



On Non-Archimedean Curves Omitting Few Components and their Arithmetic Analogues

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Abstract. Let k be an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Let D_1, \dots, D_n be effective nef divisors intersecting transversally in an n -dimensional nonsingular projective variety X . We study the degeneracy of non-Archimedean analytic maps from k into $X \setminus \bigcup_{i=1}^n D_i$ under various geometric conditions. When X is a rational ruled surface and D_1 and D_2 are ample, we obtain a necessary and sufficient condition such that there is no non-Archimedean analytic map from k into $X \setminus D_1 \cup D_2$. Using the dictionary between non-Archimedean Nevanlinna theory and Diophantine approximation that originated in earlier work with T. T. H. An, we also study arithmetic analogues of these problems, establishing results on integral points on these varieties over \mathbb{Z} or the ring of integers of an imaginary quadratic field.

1 Introduction

Let k be an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Our primary object of study is the degeneracy of non-Archimedean analytic maps from k to an n -dimensional projective variety X omitting an effective divisor with at least n irreducible components. As argued in [2], results on non-Archimedean analytic curves in this context should have arithmetic counterparts in Diophantine geometry; they should correspond to results on integral points over \mathbb{Z} or the ring of integers of an imaginary quadratic field. Thus, a second objective is to prove an appropriate arithmetic analogue of all of our results on non-Archimedean analytic curves, further illustrating and justifying the correspondence proposed in [2]. Before discussing our main results, we briefly recall some aspects of the correspondence in [2] as well as several examples of parallel non-Archimedean and arithmetic results.

To begin, we recall the connection between Nevanlinna theory, the quantitative theory that grew out of Picard's theorem, and Diophantine approximation, a quantitative theory behind many results on rational and integral points on varieties. Originating in the work of Osgood, Vojta, and Lang, it has been observed that there is a

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striking correspondence between many statements in Nevanlinna theory and statements in Diophantine approximation. A detailed “dictionary” between the two subjects has been constructed by Vojta [13]. Qualitatively, in the simplest case, holomorphic curves in a variety X should correspond to infinite sets of integral points on X (viewing X both as a variety over a number field and as a complex analytic space). Taking $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$, this implies that a non-constant entire function without zeros is analogous to an infinite set of units in some ring of integers.

In contrast to the situation in complex analysis, where the exponential function is an entire function without zeros, it is an easy fact that a non-Archimedean entire function without zeros must be constant. In view of the aforementioned analogies, this suggests that to obtain a Diophantine analogue of non-Archimedean analytic curves, we should consider only rings of integers with a finite unit group, *i.e.*, \mathbb{Z} or the ring of integers of an imaginary quadratic field. This observation was the starting point for the work in [2], where more generally it was argued that, at least for certain classes of varieties, a non-constant non-Archimedean analytic map into a variety X should correspond to an infinite set of \mathcal{O}_k -integral points on X , where $\mathcal{O}_k = \mathbb{Z}$ or the ring of integers of an imaginary quadratic field. For completeness, we also mention that this correspondence can frequently be made quantitative (see [2]), resulting in parallel statements in non-Archimedean Nevanlinna theory and Diophantine approximation.

Determining the precise class of varieties under which a correspondence should hold was left open in [2]. At the least, it seems necessary that the varieties X be affine (or close to affine) and that, in the arithmetic case, the varieties satisfy a rationality condition on the components at infinity:

(*) There exists a projective closure \tilde{X} of X nonsingular at every point in $\tilde{X} \setminus X$ and such that every (geometric) irreducible component of $\tilde{X} \setminus X$ is defined over k ,

where $k = \mathbb{Q}$ or an imaginary quadratic field. We now illustrate the correspondence by recalling several examples of parallel non-Archimedean and arithmetic results.

For curves, condition (*) yields a sufficient hypothesis under which our correspondence holds (see Section 2 for the definitions).

Theorem 1.1 *Let k be \mathbb{Q} or an imaginary quadratic field and suppose that $\text{char } \mathbf{k} = 0$. If X is an affine curve over k satisfying (*), then X contains an infinite set of \mathcal{O}_k -integral points if and only if there exists a non-constant analytic map $f: \mathbf{k} \rightarrow X$ if and only if X is rational with a single point at infinity.*

The last equivalence holds in arbitrary characteristic (we only assumed $\text{char } \mathbf{k} = 0$ to ensure that X makes sense over both k and \mathbf{k}). This follows easily from Siegel’s theorem and Berkovich’s non-Archimedean analogue of Picard’s theorem. As a special case, we have the following version of Berkovich’s Picard theorem.

Theorem 1.2A *Any analytic map from \mathbf{k} to a projective curve omitting two points must be constant.*

This corresponds to the following theorem on integral points.

Theorem 1.2B *Let k be \mathbb{Q} or an imaginary quadratic field and let C be an affine curve over k with at least two k -rational points at infinity. Then any set of \mathcal{O}_k -integral points on C is finite.*

A generalization of Theorem 1.2A to higher dimensions was obtained by Lin and Wang [10], and a refinement was given in [2] by An, Levin, and Wang as follows.

Theorem 1.3A *Let X be a nonsingular projective variety over \mathbf{k} . Let D_1, \dots, D_m be effective divisors on X with empty intersection. Let $D = \sum_{i=1}^m D_i$.*

- (i) *If $\kappa(D_i) > 0$ for all i , then the image of an analytic map $f: \mathbf{k} \rightarrow X \setminus D$ is contained in a proper subvariety of X .*
- (ii) *If D_i is big for all i , then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f: \mathbf{k} \rightarrow X \setminus D$ is contained in Z .*
- (iii) *If D_i is ample for all i , then there is no non-constant analytic map from \mathbf{k} to $X \setminus D$.*

The arithmetic analogue of Theorem 1.3A is implicit in the proof of a more general result proved in [9] using a higher-dimensional version of "Runge's method".

Theorem 1.3B (Levin) *Let k be \mathbb{Q} or an imaginary quadratic field. Let X be a nonsingular projective variety over k . Let D_1, \dots, D_m be effective divisors on X , defined over k , with empty intersection. Let $D = \sum_{i=1}^m D_i$.*

- (i) *If $\kappa(D_i) > 0$ for all i , then any set R of \mathcal{O}_k -integral points on $X \setminus D$ is contained in a proper Zariski-closed subset of X .*
- (ii) *If D_i is big for all i , then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set R of \mathcal{O}_k -integral points on $X \setminus D$, the set $R \setminus Z$ is finite.*
- (iii) *If D_i is ample for all i , then all sets R of \mathcal{O}_k -integral points on $X \setminus D$ are finite.*

Furthermore, as shown in [9], in each of the above cases, the integral points can be effectively computed.

We note that if D_1, \dots, D_m are effective divisors in general position on X , then they have empty intersection if $m \geq \dim X + 1$. Here, a collection of effective divisors D_i on a projective variety X of dimension n is said to be *in general position* if for each $1 \leq k \leq n + 1$ and each choice of indices $i_1 < \dots < i_k$, each irreducible component of $D_{i_1} \cap \dots \cap D_{i_k}$ has codimension at least k in X ; in particular, this intersection is empty when $k = n + 1$.

In view of Theorems 1.3A and 1.3B, a next natural case to study is varieties of the form $X \setminus \bigcup_{i=1}^n D_i$, where $n = \dim X$. The non-Archimedean case was studied for projective space by An, Wang, and Wong [3] (cf. [2] for a necessary and sufficient statement), and the arithmetic analogue was established in [2].

Theorem 1.4A (An, Wang, Wong) *Let D_1, \dots, D_n be nonsingular hypersurfaces in \mathbb{P}^n intersecting transversally. Then there is no non-constant analytic map from \mathbf{k} to $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$ if $\deg D_i \geq 2$ for each $1 \leq i \leq n$.*

Theorem 1.4B (An, Levin, Wang) *Let k be \mathbb{Q} or an imaginary quadratic field. Let D_1, \dots, D_n be nonsingular hypersurfaces defined over k in \mathbb{P}^n intersecting transversally. Suppose that every point in the intersection $\bigcap_{i=1}^n D_i$ is k -rational. Then any set of \mathcal{O}_k -integral points on $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$ is finite.*

More generally, when k is \mathbb{Q} or an imaginary quadratic field and D_1 and D_2 are curves over k intersecting transversally, a complete characterization of finiteness of \mathcal{O}_k -integral points on $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ was given in [2].

We now state our main results. Under various geometric conditions, we prove the degeneracy of analytic maps $f: \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$. We also prove analogous results for \mathcal{O}_k -integral points on $X \setminus \bigcup_{i=1}^n D_i$ under condition (*) and the following condition:

(**) Every point in the intersection $\bigcap_{i=1}^n D_i$ is k -rational.

Notation and definitions will be given in the next section.

Theorem 1.5A *Let D_1, \dots, D_n be effective nef divisors intersecting transversally in an n -dimensional nonsingular projective variety X over \mathbf{k} . Let K_X denote the canonical divisor on X .*

- (i) *Assume that either $D_i^n > 1$ or that $D_i^n = 1$ and $K_X \cdot D_i^{n-1} < 1 - n$ for each $1 \leq i \leq n$. Then the image of an analytic map $f: \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$ is contained in a proper subvariety of X .*
- (ii) *If $D_i^n > 1$ for all i , then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f: \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$ is contained in Z .*

Theorem 1.5B *Let k be \mathbb{Q} or an imaginary quadratic field. Let X be an n -dimensional nonsingular projective variety over k . Let D_1, \dots, D_n be effective nef divisors on X , all defined over k , intersecting transversally in X . Let K_X denote the canonical divisor on X . Suppose that every point in the intersection $\bigcap_{i=1}^n D_i$ is k -rational.*

- (i) *Assume that either $D_i^n > 1$ or that $D_i^n = 1$ and $K_X \cdot D_i^{n-1} < 1 - n$ for each $1 \leq i \leq n$. Then any set R of \mathcal{O}_k -integral points on $X \setminus \bigcup_{i=1}^n D_i$ is contained in a proper Zariski-closed subset of X .*
- (ii) *If $D_i^n > 1$ for all i , then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set R of \mathcal{O}_k -integral points on $X \setminus \bigcup_{i=1}^n D_i$, the set $R \setminus Z$ is finite.*

Combined with an appropriate version of the Hodge index theorem (Theorem 2.3), this yields the following corollaries.

Corollary 1.6A *Let X be an n -dimensional nonsingular projective variety over \mathbf{k} . Suppose that $-K_X$ is nef and $(-K_X)^n > (n - 1)^n$. Let D_1, \dots, D_n be effective nef and big divisors on X intersecting transversally. Then the image of an analytic map $f: \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$ is contained in a proper subvariety of X .*

Corollary 1.6B *Let k be \mathbb{Q} or an imaginary quadratic field. Let X be an n -dimensional nonsingular projective variety over k . Suppose that $-K_X$ is nef and $(-K_X)^n > (n - 1)^n$. Let D_1, \dots, D_n be effective nef and big divisors on X , all defined over k , intersecting transversally in X . Suppose that every point in the intersection $\bigcap_{i=1}^n D_i$ is k -rational.*

Then any set R of \mathcal{O}_k -integral points on $X \setminus \bigcup_{i=1}^n D_i$ is contained in a proper Zariski-closed subset of X .

The surfaces X satisfying the hypotheses of Corollary 1.6A can be completely classified as follows. (cf. [7, Chapter III, Exercise 3.8])

Remark 1.7 Let X be a smooth projective surface over an algebraically closed field. Then $-K_X$ is nef and $(-K_X)^2 > 1$ if and only if one of the following holds:

- (a) $X \cong \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over \mathbb{P}^1 ;
- (b) $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$;
- (c) X is obtained from \mathbb{P}^2 by successively blowing up at most 7 points.

In contrast to the case of projective space (Theorem 1.4A) and the situation for degeneracy (Theorem 1.5A), on general projective varieties, even under a transversality assumption, there is no condition on the self-intersection numbers of the divisors or the degrees of the divisors (in some fixed projective embedding) sufficient to ensure that there are no non-constant analytic maps $f: \mathbf{k} \rightarrow X \setminus (D_1 \cup \dots \cup D_n)$. A similar statement holds for integral points, but we give an example only in the non-Archimedean setting.

Example 1.8 Let m and n be positive integers. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let D_1 and D_2 be effective divisors of type $(1, m)$ and $(1, n)$, respectively. Let $D = D_1 + D_2$. Let P be a point in the intersection of the supports of D_1 and D_2 and let L be the line on X of type $(0, 1)$ through P . Then since $L \setminus D = L \setminus \{P\} \cong \mathbb{A}^1$, there exists a non-constant analytic map $f: \mathbf{k} \rightarrow X \setminus D$. Note that D_1 and D_2 are very ample and $D_1^2 = 2m, D_2^2 = 2n$. Thus, there is no condition on the self-intersection numbers of (very) ample divisors D_1 and D_2 on X that is sufficient to guarantee that there are no non-constant analytic maps from \mathbf{k} to $X \setminus (D_1 \cup D_2)$. Note also that for any embedding $X \subset \mathbb{P}^N$, the divisors D_i may have arbitrarily large degree.

In the case of rational ruled surfaces, we are able to completely classify analytic maps $f: \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$ when D_1 and D_2 are ample effective divisors intersecting transversally. We also prove the arithmetic analogue, again under conditions $(*)$ and $(**)$.

Recall that if X is a rational ruled surface, then $X \cong \mathbf{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-e)$ over \mathbb{P}^1 and $e \geq 0$ is uniquely determined. We let F denote a fiber on X and let C_0 denote a section of X such that $\mathcal{O}(C_0) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Theorem 1.9A Let X be a rational ruled surface over \mathbf{k} . Let D_1 and D_2 be effective divisors intersecting transversally in X .

- (i) If D_1 and D_2 are big, then the image of an analytic map $f: \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$ is contained in a proper subvariety of X .
- (ii) Suppose that D_1 and D_2 are ample. The image of an analytic map $f: \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$ is contained in either a fiber or a section C with $C \sim C_0$. There is no non-constant analytic map from \mathbf{k} to $X \setminus D_1 \cup D_2$ if and only if every fiber and

every section $C, C \sim C_0$, intersects $D_1 \cup D_2$ in more than one point. In particular, this holds if $D_i \cdot F \geq 2$ and $D_i \cdot C_0 \geq 2$ for $i = 1, 2$.

Theorem 1.9B Let k be \mathbb{Q} or an imaginary quadratic field. Let $\pi: X \rightarrow \mathbb{P}^1$ be a rational ruled surface over k . Let D_1 and D_2 be effective divisors intersecting transversally in X . Suppose that all the irreducible components of D_1 and D_2 are defined over k and that all of the points in the intersection $D_1 \cap D_2$ are k -rational.

- (i) If D_1 and D_2 are big, then any set R of \mathcal{O}_k -integral points on $X \setminus D_1 \cup D_2$ is contained in a proper Zariski-closed subset of X .
- (ii) Suppose that D_1 and D_2 are ample. Then any set R of \mathcal{O}_k -integral points on $X \setminus D_1 \cup D_2$ is contained in a finite union of fibers and sections C with $C \sim C_0$. Any set R of \mathcal{O}_k -integral points on $X \setminus D_1 \cup D_2$ is finite if and only if every curve $C \subset X$ over k that is linearly equivalent to C_0 or a fiber F intersects $D_1 \cup D_2$ in more than one point. In particular, this holds if $D_i \cdot F \geq 2$ and $D_i \cdot C_0 \geq 2$ for $i = 1, 2$.

2 Preliminaries

We first introduce some notation and definitions.

Let k be a number field and let M_k denote the set of places of k . Let D be an effective divisor on a nonsingular projective variety X , both defined over k . Recall that we can associate to D a height function h_D (well-defined up to $O(1)$) which for points $P \in X(k) \setminus D$ decomposes as a sum of local height functions $h_D = \sum_{v \in M_k} h_{D,v}$. Let S be a finite set of places of k . A set of points $R \subset X(k) \setminus D$ is called a set of $\mathcal{O}_{k,S}$ -integral points on $X \setminus D$ if there exist constants $c_v, v \in M_k$, such that $c_v = 0$ for all but finitely many v , and for all $v \notin S, h_{D,v}(P) \leq c_v$ for all $P \in R$. We refer the reader to [13] for further details on height functions and integral points.

Let D be a divisor on a nonsingular projective variety X , both defined over a field k . For a nonzero rational function $\phi \in k(X)$, we let $\text{div}(\phi)$ denote the divisor associated to ϕ . Then we let $L(D) = \{\phi \in k(X) \mid \text{div}(\phi) + D \geq 0\}$ and $h^0(D) = \dim H^0(X, \mathcal{O}(D)) = \dim L(D)$. If $h^0(nD) = 0$ for all $n > 0$ then we let $\kappa(D) = -\infty$. Otherwise, we define the Kodaira–Iitaka dimension of D to be the integer $\kappa(D)$ such that there exist positive constants c_1 and c_2 with

$$c_1 n^{\kappa(D)} \leq h^0(nD) \leq c_2 n^{\kappa(D)}$$

for all sufficiently divisible $n > 0$. We define a divisor D on X to be big if $\kappa(D) = \dim X$.

Definition 2.1 A divisor D on X is said to be numerically effective, or nef, if $D \cdot C \geq 0$ for any irreducible curve C on X .

We recall some basic properties of nef divisors on X ; see [6].

Lemma 2.2 Nef divisors satisfy the following:

- (i) Let $n = \dim X$. If D_1, \dots, D_n are nef divisors on X , then

$$D_1 D_2 \cdots D_n \geq 0.$$

- (ii) Let $f: X \rightarrow Y$ be a morphism and let D be a nef divisor on Y with $f(X)$ not contained in the support of D . Then $f^*(D)$ is nef on X .

We recall the following generalized Hodge index theorem; see [8, Theorem 1.6.1].

Theorem 2.3 Let D_1, \dots, D_n be nef divisors on an n -dimensional nonsingular projective variety X . Then $(D_1 \cdots D_n)^n \geq D_1^n \cdots D_n^n$.

When X is a surface and $n = 2$, this holds without the nef hypothesis as long as $D_1^2 > 0$.

For nef divisors, we have the following asymptotic Riemann–Roch formula.

Lemma 2.4 Suppose D is a nef divisor on a nonsingular projective variety X . Let $n = \dim X$. Then $h^0(mD) = \frac{D^n}{n!} \cdot m^n + O(m^{n-1})$. In particular, $D^n > 0$ if and only if D is big.

The next theorem gives a more refined bound for the dimension of the space of global sections of nef and big divisors (cf. [11] or [7, Chapter VI Theorem 2.15.9]).

Theorem 2.5 (Matsusaka) Let X be a nonsingular projective variety of dimension n and let D be a nef and big divisor on X . Then

$$h^0(mD) = \frac{D^n}{n!} m^n - \frac{K_X \cdot D^{n-1}}{2(n-1)!} m^{n-1} + O(m^{n-2}).$$

We will also make use of two basic exact sequences.

Lemma 2.6 Let D be an effective divisor on a nonsingular projective variety X with inclusion map $i: D \rightarrow X$. Let \mathcal{L} be an invertible sheaf on X . Then we have exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{L} \otimes \mathcal{O}(-D) \rightarrow \mathcal{L} \rightarrow i_*(i^*\mathcal{L}) \rightarrow 0, \\ 0 \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}(-D)) \rightarrow H^0(X, \mathcal{L}) \rightarrow H^0(D, i^*\mathcal{L}). \end{aligned}$$

Proof If D is an effective divisor on X , then a fundamental exact sequence is

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_D \rightarrow 0.$$

Tensoring with \mathcal{L} and using the projection formula, we get the first exact sequence. Taking global sections then gives the second exact sequence. ■

3 Proof of Theorems 1.5A and 1.5B

We first prove a theorem controlling the Kodaira–Itaka dimension of certain divisors on blow-ups.

Theorem 3.1 Let X be a nonsingular projective variety of dimension n over a field k and let D be a nef divisor on X . Let K_X denote the canonical divisor on X . Let $\pi: \tilde{X} \rightarrow$

X be the blow-up along m distinct points P_1, \dots, P_m of X , successively, and let $E_i := \pi^{-1}(P_i)$ be the exceptional divisor for $1 \leq i \leq m$.

- (i) If $D^n > 1$, then $\pi^*D - E_i$ is big for each $1 \leq i \leq m$.
- (ii) If $D^n = 1$ and $K_X \cdot D^{n-1} < 1 - n$, then $\kappa(\pi^*D - E_i) \geq n - 1$ for each $1 \leq i \leq m$.

Remark 3.2 It is easy to see that part (i) may be false if $D^n = 1$. If H is a hyperplane in \mathbb{P}^n and $P = P_1 \in H$, $m = 1$, then $H^n = 1$ and $\pi^*H - E$ is not big. Similarly, at least for $n = 2$, part (ii) may be false if $K_X \cdot D^{n-1} = 1 - n$. Let $P_1, \dots, P_9 \in \mathbb{P}^2$ be nine points in the plane with a unique cubic curve C passing through the nine points. Let $\pi: X \rightarrow \mathbb{P}^2$ be the blow up at P_1, \dots, P_8 , with corresponding exceptional divisors E_1, \dots, E_8 . Then $K_X = \pi^*(-3L) + \sum_{i=1}^8 E_i$, where L is a line in \mathbb{P}^2 . Let $D = \pi^*C - \sum_{i=1}^8 E_i \sim -K_X$ be the strict transform of C . Then $D^2 = 1$ and $D \cdot K_X = -1$. Let $\tilde{\pi}: \tilde{X} \rightarrow X$ be the blow up of X at P_9 and let E be the exceptional divisor. Then $\kappa(\tilde{\pi}^*D - E) = 0$.

Proof We first treat the case of blowing-up one point, *i.e.*, $m = 1$. For simplicity of notation, we let $P = P_1$ and $E = E_1$. Let $i: E \rightarrow \tilde{X}$ be the inclusion map. From Lemma 2.6, we have an exact sequence

$$0 \rightarrow H^0(\tilde{X}, \mathcal{O}(m\pi^*D - (j+1)E)) \rightarrow H^0(\tilde{X}, \mathcal{O}(m\pi^*D - jE)) \rightarrow H^0(E, i^*\mathcal{O}(m\pi^*D - jE)).$$

Now since $\pi^*D \cdot E = 0$, $i^*\mathcal{O}(m\pi^*D - jE) \cong i^*\mathcal{O}(-jE)$. Recall that we can identify E with \mathbb{P}^{n-1} . Under this identification, $i^*\mathcal{O}(-E) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. It follows that

$$\begin{aligned} \dim H^0(E, i^*\mathcal{O}(m\pi^*D - jE)) &= \dim H^0(E, i^*\mathcal{O}(-jE)) = \dim H^0(\mathbb{P}^{n-1}, \mathcal{O}(j)) \\ &= \binom{j+n-1}{n-1}. \end{aligned}$$

Thus,

$$h^0(m\pi^*D - jE) - h^0(m\pi^*D - (j+1)E) \leq \binom{j+n-1}{n-1}.$$

Then

$$\begin{aligned} h^0(m\pi^*D) - h^0(m\pi^*D - mE) &= \sum_{j=0}^{m-1} h^0(m\pi^*D - jE) - h^0(m\pi^*D - (j+1)E) \\ &\leq \sum_{j=0}^{m-1} \binom{j+n-1}{n-1} = \binom{m+n-1}{n} \\ &\leq \frac{m^n}{n!} + \frac{(n-1)n}{2} \frac{m^{n-1}}{n!} + O(m^{n-2}) \\ &\leq \frac{m^n}{n!} + \frac{1}{2} \frac{m^{n-1}}{(n-2)!} + O(m^{n-2}). \end{aligned}$$

Using Theorem 2.5 to compute $h^0(m\pi^*D)$, we find that

$$h^0(m\pi^*D - mE) \geq \frac{D^n}{n!} m^n - \frac{K_X \cdot D^{n-1}}{2(n-1)!} m^{n-1} - \left(\frac{m^n}{n!} + \frac{1}{2} \frac{m^{n-1}}{(n-2)!} \right) + O(m^{n-2}).$$

Simplifying, we get

$$h^0(m\pi^*D - mE) \geq \frac{D^n - 1}{n!}m^n - \frac{K_X \cdot D^{n-1} + (n-1)}{2(n-1)!}m^{n-1} + O(m^{n-2}),$$

proving the theorem for the case of blowing-up one point.

For the general case, we use the following notation. Let $\tilde{\pi}_i: X_i \rightarrow X$ be the blow-up at P_i , and let $\epsilon_i: \tilde{X} \rightarrow X_i$ be such that $\pi = \tilde{\pi}_i \circ \epsilon_i$. Let $\tilde{E}_i = \tilde{\pi}_i^{-1}(P_i)$ be the exceptional divisor of $\tilde{\pi}_i$ and note that $E_i = \epsilon_i^* \tilde{E}_i$. Then we have the following:

$$h^0(m(\pi^*D - E_i)) = h^0(m(\epsilon_i^*(\tilde{\pi}_i^*D) - \epsilon_i^*\tilde{E}_i)) \geq h^0(m(\tilde{\pi}_i^*D - \tilde{E}_i)).$$

Thus, the general case follows from the previously proved one-point case applied to the blow-ups $\tilde{\pi}_i: X_i \rightarrow X, i = 1, \dots, m$. ■

Proof of Theorems 1.5A and 1.5B Since a major part of the proofs of the two theorems is the same, we will largely do the proofs together, giving separate arguments when necessary. By Lemma 2.4, the assumption that D_i is nef and $D_i^n \geq 1$ for all i implies that D_i is big for each $1 \leq i \leq n$. We first observe that it suffices to consider when $\bigcap_{i=1}^n D_i$ is nonempty. Indeed, suppose that $\bigcap_{i=1}^n D_i$ is empty. Then in the non-Archimedean case, by Theorem 1.3A there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f: \mathbf{k} \rightarrow X \setminus D$ is contained in Z . While in the arithmetic case, by Theorem 1.3B there exists a proper Zariski-closed subset $Z \subset X$ such that for any set R of \mathcal{O}_k -integral points on $X \setminus \bigcup_{i=1}^n D_i$, the set $R \setminus Z$ is finite. Therefore, we now assume that $\bigcap_{i=1}^n D_i$ is not empty.

Since D_1, \dots, D_n intersect transversally, their intersection contains only points. Let $\bigcap_{i=1}^n D_i = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Let $\pi: \tilde{X} \rightarrow X$ be the blow-up along $\mathfrak{p}_1, \dots, \mathfrak{p}_m$, successively, and let $E_j := \pi^{-1}(\mathfrak{p}_j)$ be the exceptional divisor for $1 \leq j \leq m$. For $1 \leq i \leq n$ and $1 \leq j \leq m$, we let $G_{ij} := \pi^*(D_i) - E_j = \tilde{D}_i + E_1 + \dots + E_{j-1} + E_{j+1} + \dots + E_m$, where \tilde{D}_i is the strict transform of D_i under π . Clearly, G_{ij} is effective and by Theorem 3.1, $\kappa(G_{ij}) > 0$. It is clear from the construction and our transversality assumptions that

$$\bigcap_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} G_{ij} = \emptyset \quad \text{and} \quad \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} G_{ij} = \bigcup_{i=1}^n \pi^{-1}(D_i).$$

In the non-Archimedean case, the analytic map $f: \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$ lifts to

$$\tilde{f}: \mathbf{k} \rightarrow \tilde{X} \setminus \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} G_{ij}$$

with $f = \pi \circ \tilde{f}$. Theorem 1.3A implies that the image of \tilde{f} is contained in a proper subvariety of \tilde{X} , hence the image of f is contained in a proper subvariety of X . If $D_i^n > 1$ for all i , then G_{ij} is big for all i and j by Theorem 3.1. Now applying Theorem 1.3A as before yields the desired result.

In the arithmetic case, every point in the intersection $\bigcap_{i=1}^n D_i = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ is k -rational by assumption. Then \tilde{X} and all the G_{ij} are defined over k , since $\pi: \tilde{X} \rightarrow X$ is blown up over k -rational points. Under this construction, any set R of \mathcal{O}_k -integral

points on $X \setminus \cup_{i=1}^n D_i$ lifts to a set \tilde{R} of \mathcal{O}_k -integral points on

$$\tilde{X} \setminus \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} G_{ij}.$$

By Theorem 1.3B, \tilde{R} is contained in a proper Zariski-closed subset \tilde{Z} of \tilde{X} , and hence R is contained in the proper Zariski-closed subset $\pi(\tilde{Z})$ of X . If $D_i^n > 1$ for all i , then G_{ij} is big for all i and j by Theorem 3.1. Now applying Theorem 1.3B as before yields the desired result. ■

Proof of Corollaries 1.6A and 1.6B By Theorem 2.3, for all i ,

$$(-K_X \cdot D_i^{n-1})^n \geq (-K_X)^n (D_i^n)^{n-1} \geq (-K_X)^n > (n-1)^n.$$

Since $-K_X \cdot D_i^{n-1} \geq 0$ for all i , this immediately implies that $K_X \cdot D_i^{n-1} < 1 - n$ for all i . ■

4 Proof of Theorems 1.9A and 1.9B

We first recall the definition and basic properties of rational ruled surfaces (cf. [5, Chapter V.2] and [7, Chapter IV.1]).

Definition 4.1 Let k be a field and let \bar{k} be the algebraic closure of k . A *rational ruled surface* is a surface X (over \bar{k}), together with a surjective morphism $\pi: X \rightarrow \mathbb{P}^1$, such that the fiber X_y is isomorphic to \mathbb{P}^1 for every point $y \in \mathbb{P}^1$. We say $\pi: X \rightarrow \mathbb{P}^1$ is a *rational ruled surface defined over k* if both X and the morphism π are defined over k and X is a rational ruled surface over \bar{k} .

Proposition 4.2 A rational ruled surface X (over \bar{k}) is isomorphic to

$$X_e := \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$$

over \mathbb{P}^1 for some nonnegative integer e . Let F denote a fiber on X and let C_0 denote a section of X such that $\mathcal{O}(C_0) \cong \mathcal{O}_{X_e}(1)$.

- (a) $\text{Pic } X \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $C_0 \subset X$ and F with $C_0^2 = -e$, $F^2 = 0$, and $C_0 \cdot F = 1$.
- (b) Let K_X be the canonical divisor on X . Then $K_X \sim -2C_0 - (2 + e)F$. In particular, $K_X^2 = 8$.
- (c) Let D be a divisor on X equivalent to $aC_0 + bF$ in $\text{Pic } X$.
 - (i) If D is an irreducible curve $\not\sim C_0, F$, then $a, b > 0$, $b \geq ae$, and $D^2 > 0$.
 - (ii) D is big if and only if $a > 0$ and $b > 0$.
 - (iii) D is ample if and only if $a > 0$ and $b > ae$.

Remark 4.3 When $e = 0$, X_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; when $e = 1$, $X_e = X_1$ is \mathbb{P}^2 blown up at one point.

Proof of Theorem 1.9A A rational ruled surface X is isomorphic to X_e with $e \geq 0$ as described in Proposition 4.2. Let D_1 and D_2 be big effective divisors on X . Suppose first that D_1 and D_2 have irreducible components E_1 and E_2 , respectively, with $E_i \not\sim C_0, F$, for $i = 1, 2$. Then by Proposition 4.2, $E_i^2 \geq 1$ for $i = 1, 2$. Let K_X be

the canonical divisor on X . By the Hodge index theorem (or direct calculation), $(K_X.E_i)^2 \geq K_X^2 E_i^2 \geq K_X^2 = 8$ and $K_X.E_i \leq -3$. Then by Theorem 1.5A, the image of an analytic map $f: \mathbf{k} \rightarrow X \setminus (D_1 \cup D_2) \subset X \setminus (E_1 \cup E_2)$ is contained in a proper subvariety of X . Suppose now that, say, D_1 has every irreducible component linearly equivalent to either C_0 or F . Since D_1 is big, by Proposition 4.2, D_1 must contain at least two irreducible components C and F' with $C \sim C_0$ and F' a fiber. Since D_2 is linearly equivalent to a positive integral linear combination of C and F' , there exists a non-constant function $\phi \in \mathbf{k}(X)^*$ with poles and zeros only in the support of D_1 and D_2 . Consider an analytic map $f: \mathbf{k} \rightarrow X \setminus (D_1 \cup D_2)$. Then $\phi \circ f: \mathbf{k} \rightarrow \mathbb{A}^1 \setminus \{0\}$ is analytic, and hence constant. It follows that the image of f is contained in a proper subvariety of X .

Assume further that D_1 and D_2 are ample. By (a), the image of a non-constant analytic map $f: \mathbf{k} \rightarrow X \setminus (D_1 \cup D_2)$ is contained in a curve in X . Let C be the Zariski closure of the image of f in X . If $C \cap (D_1 \cup D_2)$ contains more than one point, then f must be constant. On the other hand, $D_i \cap C \neq \emptyset$ for $i = 1, 2$, since D_1 and D_2 are ample. Therefore, we only need to consider when $C \cap D_1 = C \cap D_2 = \{x\}$ for some $x \in X$.

Let $\pi: \tilde{X} \rightarrow X$ be the blow-up of X at x with exceptional divisor E . Let \tilde{C} be the strict transform of C and \tilde{D}_i the strict transform of D_i , $i = 1, 2$. Then $f: \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$ lifts to $\tilde{f}: \mathbf{k} \rightarrow \tilde{X} \setminus \tilde{D}_1 \cup \tilde{D}_2$ with $f = \pi \circ \tilde{f}$ and the image of \tilde{f} is contained in \tilde{C} . Denote by $m = m_x(C)$ the multiplicity of C at x . If $(C.D_i)_x \geq m = m_x(C) \cdot m_x(D_i)$ for $i = 1, 2$, then each \tilde{D}_i must intersect \tilde{C} at some point on \tilde{X} lying above x . Since D_1 and D_2 intersect transversally, $\cap_{i=1}^2 \tilde{D}_i \cap E = \emptyset$. Thus, there must be at least two points on \tilde{C} lying above x . Consequently, $\tilde{f}: \mathbf{k} \rightarrow \tilde{X} \setminus \tilde{D}_1 \cup \tilde{D}_2$ is constant by Theorem 1.2A, and hence f is also constant. Therefore, it remains to consider when $(C.D_i)_x = m$ for $i = 1$ or $i = 2$.

Without loss of generality, let $(C.D_1)_x = m$. Suppose that C is linearly equivalent to $cC_0 + dF$ and D_1 is linearly equivalent to $aC_0 + bF$. Proposition 4.2 implies that the intersection multiplicity is given by

$$(C.D_1)_x = C.D_1 = ad + c(b - ae).$$

Assume that C is not linearly equivalent to C_0 or F . From Proposition 4.2, we have that $a, b - ae, c, d > 0$. By taking F to be the fiber passing through x , we see that $c = C.F \geq m$. Then

$$m = C.D_1 = ad + c(b - ae) > c \geq m,$$

a contradiction. This proves the first statement in (ii). The second statement follows from Theorem 1.2A and the observation that if some fiber or some section $C \sim C_0$ intersects $D_1 \cup D_2$ in exactly one point, then since every fiber and every section is isomorphic to \mathbb{P}^1 and \mathbb{A}^1 admits a non-constant analytic map, there exists a non-constant analytic map from \mathbf{k} to $X \setminus D_1 \cup D_2$.

For the last statement, note that any fiber or section must intersect one of D_1 or D_2 transversally at x , as any fiber or section is nonsingular and D_1 and D_2 intersect transversally. If $D_i.F \geq 2$ and $D_i.C_0 \geq 2$ for $i = 1, 2$, then this implies that every fiber and every section intersects $D_1 \cup D_2$ in more than one point. ■

Proof of Theorem 1.9B We will use the same notation as in the proof of Theorem 1.9A and only give arguments that are different from the non-Archimedean case. We note that section C_0 may not be defined over k (and consequently, we work with linear equivalence over \bar{k} below). Let D_1 and D_2 be big effective divisors on X . Suppose first that D_1 and D_2 have irreducible components E_1 and E_2 , respectively, with $E_i \sim C_0, F$, for $i = 1, 2$. We have shown that $E_i^2 \geq 1$ for $i = 1, 2$ and $K_X \cdot E_i \leq -3$. Then by Theorem 1.5B, any set R of \mathcal{O}_k -integral points on $X \setminus (D_1 \cup D_2) \subset X \setminus (E_1 \cup E_2)$ is contained in a proper Zariski-closed subset of X . Suppose now that, say, D_1 has every irreducible component linearly equivalent to either C_0 or F . Since D_1 is big, by Proposition 4.2, D_1 must contain at least two irreducible components C and F' with $C \sim C_0$ and F' a fiber. Since D_2 is linearly equivalent to a positive integral linear combination of C and F' , there exists a non-constant function $\phi \in \mathbb{Q}(X)^*$ with poles and zeros only in the support of D_1 and D_2 . Since all the irreducible components of D_1 and D_2 are defined over k , we may take ϕ to be defined over k . Then $\phi(R)$ is a set of \mathcal{O}_k -integral points on $\mathbb{A}^1 \setminus \{0\}$ which is finite. Consequently, we can write $\phi(R) = \{\alpha_1, \dots, \alpha_m\} \subset k \setminus \{0\}$. It follows that the image of R is contained in a proper subvariety of X given by $\bigcup_{i=1}^m \{\mathfrak{p} \in X \mid \phi(\mathfrak{p}) = \alpha_i\}$.

Assume furthermore that D_1 and D_2 are ample. Suppose that R is an infinite set of \mathcal{O}_k -integral points on $X \setminus (D_1 \cup D_2)$. By (a), one can find a Zariski closed curve C containing an infinite subset R' of R . Then C must be defined over k and $C \cap (D_1 \cup D_2)$ contains at most two points by Siegel's theorem. If $C \cap (D_1 \cup D_2)$ contains exactly two points $\mathfrak{p}, \mathfrak{q}$, then either \mathfrak{p} and \mathfrak{q} are k -rational points or they are conjugate to each other. Since C, D_1 , and D_2 are defined over k , the second case implies that $C \cap D_1 = C \cap D_2 = \{\mathfrak{p}, \mathfrak{q}\}$ and hence $\{\mathfrak{p}, \mathfrak{q}\} \subset D_1 \cap D_2$, contradicting our assumption that $D_1 \cap D_2$ contains only k -rational points. The remaining case that \mathfrak{p} and \mathfrak{q} are both k -rational points is impossible due to Theorem 1.3B(iii). Therefore, $C \cap (D_1 \cup D_2)$ contains at most one point. On the other hand, $D_i \cap C \neq \emptyset$ for $i = 1, 2$, since D_1 and D_2 are ample. Therefore, we only need to consider when $C \cap D_1 = C \cap D_2 = \{x\}$ for some $x \in X$. Moreover, x is a k -rational point, since it is a point in $D_1 \cap D_2$.

Let $\pi: \tilde{X} \rightarrow X$ be the blow-up of X at x with exceptional divisor E . Let \tilde{C} be the strict transform of C and \tilde{D}_i the strict transform of $D_i, i = 1, 2$. We note that \tilde{X}, \tilde{C} , and \tilde{D}_i are all defined over k , since x is k -rational. Then this infinite set R' of \mathcal{O}_k -integral points on $X \setminus (D_1 \cup D_2)$ lifts to an infinite set \tilde{R}' of \mathcal{O}_k -integral points on $\tilde{X} \setminus (\tilde{D}_1 \cup \tilde{D}_2)$ with $R' = \pi(\tilde{R}')$. Denote by $m = m_x(C)$ the multiplicity of C at x . If $(C \cdot D_i)_x > m = m_x(C) \cdot m_x(D_i)$ for $i = 1, 2$, then each \tilde{D}_i must intersect \tilde{C} at some point on \tilde{X} lying above x . Since D_1 and D_2 intersect transversally, $\bigcap_{i=1}^2 \tilde{D}_i \cap E = \emptyset$. Thus, there must be at least two points on \tilde{C} lying above x . As before, by Siegel's theorem, it suffices to consider when there are exactly two points in $\tilde{C} \cap (\tilde{D}_1 \cup \tilde{D}_2)$. The above construction shows that this can only happen if $\tilde{C} \cap \tilde{D}_1 = \{\mathfrak{p}\}, \tilde{C} \cap \tilde{D}_2 = \{\mathfrak{q}\}$, and $\mathfrak{p} \neq \mathfrak{q}$. Since \tilde{C} and $\tilde{D}_i, i = 1, 2$, are defined over k , \mathfrak{p} and \mathfrak{q} must be k -rational points. However, Theorem 1.3B(iii) implies that any set of \mathcal{O}_k -integral points on $\tilde{C} \setminus (\tilde{D}_1 \cup \tilde{D}_2)$ is finite, which yields a contradiction. Therefore, it remains to consider when $(C \cdot D_i)_x = m$ for $i = 1$ or $i = 2$. The same argument as in the proof of Theorem 1.9A shows that C is linearly equivalent to C_0 or F under this assumption. This proves the first statement in (ii).

For the second statement, we continue our argument from above and consider when C is linearly equivalent to C_0 or F . Suppose that C intersects $D_1 \cup D_2$ in at least two points. If there are two k -rational points or more than two points in $C \cap \{D_1 \cup D_2\}$, then we get a contradiction again by Theorem 1.3B(iii) or by Siegel's theorem. The only case left is $C \cap \{D_1 \cup D_2\} = \{p, q\}$, and one of the points is not k -rational. Since C and the D_i are defined over k , this implies that p and q are conjugate over k and $C \cap D_1 = C \cap D_2 = \{p, q\}$. Then $\{p, q\} \subset D_1 \cap D_2$, which contradicts our assumption that the intersection points of D_1 and D_2 are k -rational. Therefore, the cardinality of R must be finite. For the converse direction of the second statement, we first observe that if an irreducible curve C defined over k intersects $D_1 \cup D_2$ in exactly one point, then this point must be k -rational. Furthermore, if C is linearly equivalent to F or C_0 then C is a fiber or a section which is isomorphic to \mathbb{P}^1 . Indeed, C is k -isomorphic to \mathbb{P}^1 since C is defined over k and $C(k)$ is not empty. Since \mathbb{A}^1 admits infinitely many \mathcal{O}_k -integral points, there exists infinitely many \mathcal{O}_k -integral points on $C \setminus (D_1 \cup D_2) \subset X \setminus (D_1 \cup D_2)$.

For the last statement, the proof is the same as in Theorem 1.9A. ■

References

- [1] T. T. H. An, *A defect relation for non-Archimedean analytic curves in arbitrary projective varieties*. Proc. Amer. Math. Soc. 135(2007), 1255–1261. <http://dx.doi.org/10.1090/S0002-9939-06-08591-1>
- [2] T. T. H. An, A. Levin, and J. T.-Y. Wang *A p -adic Nevanlinna-Diophantine correspondence*. Acta Arith. 146(2011), no. 4, 379–397. <http://dx.doi.org/10.4064/aa146-4-5>
- [3] T. T. H. An, J. T.-Y. Wang, and P.-M. Wong, *Non-Archimedean analytic curves in the complements of hypersurface divisors*. J. Number Theory 128(2008), no. 8, 2275–2281. <http://dx.doi.org/10.1016/j.jnt.2007.10.011>
- [4] W. Fulton, *Intersection theory*. Second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Springer-Verlag, Berlin, 1998.
- [5] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, 52, Springer-Verlag, New York, 1977.
- [6] S. L. Kleiman, *Toward a numerical theory of ampleness*. Ann. of Math. 84(1966), 293–344. <http://dx.doi.org/10.2307/1970447>
- [7] J. Kollár, *Rational curves on algebraic varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 32, Springer-Verlag, Berlin, 1996.
- [8] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*. A Series of Modern Surveys in Mathematics vol. 48, Springer-Verlag, Berlin, 2004.
- [9] A. Levin, *Variations on a theme of Runge: effective determination of integral points on certain varieties*. J. Théor. Nombres Bordeaux 20(2008), no. 2, 385–417. <http://dx.doi.org/10.5802/jtnb.634>
- [10] C.-W. Lin and J. T.-Y. Wang, *Generalizations of rigid analytic Picard theorems*. Proc. Amer. Math. Soc. 138(2010), 133–139. <http://dx.doi.org/10.1090/S0002-9939-09-10038-2>
- [11] T. Matsusaka, *On numerically effective divisors with positive self-intersection numbers*. J. Fac. Sci. Univ. Tokyo 38(1991), no. 1, 165–183.
- [12] J.-P. Serre, *Lectures on the Mordell-Weil theorem*. In: Aspects of Mathematics, third ed., Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [13] P. Vojta, *Diophantine approximations and value distribution theory*. Lecture Notes in Mathematics, 1239, Springer-Verlag, Berlin, 1987.

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