\#(N_W)$. Then the naturality of boundary forms [6] shows that

$$\eta_k(f^\#(T^*W), f^\#(N_W)) = \tilde{f}^*(\eta_k(T^*W, N_W)) , \tag{4.8}$$

where $\tilde{f} : f^\#(T^*W) \to T^*W$ is the map defined by

$$\tilde{f}(p, v) = v \qquad \text{for } (p, v) \in f^\#(T^*W) .$$

Furthermore let $N^\#$ be any hermitian norm on $f^\#(T^*W)$. Then we have the following homotopy lemma of boundary forms.

LEMMA 4.1. *With the above notations,*

$$\begin{aligned} &\eta_k(f^\#(T^*W), N^\#) - \eta_k(f^\#(T^*W), f^\#(N_W)) \\ &\quad = \pi_{\mathrm{I}}^*(\xi_k) + d\eta_k , \qquad \textit{on } (f^\#(T^*W))_0 . \end{aligned} \tag{4.9}$$

Proof. It is easy to see from (2.8) and the homotopy lemma of Chern forms [1] that for some $\xi_k \in A^{2k-1}(V)$,

$$d\eta_k(f^\#(T^*W), N^\#) = d\eta_k(f^\#(T^*W), f^\#(N_W)) = \pi_{\mathrm{I}}^*(\xi_k) . \tag{4.10}$$

But since the fibre integral of any boundary form of a holomorphic vector bundle is equal to -1, we can prove (4.9), combining (4.10) with Lemma 3.2. Q.E.D.

Applying here $\eta_k(f^\#(T^*W))$ to (4.9), we obtain from (4.9) and $\tilde{f}\circ f^\#(\omega) = \omega\circ f$,

$$(4.9)' \quad \begin{aligned}&(f^\#(\omega))^*\eta_k(f^\#(T^*W)) \\ &\qquad = f^*\omega^*\eta_k(T^*W, N_W) + (f^\#(\omega))^*\pi_1^*\xi_k + df^\#(\omega)^*\eta_k .\end{aligned}$$

Let us put

$$\mathrm{I}_1 = \lim_{\varepsilon\to 0}\int_{\partial f^{-1}(U_\varepsilon(q))} (f^\#\omega)^*\eta_k(f^\#(T^*W_0)) \wedge C_{n-k}(Q(V_0, W_0)) .$$

Then using $dC_{n-k}(Q(V_0, W_0)) = 0$, we have from (4.9)'

$$\mathrm{I}_1 = \lim_{\varepsilon\to 0}\int_{\partial f^{-1}(U_\varepsilon(q))} f^*\circ\omega^*\eta_k(T^*W_0, N_W) \wedge C_{n-k}(V_0, W_0)) ,$$

so that from $f^{-1}(U_\varepsilon(q)) \cong U_\varepsilon(q)\times f^{-1}(q)$,

$$(4.11) \quad \mathrm{I}_1 = \lim_{\varepsilon\to 0}\int_{\partial U_\varepsilon(q)\ni q'}\left(\int_{f^{-1}(q')} C_{n-k}(Q(V_0, W_0))\right)\cdot(\omega^*\eta_k(T^*W_0, N_W))_{q'} .$$

Here we have the following

LEMMA 4.2. *For any $q'\in\partial U_\varepsilon(q)$, we get*

$$(4.12) \quad \int_{f^{-1}(q')} C_{n-k}(Q(V_0, W_0)) = (-1)^{n-k}\chi(F) ,$$

where F is a general fibre of $f: V\to W$.

Proof. Let $i_{q'}$ be the inclusion of $f^{-1}(q')$ into V. Then as f is fibre map on $f^{-1}(U_{\varepsilon_0}(q))$ $(\varepsilon<\varepsilon_0)$ and $Q(V_0, W_0) = T^*V_0/T(f)(f^\#(T^*W_0))$, it is clear that $\iota_{q'}^\# Q(V_0, W_0)$ is isomorphic to $T^*(f^{-1}(q'))$. Therefore we find

$$\hat{C}_{n-k}(\iota_{q'}^\# Q(V_0, W_0) = \hat{C}_{n-k}(T^*(f^{-1}(q')) ,$$

where ^ represents the cohomology class. And also by [2] it follows that

$$\hat{C}_{n-k}(\iota_{q'}^\# Q(V_0, W_0)) = \iota_{q'}^*\hat{C}_{n-k}(Q(V_0, W_0)) ,$$

so that we have

$$(4.13) \quad \int_{f^{-1}(q')} C_{n-k}(Q(V_0, W_0)) = (-1)^{n-k}\chi(f^{-1}(q')) .$$

On the other hand recall that Σ is the singular set of f. Then $f|_{(V-\Sigma)}: (V-\Sigma)\to(W-f(\Sigma))$ is the fibre bundle and $f(\Sigma)$ is the analytic set of W with $\dim_C f(\Sigma)\leqq k-1$. Thus $(W-f(\Sigma))$ being connected, it follows that $f^{-1}(q_1)$ is diffeomorphic to $f^{-1}(q_2)$ for any $q_i \in W - f(\Sigma)$, $i = 1, 2$. Hence (4.12) is trivial from (4.13). Q.E.D.

Moreover we know in virtue of [1] that

$$\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(q)} \omega^* \eta_k(T^* W_0, N_W) = -\mathrm{I}_q(\omega) \, , \tag{4.14}$$

where $\mathrm{I}_q(\omega)$ is the degree of zeros of ω at q. From (4.11), (4.12) and (4.14) it follows

$$\mathrm{I}_1 = (-1)^{n-k} \chi(F) \mathrm{I}_q(\omega) \, .$$

Thus

$$\begin{aligned} \sum_{j=1}^{m} \lim_{\varepsilon \to 0} \int_{\partial f^{-1}(U_\varepsilon(q_j))} & (f^*\omega)^* \eta_n(T^* V) \\ & = (-1)^{n-k+1} \chi(F) \Big\{ \sum_{j=1}^{m} \mathrm{I}_{q_j}(\omega) \Big\} \\ & = (-1)^{n+1} \chi(F) \chi(W) \, , \end{aligned} \tag{4.15}$$

here we used $\chi(W) = (-1)^k \int_W C_k(T^* W) = (-1)^k \Big\{ \sum_{j=1}^{m} I_{q_j}(\omega) \Big\}$.

(B) Calculation of $\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p_j)} (f^*\omega)^* \eta_n(T^* V)$.

Let us put $p = p_j$ and $f(p) = q$. Remember that p is a generic point in the singular set Σ of f. As $\mathrm{rank}_p f = k - 1$, we are able to choose a holomorphic chart $(\{z^j\}_{j=1}^k, U_q)$ at q with $z^j(q) = 0$, $(j = 1, \cdots, k)$ such that $(dz^i \circ f)_p, \cdots$, and $(dz^{k-1} \circ f)_p$ are linearly independent, and so take a holomorphic chart $(\{w^j\}_{j=1}^n, U_p)$ at p such that $w^j = z^j \circ f$ $(1 \le j \le k-1)$ and $w^j(p) = 0$ $(1 \le j \le n)$. Suppose that $f(U_p) \subset U_q$ and that ω never vanishes on U_q because of $\omega_q \ne 0$. Now let

$$\omega = \sum_{j=1}^{k} a_j dz^j \qquad \text{on } U_q \, . \tag{4.16}$$

Then from the choice of holomorphic coordinates $\{z^i\}_{i=1}^k$ and $\{w^j\}_{j=1}^n$ it is clear that on U_p

$$f^*\omega = \sum_{i=1}^{k-1} \Big\{ a_i \circ f + (a_k \circ f) \frac{\partial f^k}{\partial w^i} \Big\} dw^i + \sum_{j=k}^{n} (a_k \circ f) \frac{\partial f^k}{\partial w^\alpha} dw^\alpha \, , \tag{4.17}$$

where $f^k = z^k \circ f$.

Here let

$$\mathrm{I}_2 = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (f^*\omega)^* \eta_n(T^* V) \, .$$

Then we know already that

$$I_2 = -I_p(f^*\omega) . \tag{4.18}$$

To compute $I_p(f^*\omega)$ let us first examine local properties of the singular set Σ about p. From $w^j = z^j \circ f$ $(1 \leq j \leq k-1)$ and rank $f \geqq k-1$, we observe

$$U_p \cap \Sigma = \left\{\frac{\partial f^k}{\partial w^k} = 0, \cdots, \frac{\partial f^k}{\partial w^n} = 0\right\} . \tag{4.19}$$

Set $\Sigma_p = U_p \cap \Sigma$. Since p is generic, we can assume that Σ_p is a complex manifold of dimension n. Therefore there exists an $(n-k)$-dimensional complex submanifold of V such that $\Sigma_p \cap \Sigma_p^\perp = \{p\}$ (transversal at p) and $\partial \Sigma_p^\perp \cong S^{2k-1}$, called a complemental submanifold of Σ_p at p.

At first we see that if U_p is a sufficiently small neighborhood of p,

$$(a_k \circ f)(p') \neq 0 \qquad \text{for any } p' \in U_p . \tag{4.20}$$

Indeed from $f_p^*\omega = 0$ and (4.17) it follows that

$$\begin{cases} (a_i \circ f)(p) + a_k \circ f(p) \dfrac{\partial f^k}{\partial w^i}(P) = 0 , & (i = 1, \cdots, k-1) , \\ (a_k \circ f)(p) \dfrac{\partial f^k}{\partial w^\alpha}(P) = 0 & (\alpha = k, \cdots, n) . \end{cases}$$

However as $\omega_q \neq 0$, it is easy to see that

$$(a_k \circ f)(p) = a_k(q) \neq 0 .$$

This means (4.20).

Now let $v = (a_k \circ f)\left(\dfrac{\partial f^k}{\partial w^k}, \cdots, \dfrac{\partial f^k}{\partial w^n}\right)$ be the holomorphic map of U_p into C^{n-k+1}, related with (4.17). Then we have by (4.19) and (4.20).

$$\text{zeros of} \quad v = \Sigma_p , \tag{4.21}$$

and so

$$\text{zeros of} \quad v \,|\, \Sigma_p^\perp = \{p\} . \tag{4.21$'$}$$

LEMMA 4.3. *Let v be as above and let $\mu_f(p)$ be the obstruction number of f at the generic point p (Def. 13). Then we have*

$$I_p(v|_{\Sigma_p^\perp}) = \mu_f(p) . \tag{4.22}$$

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Proof. First we shall recall the definition of $\mu_f(p)$. Let $\partial f/\partial w$ be the Jacobian matrix of f with respect to the above coorinates $\{w^i\}_{i=1}^n$ and $\{z^i\}_{i=1}^k$. Then

$$\frac{\partial f}{\partial w} = \left\| \begin{array}{c|c} 1_{k-1} & 0 \\ \hline \frac{\partial f^k}{\partial w^1}, \cdots, \frac{\partial f^k}{\partial w^{k-1}} & \frac{\partial f^k}{\partial w^k}, \cdots, \frac{\partial f^k}{\partial w^n} \end{array} \right\| .$$

And here

$$\mu_f(p) = \int_{\partial \Sigma_p} \left(\frac{\partial f}{\partial w}\right)^* \eta_{n,k} ,$$

where $\hat{\eta}_{n,k}$ is the generator of the $(2n-k)+1$-th cohomology group $H^{2(n-k)+1}(V(n,k;C);Z)$ of the Stiefel manifold as in § 1.

Now let Φ be the holomorphic map on U_p defined by

$$\Phi = \left\| \begin{array}{c|c} 1_{k-1} & 0 \\ \hline 0 & v \end{array} \right\| .$$

Then $\Phi|_{\Sigma_p^\perp\{p\}}$ is the map of $\Sigma_p^\perp - \{p\}$ into $V(n,k;\boldsymbol{C})$. Moreover we find, using $a_k \circ f \neq 0$ on U_p, that $\left.\frac{\partial f}{\partial w}\right|_{\Sigma_p^\perp - \{p\}}$ is homotopic to $\Phi|_{\Sigma_p^\perp\{p\}}$. Therefore we get

$$\mu_p(f) = \int_{\partial \Sigma_p^\perp} \Phi^* \eta_{n,k} . \tag{4.23}$$

But in terms of Lemma 3.7 in [6] we have

$$\int_{\partial \Sigma_p^\perp} \Phi^* \eta_{n,k} = \int_{\partial \Sigma_p^\perp} v^* \eta_{n,1} = \mathrm{I}_p(v|_{\Sigma_p^\perp}) .$$

These facts show (4.22). Q.E.D.

Next let $u = \left(a \circ f + (a_k \circ f)\frac{\partial f^k}{\partial w^1}, \cdots, a_{k-1} \circ f + (a_k \circ f)\frac{\partial f^k}{\partial w^{k-1}}\right)$ be the map of U_p into $\boldsymbol{C}^{k-1}$ similar with v. It follows from (zeros of $f^*\omega) \cap U_p = \{p\}$ and (4.21) that

$$\text{zeros of} \quad u|_{\Sigma_p} = \{p\} . \tag{4.24}$$

In order to compare $\mathrm{I}_p(u|_{\Sigma_p})$ with the restricted index $\tilde{I}_p(\omega)$ of ω at p (Definition 1.6) let $\tilde{}$ be the projection of $f^\#(T^*W)|_\Sigma$ onto $Q(\Sigma) = f^\#(T^*W)/\operatorname{Ker} T(f)$ as denoted in § 1. Then by (4.16) it is trivial that

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$$\widetilde{f^{\#}(\omega)}|_{\Sigma_p} = \sum_{i=1}^{k} (a_i \circ f)\widetilde{f^{\#}(dz^i)}|_{\Sigma_p} . \tag{4.25}$$

LEMMA 4.4. *Notations being as above, it follows that*

$$\widetilde{f^{\#}(\omega)}|_{\Sigma_p} = \sum_{j=1}^{k-1} \left(a_j \circ f + (a_k \circ f)\frac{\partial f^k}{\partial w^j} \right) \widetilde{f^{\#}(dz^j)}|_{\Sigma_p} \tag{4.26}$$

and that $\{f^{\#}(dz^j)|_{\Sigma_p}\}_{j=1}^{k-1}$ *is a base of* $Q_k)\Sigma)|_{\Sigma_p}$.

Proof. Clearly $\{f^{\#}(dz^i)\}_{i=1}^{k}$ is the base of $f^{\#}(T^*W)|U_p$. Here let $\theta = \Sigma\, b_i f^{\#}(dz^i)$ be any section of $f^{\#}(T^*W)$ on U_p. Then

$$T(f)(\theta) = \sum_{j=1}^{k-1} \left(b_j + b_k \frac{\partial f^k}{\partial w^j} \right) dw^j + b_k \left(\frac{\partial f^k}{\partial w^k} dw^k + \cdots + \frac{\partial f^k}{\partial w^n} dw^n \right) .$$

But since $\dfrac{\partial f^k}{\partial w^\alpha} = 0$ on Σ_p $(\alpha = k, \cdots, n)$, $T(f)(\theta) = 0$ means that $b_j = -b_k \cdot \dfrac{\partial f^k}{\partial w^j}$ on Σ_p for $j = 1, \cdots, k-1$. Thus we see that $\sum_{j=1}^{k-1} \dfrac{\partial f^k}{\partial w^i} f^{\#}(dz^i) - f^{\#}(dz^k)$ is the base of $\operatorname{Ker} T(f)|\Sigma_p$. Here the second statement in Lemma 4.4 is proved. Moreover as

$$\widetilde{f^{\#}(dz^k)} = \sum_{j=1}^{k-1} \frac{\partial f^k}{\partial w^j} \widetilde{f^{\#}(dz^j)} \qquad \text{on } \Sigma_p ,$$

we can prove directly (4.26). Q.E.D.

From the above lemma we obtain the following

COROLLARY 4.5.

$$\tilde{I}_p(\omega) = I_p(u|_{\Sigma_p}) . \tag{4.27}$$

Finally let us put

$$\Phi = (v, u) .$$

Then it is trivial from (4.17) and (4.18) that

$$I_p(f^*\omega) = I_p(\Phi) . \tag{4.28}$$

Under the above preparations we are able to prove the following

PROPOSITION 4.6.

$$I_p(f^*\omega) = \mu_f(p)\tilde{I}_p(\omega) . \tag{4.29}$$

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Proof. From (4.22), (4.27) and (4.28) it is enough to prove

$$I_p(v, u) = I_p(v|_{\Sigma_p}) \cdot I_p(u|_{\Sigma_p^{\perp}}) . \tag{4.30}$$

However as we can assume $U_p \cong \Sigma_p \times \Sigma_p^{\perp}$, we observe from (4.21)′ and (4.24) that

$$(v, u) \quad \text{is homotopic to} \quad (v|_{\Sigma_p}, u|_{\Sigma_p^{\perp}}) .$$

And from elementary facts of Topology we have (4.30). Q.E.D.

The above proposition shows in terms of (4.18)

$$\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (f^*\omega)^* \eta_n(T^*V) = -\mu_f(p)\tilde{I}_p(\omega) . \tag{4.31}$$

Now we shall complete the proof of Theorem 1.9. First of all it holds from (4.2), (4.15) and (4.29) that

$$\int_V C_n(T^*V) = (-1)^n \chi(F)\chi(W) + \sum_{j=1}^{\ell} \mu_f(p_j)\tilde{I}_{p_j}(\omega) ,$$

and so from (4.1) we can prove (1.4) in Theorem 1.9.

§ 5. Appendix

In this section we shall prove Proposition 1.8 in § 1. Before proving this fact we review definition in [5]. Let N be a smooth manifold and $T^R N$ be the real tangent bundle of N.

DEFINITION 5.1. Let $S = \{S_i\}_{i \in I}$ be a partition of N, that is, $N = \bigcup_{i \in I} S_i$ ($S_i \cap S_j = \phi$ if $i \neq j$). Then the partition S is called a *stratification* of N when the following conditions are satisfied;

a) I is countable,

b) each S_i which is called a strata is a regular submanifold of N,

c) if for any non-negative integer p we put

$$I(p) = \{i \in I ; \dim S_i \leq p\} ,$$

then union $\bigcup_{j \in I(p)} S_j$ is closed in N.

One notices that by conditions a) and c) S is locally finite.

Let $S = \{S_i\}_{i \in I}$ be a stratification and let $J \subset I$. Put $S_J = \{S_i ; i \in J\}$. Then we set

$$|S_J| = \bigcup_{j \in J} S_j .$$

Now let S be a stratification of N. Then for any $x \in N$, $T_x(S)$ is defined by

$T_xS =$ the tangent space T_xS_i of a strata S_i containing x. With this notation we state the following

DEFINITION 5.2. Let E be a C^∞-vector bundle over N, and let $\Gamma_\infty(E)$ be the set of all smooth sections of E. Suppose that a stratification $S = \{S_i\}_{i \in I}$ of E is given. Then a section $\omega \in \Gamma_\infty(E)$ is called *transversal* to S if and only if

$$\omega_*(T_xN) + T_{\omega(x)}S = T_{\omega(x)}E\ . \tag{5.1}$$

We denote by $\mathfrak{h}(S)$ the set of all transversal cross-section of E.

Now let us return to the proof of Proposition 1.8. Let V and W be compact complex manifolds of $\dim_C V = n$ and $\dim_C W = k$, and let f be an (IF)-holomorphic mapping of V into W with $\operatorname{rank} f \geqq k-1$ as in §1. When $\tilde{f}$ is the linear map of $f^\#(T^*W)$ onto T^*W defined by $\tilde{f}(x, v) = v$ for any $(x, v) \in f^\#(T^*W)$, we observe that $\tilde{f}$ is proper, because V is compact. Here let Σ be the singular set of f. Since Σ is the analytic set in V of $\dim_C \Sigma = k-1$ and f is of $\operatorname{rank} k-1$ on Σ, the closed subset $\operatorname{Ker} T(f)|_\Sigma$ of $f^\#(T^*W)$ becomes the k-dimensional analytic set, where $T(f)\colon f^\#(T^*W) \to T^*V$ is defined as follows; for any $(p, v) \in f^\#(T^*W)$, $T(f)(p, v) = f_p^*v$. For simplicity set

$$L(\Sigma) = \tilde{f}(\operatorname{Ker} T(f)|_\Sigma)\ . \tag{5.2}$$

Then as $\tilde{f}$ is proper, we find from the proper mapping theorem ([6]) that $L(\Sigma)$ is the analytic set of T^*W such that

$$\dim_C L(\Sigma) \leq k\ . \tag{5.3}$$

The next proposition is due to Whitny [8].

PROPOSITION 5.3. *Let M be a complex manifold and Σ' be an analytic set of M. Then M has a stratification $S = \{S_i\}_{i \in I}$ such that $M - \Sigma'$ is a strata of S and $\bar{S}_i - S_i \subset \bigcup_{j \in I(p)} S_j$ $(p \leq \dim_R S_i)$ for each $i \in I$. We call stratification in this proposition Σ'-stratification of N.*

At first we get the following

LEMMA 5.4. *All notations are as before. Let Σ_s be the singular set of Σ and let $L(\Sigma_s) = \tilde{f}(\operatorname{Ker} T(f)|_{\Sigma_s})$. Further let $S(L(\Sigma_s))$ be a $L(\Sigma_s)$-*

stratification of T^*W. Then for any $\omega \in \mathfrak{M}(S(L(\Sigma_s)))$ we obtain

(5.4) $$\omega^{-1}(L(\Sigma_s)) = \phi .$$

Proof. Assume $q \in \omega^{-1}(L(\Sigma_s))$. Then it follows from (5.1) that

(5.5) $$\omega_*(T^{\mathbf{R}}_q W) + T^{\mathbf{R}}_{\omega(q)} S_i = T^{\mathbf{R}}_{\omega(q)}(T^*W) ,$$

where $S_i(\ni \omega(q))$ is a strata of $S(L(\Sigma_s))$.

However S_i is contained in $L(\Sigma_s)$ with $\dim_{\mathbf{R}} L(\Sigma_s) \leq 2(k-1)$, and so $\dim T^{\mathbf{R}}_{\omega(q)} S_i \leq 2(k-1)$. This is contrary to (5.5), because W is of real $2k$-dimension. Q.E.D.

Secondly it follows the following

LEMMA 5.5. *Let $S(L(\Sigma)) = \{S_i\}_{i \in I}$ be an $L(\Sigma)$-stratification of T^*W and let us denote by $S^j(L(\Sigma))$ the set of stratum S_i such that $\dim_R S_i \geq j$. Then we find that for any $\omega \in \mathfrak{M}(S(L(\Sigma)))$,*

(5.6) $$\omega^{-1}(|S^{2k}(L(\Sigma))|) \quad \textit{is a finite point set.}$$

(As $\dim_{\mathbf{R}} L(\Sigma) \leq 2k$ and S is $L(\Sigma)$-stratification, $|S^{2k}(L(\Sigma))|$ coincides with $L(\Sigma)$.)

Proof. Take an arbitrary strata S_i of real dimension $2k$. Then $\omega^{-1}(S_i)$ is discrete and without accumulating points. Indeed discreteness is trivial, since ω is transversal to S_i. On the other hand suppose $\{q_\alpha\} \subset \omega^{-1}(S_i)$ converges to a point q_0. Then $\omega(q_p) \in \bar{S}_i$ and so $\omega(q_0) \in \bar{S}_i - S_i$. And from the definition of $L(\Sigma)$-stratification, $\omega(q_0) \in |S^{2k-1}(L(\Sigma))|$.

But we can prove similarly as Lemma 5.4 that

$$\omega^{-1}(|S^{2k-1}(L(\Sigma))|) = \phi .$$

This show $\omega^{-1}(S_i)$ has not accumlating points. Next recall S is locally finite. Noticing W is compact, we see that $\{i; \omega^{-1}(S_i) \neq \phi\}$ is finite. Therefore Lemma 5.5 is proved. Q.E.D.

Now, $f(\Sigma)$ being the analytic set of W with $\dim_R f(\Sigma) \leq 2k-2$, the zero-section of $T^*(W)|_{f(\Sigma)}$ is also analytic set. We write for $f(\Sigma)$ this section without confusion. Then with respect to $f(\Sigma)$-stratification $S(f(\Sigma))$ we can also prove that

(5.7) $$\omega|_{f(\Sigma)} \text{ is non-zero} \qquad \text{for all } \omega \in \mathfrak{M}(S(f(\Sigma))) .$$

On the other hand we know transversality theorem in [5] that

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$$D(\Sigma) = \mathscr{A}(S(f(\Sigma))) \cap \mathscr{A}(S(L(\Sigma_s))) \cap \mathscr{A}(S(L(\Sigma)))$$

is empty.
Here let us take $\omega \in D(\Sigma)$. Then from the above results it follows that

(5.8) ω is non-zero on $f(\Sigma)$,

(5.9) $\omega(f(\Sigma_s)) \not\subset L(\Sigma_s)$,

and

(5.10) $\omega^{-1}(L(\Sigma))$ is finite, say $\{q_1, \cdots, q_m\}$, note $(q_i \in f(\Sigma))$.

Now Proposition 1.8 is trivial. Indeed let $q_i \in \omega^{-1}(L(\Sigma))$. This means that $T(f)(f^{\#}(\omega)(p)) = 0$ for any $p \in \Sigma \cap f^{-1}(q_i)$. Since from (5.9), $q_i \notin f(\Sigma_s)$, we observe that $\Sigma \cap f^{-1}(q_i)$ is in the regular points set of Σ and finite, because f has only isolated singularities. Moreovər it is clear from (5.8) that $\omega(q_i) \neq 0$ $(i = 1, \cdots, m)$. Here ω satisfy condition i) and ii) of f-forms (cf. Definition 1.7). This completes the proof of Proposition 1.8.

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