Glasgow Mathematical Journal (2025), **67**, pp. 228–231
doi:10.1017/S0017089524000363

Glasgow Mathematical Journal

RESEARCH ARTICLE

Actions of nilpotent groups on nilpotent groups

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Received: 31 January 2024; **Revised:** 10 November 2024; **Accepted:** 5 December 2024;
First published online: 16 January 2025

Keywords: nilpotent-by-nilpotent group actions; primary decompositions in cohomology; fixed points of non-coprime actions

2020 Mathematics Subject Classification: *Primary* - 05E18, 20D15, 20J06; *Secondary* - 20D45, 55N45

Abstract
For finite nilpotent groups J and N, suppose J acts on N via automorphisms. We exhibit a decomposition of the first cohomology set in terms of the first cohomologies of the Sylow p-subgroups of J that mirrors the primary decomposition of $H^1(J, N)$ for abelian N. We then show that if $N \rtimes J$ acts on some non-empty set Ω, where the action of N is transitive and for each prime p a Sylow p-subgroup of J fixes an element of Ω, then J fixes an element of Ω.

1. Introduction

Given a finite nilpotent group J acting on a finite nilpotent group N via automorphisms, crossed homomorphisms are maps $\varphi : J \to N$ satisfying $\varphi(jj') = \varphi(j)\varphi(j')^{j^{-1}}$ for all $j, j' \in J$. Two such maps φ and φ' are cohomologous if there exists $n \in N$ such that $\varphi'(j) = n^{-1}\varphi(j)n^{j^{-1}}$ for all $j \in J$; in this case, we write $\varphi \sim \varphi'$. We define the first cohomology $H^1(J, N)$ to be the pointed set $Z^1(J, N)$ of crossed homomorphisms modulo this equivalence relation where the distinguished point corresponds to the class containing the map taking each element of J to the identity of N.

We first show that the cohomology set $H^1(J, N)$ decomposes in terms of the first cohomologies of the Sylow p-subgroups J_p of J as follows:

Lemma 1. *For finite nilpotent groups J and N, suppose J acts on N via automorphisms. Then the map $\varphi \mapsto \times_{p\in\mathcal{D}}\varphi|_{J_p}$ for $\varphi \in H^1(J, N)$ induces an isomorphism $H^1(J, N) \cong \times_{p\in\mathcal{D}} H^1(J_p, N)^{J'_p}$ of pointed sets, where $\mathcal{D}$ denotes the shared prime divisors of $|J|$ and $|N|$, and for each p, J_p is the Sylow p-subgroup of J and J'_p is the Hall p'-subgroup of J.*

This parallels the well-known primary decomposition of $H^1(J, N)$ for abelian N (see Section 3 for details). As the bijective correspondence between $H^1(J, N)$ and the N-conjugacy classes of complements to N in $N \rtimes J$ continues to hold for nonabelian N [6, Exer. 1 in §I.5.1], Lemma 1 provides an alternate proof of a result of Losey and Stonehewer [5]:

Proposition 2 (Losey and Stonehewer). *Two nilpotent complements of a normal nilpotent subgroup in a finite group are conjugate if and only if they are locally conjugate.*

Here, two subgroups $H, H' \leq G$ are locally conjugate if a Sylow p-subgroup of H is conjugate to a Sylow p-subgroup of H' for each prime p. It also readily follows that:

Proposition 3. *Let G be a finite split extension over a nilpotent subgroup N such that G/N is nilpotent. If for each prime p, there is a Sylow p-subgroup S of G such that any two complements of $S \cap N$ in S are conjugate in G, then any two complements of N in G are conjugate.*

We then establish a fixed point result for nilpotent-by-nilpotent actions in the style of Glauberman:

Theorem 4. *For finite nilpotent groups J and N, suppose J acts on N via automorphisms and that the induced semidirect product $N \rtimes J$ acts on some non-empty set Ω where the action of N is transitive. If for each prime p, a Sylow p-subgroup of J fixes an element of Ω, then J fixes an element of Ω.*

Glauberman showed that this result holds whenever the orders of N and J are coprime, without any further restrictions on N or J [4, Thm. 4]. Thus, this result is only interesting when $|N|$ and $|J|$ share one or more prime divisors (i.e. when the action is *non-coprime*). Analogous results hold if N is abelian or if N is nilpotent and $N \rtimes J$ is supersoluble [2].

1.1. Outline

In the remainder of this section, we introduce some notation. We then prove the results in Section 2 and conclude in Section 3.

1.2. Notation

All groups in this note are finite. For a nilpotent group J, we let $J_p \in \mathrm{Syl}_p(J)$ denote its unique Sylow p-subgroup and J'_p denote its Hall p'-subgroup so that $J \cong J_p \times J'_p$. We let $g^\gamma = \gamma^{-1} g \gamma$ for $g, \gamma \in G$. We otherwise use standard notation from group theory that can be found in Doerk and Hawkes [3].

For a subgroup $K \leq J$, we let $\varphi|_K$ denote the restriction of $\varphi \in Z^1(J,N)$ to K and let $\mathrm{res}^J_K : H^1(J,N) \to H^1(K,N)$ be the map induced in cohomology. For $\varphi \in Z^1(K,N)$ and $j \in J$, define $\varphi^j(x) = \varphi(x^{j^{-1}})^j$. We say φ is J-invariant if $\mathrm{res}^K_{K\cap K^j}\varphi \sim \mathrm{res}^{K^j}_{K\cap K^j}\varphi^j$ for all $j \in J$ and let $\mathrm{inv}_J H^1(K,N)$ be the set of J-invariant elements in $H^1(K,N)$. For any $\varphi \in Z^1(J,N)$, we have $\varphi^j(x) = n^{-1}\varphi(x)n^{x^{-1}}$ where $n = \varphi(j^{-1})$ so that $\varphi^j \sim \varphi$. Consequently, $\mathrm{res}^J_K H^1(J,N) \subseteq \mathrm{inv}_J H^1(K,N)$.

For nilpotent J and $\varphi \in Z^1(J_p,N)$, any $j \in J$ may be written $j = j_p \times j'_p$ for $j_p \in J_p$ and $j'_p \in J'_p$ so that $\varphi^j(x) = \varphi(x^{j_p^{-1}})^{j_p j'_p} = \varphi'(x)^{j'_p}$ for some $\varphi' \sim \varphi$. It follows that $\mathrm{inv}_J H^1(J_p,N) = H^1(J_p,N)^{J'_p}$, that is, the J-invariant elements of $H^1(J_p,N)$ are those fixed under conjugation by J'_p.

To each complement K of N in NJ, we associate $\varphi_K \in Z^1(J,N)$ as follows. For $j \in J$, we have $j = n_j^{-1}k_j$ for unique $n_j \in N$ and $k_j \in K$; we then let $\varphi_K(j) = n_j = k_j j^{-1}$. Conversely, for any $\varphi \in Z^1(J,N)$, the subgroup $F(\varphi) = \{\varphi(j)j\}_{j\in J}$ complements N in NJ. In particular, $F(\varphi_K) = K$. Furthermore, $F(\varphi)$ and $F(\varphi')$ are N-conjugate in NJ if and only if $\varphi \sim \varphi'$ so that F induces a correspondence between $H^1(J,N)$ and the N-conjugacy classes of complements to N in NJ. See Serre [6, Ch. I §5] for further details on nonabelian group cohomology.

2. Proofs of results

We begin by establishing Lemma 1.

Proof of Lemma 1. As N is nilpotent, the natural projection maps $N \to N_p$ induce an isomorphism:

$$H^1(J,N) \cong \times_{p\in\mathcal{D}} H^1(J,N_p), \tag{1}$$

where terms $p \notin \mathcal{D}$ drop by the Schur–Zassenhaus theorem [3, Thm. A.11.3]. Thus, we may focus our attention on $H^1(J,N_q)$ for some prime q. If J is also a q-group, we are done. Otherwise, we may consider the inflation-restriction exact sequence [6, Sec. I.5.8]:

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$$1 \to H^1(J'_q, N_q^{J_q}) \to H^1(J, N_q) \xrightarrow{\mathrm{res}^J_{J_q}} H^1(J_q, N_q)^{J'_q}.$$

As $H^1(J'_q, N_q^{J_q})$ is trivial, $\mathrm{res}^J_{J_q}$ is injective. For any $\varphi \in H^1(J_q, N_q)^{J'_q}$, we may define $\tilde{\varphi}$ in terms of a representative crossed homomorphism as $\tilde{\varphi}(j'j) = \varphi(j)$ for $j \in J_q$ and $j' \in J'_q$. It is straightforward to verify that $\tilde{\varphi} \in Z^1(J, N_q)$ and $\tilde{\varphi}|_{J_q} \sim \varphi$. Thus, $\mathrm{res}^J_{J_q}$ is also surjective and thus an isomorphism. Let $\nu : H^1(J_q, N_q) \to H^1(J_q, N)$ denote the map induced by inclusion. From the decomposition (1), $H^1(J_q, N_q) \cong H^1(J_q, N)$, and as

$$H^1(J_q, N_q) \xrightarrow{\nu} H^1(J_q, N) \to H^1(J_q, N'_q)$$

is exact [6, Prop. I.38] where $H^1(J_q, N'_q)$ is trivial, it follows that ν is surjective and hence an isomorphism. As $\varphi \in H^1(J_p, N_p)^{J'_p}$ if and only if $\nu(\varphi) \in H^1(J_p, N)^{J'_p}$, it follows that $\Phi : H^1(J, N) \to \times_{p\in\mathcal{D}} H^1(J_p, N)^{J'_p}$ given by the composition $\varphi \mapsto \times_{p\in\mathcal{D}} \varphi|_{J_p}$ induces the desired isomorphism:

$$H^1(J, N) \cong \times_{p\in\mathcal{D}} H^1(J_p, N_p)^{J'_p} \cong \times_{p\in\mathcal{D}} H^1(J_p, N)^{J'_p}. \tag{2}$$

□

We now show how Propositions 2 and 3 follow from Lemma 1.

Proof of Proposition 2. Suppose J and J' each complement $N \lhd G$ as described in the hypotheses of the proposition. Let φ be a crossed homomorphism representing J' in $H^1(J, N)$. By hypothesis, $\varphi|_{J_p} \sim 1|_{J_p}$ for every prime p, where $1 \in Z^1(J, N)$ represents the distinguished point. Lemma 1 implies $\varphi \sim 1$ so that J and J' are conjugate. □

Proof of Proposition 3. Suppose $G \cong N \rtimes J$ satisfies the hypotheses of the proposition. Fix a prime p. Without loss, we may suppose any two complements of N_p in $S = J_p N_p$ are conjugate in G. If J'_p is such a complement, then $J'_p = (J_p)^g$ for some $g \in G$ so that $J'_p = (J_p)^{jn} = (J_p)^n$ for some $j \in J$ and $n \in N$. In particular, J'_p is conjugate to J_p in $J_p N$. Thus, $H^1(J_p, N)$ is trivial. As the choice of prime p was arbitrary, Lemma 1 implies that $H^1(J, N)$ is also trivial, allowing us to conclude. □

To prove Theorem 4, we also require:

Proposition 5. *Let H be a subgroup of $G \cong N \rtimes J$ where N and J are nilpotent. If for each prime p, H contains a conjugate of some $J_p \in \mathrm{Syl}_p(J)$, then H contains a conjugate of J.*

Proof of Proposition 5. It follows from the hypotheses that H supplements N in G. We induct on the order of G. If H is a p-group or all of G, the result is immediate. If multiple primes divide $|N|$, we have the nontrivial decomposition $N \cong N_p \times N'_p$ for some prime p. Induction in G/N_p implies $J^{n_0} \le HN_p$ for some $n_0 \in N'_p$. Induction in G/N'_p implies $J^{n_1} \le HN'_p$ for some $n_1 \in N_p$. Thus, $J^{n_0 n_1} \le HN_p \cap HN'_p = H$, where the last equality proceeds from the following argument of Losey and Stonehewer [5]. Suppose $g \in HN_p \cap HN'_p$ so that $g = h_0 n_0 = h_1 n_1$ for some $h_0, h_1 \in H$, $n_0 \in N_p$ and $n_1 \in N'_p$. Then $(h_1)^{-1} h_0 = n_1 (n_0)^{-1} \in H$. As n_0 and n_1 commute and have coprime orders, it follows that $n_0, n_1 \in H$ so $g \in H$.

We now proceed under the assumption that $N = N_q$ for some prime q. Upon switching to a conjugate of H if necessary, we may suppose that $J_q \le H$. Let Z denote the center of N. If $Z \cap H$ were nontrivial, then induction in $G/(Z \cap H)$ would allow us to conclude. Otherwise, in G/Z, induction implies that $J^g Z/Z \le ZH/Z$ for some $g \in G$. Let $\psi : H \to HZ/Z$ denote the isomorphism between H and HZ/Z. Then $K = \psi^{-1}(J^g Z/Z)$ complements $N \cap H$ in H and N in G. Let $\varphi \in Z^1(K, N)$ correspond to J. Then $\varphi|_{K_q}$ corresponds to J_q where $[\varphi|_{K_q}] \in H^1(K_q, H \cap N)^{K'_q} \cong H^1(K, H \cap N)$. In particular, there exists a complement, say L, to $H \cap N$ in H that contains J_q. L will also complement N in G, and as $\mathrm{Syl}_p(L) \subseteq \mathrm{Syl}_p(G)$ for all primes $p \ne q$, we may apply Proposition 2 to conclude that $J^{g'} = L \le H$ for some $g' \in G$. □

With this, we are prepared to prove Theorem 4.

Proof of Theorem 4. Given J, N, and Ω as described in the hypotheses of the theorem, let $G = N \rtimes J$ denote the induced semidirect product and consider the stabilizer subgroup G_α for some $\alpha \in \Omega$. As N

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acts transitively, $G = NG_\alpha$. For each prime p, the hypotheses of the theorem imply $(J_p)^{n_p} \leq G_\alpha$ for some $J_p \in \mathrm{Syl}_p(J)$ and $n_p \in N$ so that Proposition 5 allows us to conclude $J^g \leq G_\alpha$ for some $g \in G$. It follows that J fixes $g \cdot \alpha$. □

3. Conclusion

We conclude with a brief discussion of analogous results in the abelian case. For arbitrary J acting on abelian N, the restriction map $\mathrm{res}^J_{J_p} : H^1(J, N)_{(p)} \xrightarrow{\cong} \mathrm{inv}_J H^1(J_p, N)$ induces an isomorphism for each prime p, where $H^1(J, N)_{(p)}$ is the p-primary component of $H^1(J, N)$ and $J_p \in \mathrm{Syl}_p(J)$. Consequently, it follows from the primary decomposition of $H^1(J, N)$ that [1, Thm. III.10.3]:

$$H^1(J, N) \cong \oplus_{p \in \mathcal{D}} \mathrm{inv}_J H^1(J_p, N). \tag{3}$$

Furthermore, for abelian N, suppose $G = N \rtimes J$ acts on some non-empty set Ω, where the action of N is transitive, and for each prime p a Sylow p-subgroup of J fixes an element of Ω. Then, for arbitrary $\alpha \in \Omega$, the stabilizer G_α splits over $G_\alpha \cap N$ by Gaschütz's theorem [3, Thm. A.11.2] and is locally conjugate and thus conjugate to J by an argument analogous to the proof of Proposition 5. In particular, J fixes an element of Ω. In this note, we find that the decomposition (3) and fixed point result continue to hold for nilpotent N if J is also nilpotent.

Acknowledgments. The author thanks the editor C. M. Roney-Dougal and an anonymous reviewer for detailed feedback which considerably improved the manuscript.

Competing interests. The author declares none.

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