# NOTE ON GENERALIZED SCHREIER EXTENSIONS OF GROUPS

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By a (generalized) Schreier extension we mean a group $G$ decomposed into a subinvariant series $G_n \rightarrowtail G_{n-1} \rightarrowtail G_{n-2} \rightarrowtail \dots \rightarrowtail G_1 \rightarrowtail G_0 = G$, where $G_n$ is anti-invariant in $G$, i.e. the only subgroup of $G_n$ which is normal in $G$ is the trivial one. ("$\rightarrowtail$" denotes a group monomorphism, i.e. an injection homomorphism.) As is well known, such groups $G$ can be embedded into the repeated wreath product $F_{n-1} \wr (F_{n-2} \wr \dots \wr (F_2 \wr (F_1 \wr F_0))\dots)$, where $F_i \cong G_i/G_{i+1}$ (cf. [2], notation of M. Hall [1], p. 81).

In this note we re-establish this embedding for finite $G$, by making use of the theory of invariants of groups. The embedding we construct however is not the same as the one constructed in [2]. The proof is by induction, the induction step being provided by the following theorem.

THEOREM. Let $G_2 \rightarrowtail G_1 \rightarrowtail G_0 = G$ be a Schreier extension; $F_0 \cong G_0/G_1$, $F_1 \cong G_1/G_2$. Then there exists a monomorphism $\mu$ of $G$ into $F_1 \wr F_0$, turning $G_1$ into a subdirect product of the normal divisor $F_1^{F_0}$ of $F_1 \wr F_0$, and making the following diagram (which has exact rows) commutative:



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$$
\begin{array}{ccccccccc}
1 & \to & G_1 & \longrightarrow & G & \longrightarrow & F_0 & \to & 1 \\
 & & \downarrow \mu & & \downarrow \mu & & \| & & \\
1 & \to & F_1^{F_0} & \to & F_1 \wr F_0 & \to & F_0 & \to & 1
\end{array}
$$

Proof. One can always find a galoisian field extension $N/K$ ($K$ infinite) with Galois group $G$. Let $L \subset M$ be the intermediate fields of $K$ and $N$, which correspond to the subgroups $G_1$ and $G_2$ of $G$, respectively.

Let $L = K(\beta)$ and $M = L(\alpha)$. Take $c \in K$ such that $\gamma = \alpha + c\beta$ with $M = K(\gamma)$. Let $\tau_i$ be a K-automorphism of $L$ with $\tau_i \beta = \beta_i$ ($\beta_1 = \beta$). Let, for every $i$, $\bar{\tau}_i$ be a K-automorphism of $N$, extending $\tau_i$ ($\bar{\tau}_1 = 1 \in G$). Define $\bar{\tau}_i \alpha_j = \alpha_{ij}$, where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_a$ in some enumeration of the conjugates of $\alpha$, and let $\gamma_{ij} = \alpha_{ij} + c\beta_i$. Then, for every $i$ ($i = 1, \ldots, b$; $b = |F_0|$) one has $f_i = \mathrm{Irr}(\gamma_{ij}, L) =$

$$= \prod_{j=1}^{a} (X - \gamma_{ij}), \text{ where } a = |F_1|. \text{ One has also}$$

$f = \mathrm{Irr}(\gamma_{ij}, K) = f_1 \ldots f_b$, and obviously $N$ is the splitting field of $f$ over $K$. So $G$ has a representation as an imprimitive permutation group on $M = \{\gamma_{11}, \ldots, \gamma_{ba}\}$, with domains of imprimitivity $M_i = \{\gamma_{i1}, \ldots, \gamma_{ia}\}$ ($i = 1, \ldots, b$). $F_0$ and $F_1$ are permutation groups on the sets $\{\beta_1, \ldots, \beta_b\}$ and $\{\alpha_1, \ldots, \alpha_a\}$, respectively. $F_0$ permutes the system $M_i$ in the obvious way, but does not necessarily leave the second indices of the $\gamma_{ij}$ unaltered.

Applying a trick from field theory (cf. [3], § 61) we show that the restriction of the Galois group $G_1$ of $N/L$ to $M_i$ is precisely equal to the permutation group $F_1$ (as a permutation group of the $\gamma_{i1}, \ldots, \gamma_{ia}$, instead of the $\alpha_1, \ldots, \alpha_a$, respectively), on the understanding that some element $\sigma \in G_1$ may very well give rise to different permutations in the sets $M_i$. Denote this restriction by $F_1^{(i)}$ $(F_1^{(1)} = F_1)$. Then, to finish the proof, it is shown that every $g \in G$ gives rise to a permutation of $M$, which can be split into a product of two permutations (which do not necessarily define automorphisms of $L$ or $M$), one of which permutes the systems $M_i$ according to $F_0$, while leaving the second indices of $\gamma_{ij}$ invariant; the other one is a permutation of the direct product $\prod_{i=1}^{b} F_1^{(i)} = F_1^{F_0}$.

Let $t_1, \ldots, t_a$ denote indeterminates upon which $G$ acts trivially, and form the expressions $y_{11} = t_1\gamma_{11} + \ldots + t_a\gamma_{1a}$, $\sigma y_{11}$ with $\sigma \in F_1^{(1)}$. Note that $\sigma$ acts on $\gamma_{11}, \ldots, \gamma_{1a}$ exactly in the same way as it acts on $\alpha_1 = \alpha_{11}, \ldots, \alpha_a = \alpha_{1a}$, respectively. The set $\{\sigma y_{11} | \sigma \in F_1\}$ is a full set of conjugates of $y_{11}$ with respect to $L_t = L(t_1, \ldots, t_n)$. The coefficients of $f_{1t} = \prod_{\sigma \in F_1} (X - \sigma y_{11}) = \mathrm{Irr}(y_{11}, L_t)$ can be written uniquely in the form

$$(2) \qquad a_0(t) + a_1(t)\beta_1 + \ldots + a_{b-1}(t)\beta_1^{b-1}$$

with $a_i(t) \in K_t$. Now (loc. cit. [4]), the group of all permutations of $t_1, \ldots, t_a$ that leave the joint elements $a_i(t)$, thus obtained from all the coefficients of $f_{1t}$, invariant, is exactly the same as the permutation group $F_1^{(1)}$ (of $t_1, \ldots, t_a$ instead of $\gamma_{11}, \ldots, \gamma_{1a}$, respectively). This group does not change if a K-automorphism of $L$ is applied. For, let $\bar{\tau}_i f_{1t} = f_{it}$; then the corresponding coefficients of $f_{it}$ are

(3) $$a_0(t) + a_1(t)\beta_i + \dots + a_{b-1}(t)\beta_i^{b-1},$$

while a zero of $f_{it}$ is $\bar{\tau}_i y_{11} = t_1\gamma_{i1} + \dots + t_i\gamma_{ia}$. So (loc. cit. [4]), the Galois group $F_1^{(i)}$ of $f_{it}$ is $F_1$ (as a permutation group of $\gamma_{i1}, \dots, \gamma_{ia}$ instead of $t_1, \dots, t_a$).

Finally, let $g \in G$, then $g$ permutes the factors $f_i$ (and the corresponding domains $M_i$) according to $F_0$. Let $gM_1 = M_i$, then $g$ can be written $g = \pi_\gamma \rho$ where $\rho: \gamma_{1j} \to \gamma_{ij}$ $(j = 1, \dots, a)$ and $\pi_\gamma$ is some permutation of $\gamma_{i1}, \dots, \gamma_{ia}$. One has $\pi_\gamma \rho f_{1t} = f_{it}$, or $\pi_\gamma \rho f_{1t} = \pi_\gamma f_{it} = \pi_t^{-1} f_{it} = f_{it}$, where $\pi_t$ is the same permutation of $t_1 \dots t_a$ as $\pi_\gamma$ is of $\gamma_{i1}, \dots, \gamma_{ia}$. As there are no permutations of $t_i$ turning $f_{it}$ into itself other than those in $F_1$ one obtains $\pi_t^{-1} \in F_1$ and $\pi_\gamma \in F_1^{(i)}$. Let it be recalled that the full group of permutations of $M$ generated by the $\rho$'s and those in $\prod_{i=1}^{b} F^{(i)}$ is just the wreath product $F_1 \wr F_0$.

The embedding theorem follows from the following functorial property of $F_1 \wr F_0$. Let $F_1^1 \rightarrowtail F_1$ and $F_0^1 \rightarrowtail F_0$, be monomorphisms, then $F_1^1 \wr F_0^1 \rightarrowtail F_1 \wr F_0$.

Proof of the embedding theorem. Let $G_n \rightarrowtail G_{n-1} \rightarrowtail \dots \rightarrowtail G$ be a Schreier extension. Define $G_2^* = \bigcap_{x \in G} x G_2 x^{-1}$. We obtain a Schreier extension $G_2/G_2^* \rightarrowtail G_1/G_2^* \rightarrowtail G/G_2^*$. Applying the theorem gives $G/G_2^* \rightarrowtail F_1 \wr F_0$. The next step is carried out as follows. Let $G_3^* = G_2^* \cap G_3$. Then it is readily seen that $G_2^*/G_3^* \rightarrowtail G_2/G_3 = F_2$. Define $G_3^{**} = \bigcap_{x \in G} x G_3^* x^{-1}$. The sub-invariant series $G_3^{**} \rightarrowtail G_3^* \rightarrowtail G_2^* \rightarrowtail G$ gives rise to a Schreier extension $G_3^*/G_3^{**} \rightarrowtail G_2^*/G_3^{**} \rightarrowtail G/G_3^{**}$, from which $G/G_3^{**} \rightarrowtail F_2 \wr G/G_2^* \rightarrowtail F_2 \wr F_1 \wr F_0$ follows. This process ends when for some $i$, $G_i^{**} = \{1\}$. If $i < n-1$, we have an even more economical embedding than the one stated above.

Remark. Professor B.H. Neumann pointed out to me that, by a modification of a method of his ([3], theorem 3.5, p. 48), one can also establish the embedding of $G$ into $F_{n-1} \wr \ldots \wr F_0$. This method lends itself to an extension to the infinite case.

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