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THE MOD 2 K-HOMOLOGY OF $\Omega^3 S^3 X$

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1. Introduction. In order to compute the group $K_*(\Omega^3 S^3 X; \mathbf{Z}/2)$ when X is a finite, torsion free CW-complex we apply the techniques developed by Snaith in [**38**], [**39**], [**40**], [**41**] which were used in [**42**] to determine the Atiyah-Hirzebruch spectral sequence ([**11**], [**1**, Part III])

$$H_*(\Omega^2 S^3 X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^2 S^3 X; \mathbf{Z}/2)$$

for X as above. Roughly speaking the method consists in defining certain classes in $K_*(\Omega^3 S^3 X; \mathbf{Z}/2)$ via the π-equivariant mod 2 K-homology of $S^2 \times Y^2$,

$$K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2),$$

([**35**]), π the cyclic group of order 2 (acting antipodally on S^2, by permuting factors in Y^2, and diagonally on $S^2 \times Y^2$), Y a finite subcomplex of $\Omega^3 S^3 X$, and then showing that the classes so produced map under the edge homomorphism to cycles (in the E_1-term of the Atiyah-Hirzebruch spectral sequence for

$$K_*\left(S^2 \underset{\pi}{\times} (\Omega^3 S^3 X)^2; \mathbf{Z}/2)\right)$$

which determine certain homology classes of $H_*(\Omega^3 S^3 X; \mathbf{Z}/2)$, thus exhibiting these as infinite cycles of the spectral sequence

$$H_*(\Omega^3 S^3 X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^3 S^3 X; \mathbf{Z}/2).$$

Use of the infinite cycles so produced and of homotopy properties of the iterated loop spaces [**37**] will reduce the determination of the Atiyah-Hirzebruch spectral sequence for $K_*(\Omega^3 S^3 X; \mathbf{Z}/2)$ to homological algebra.

The main technicality required by the procedure outlined above is the Rothenberg-Steenrod spectral sequence for K-theory introduced by Hodgkin, [**25**], and exploited by Snaith in the papers above to make computations in K-theory. Our calculations heavily rely on the work of this last author.

We also compute the algebra structure of $K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$ making use of the rich knowledge in existence on the stable splitting of double

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loops of spheres [**15**], [**17**], [**20**], and by applying a useful result of F. R. Cohen [**19**, P. III] concerning the torsion in the homology of the double loop of a space, Ω^2S^2X, which allows us to conveniently relate the mod 2 exact couples [**32**] of ordinary homology and K-homology through the Atiyah-Hirzebruch spectral sequence. $K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$, as a vector space, was known by the result of Snaith, [**42**], on $K_*(\Omega^2S^3X; \mathbf{Z}/2)$ mentioned above.

Underlying the computation of the Atiyah-Hirzebruch spectral sequences above is the description of $H_*(\Omega^nS^nX; \mathbf{Z}/2)$ in terms of homology operations, [**14**], to which we dedicate the first section.

The paper is arranged as follows. In Section 2 we briefly record the definition of the mod 2 Dyer-Lashof operations in the manner of Browder [**14**], which will be suitable for our purposes in later sections, and we then list the properties of the operations for finite loop spaces following here F. R. Cohen [**19**, Part III] from whom we also take a result on the torsion in $H_*(\Omega^2S^2Y)$ useful for our computations in Section 6. In Section 3 we collect the necessary notions on $K\mathbf{Z}/2$ and $K^\pi\mathbf{Z}/2$-theory we will require in coming sections. Section 4 contains the technical features of the Rothenberg-Steenrod spectral sequence necessary for our computations in Section 5; these results are suitable analogs of some propositions of [**41**]. Section 5 consists of the proofs of the main theorems determining the Atiyah-Hirzebruch spectral sequences for

$$K(\Omega^\nu S^3X; \mathbf{Z}/2), \quad \nu = 1, 2.$$

Finally in Section 6 we determine the algebra structure of $K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$, after a quick review on the stable splitting of Ω^nS^nX due to Snaith [**37**] and further improved in [**21**], [**17**].

The main result of Section 5, namely the determination of the Atiyah-Hirzebruch spectral sequence for Ω^3S^3X, X a connected CW-complex, has been proved by J. McLure, (private communication), using more sophisticated techniques than ours.

2. Dyer-Lashof operations in mod 2 homology. In this section we introduce the Dyer-Lashof operations mod 2 in the manner of Browder [**14**]. Although May's theory of operads [**30**] is essential for the study of the properties of the homology operations [**19**, Parts I and III] the simplicity of Browder's exposition in [**14**] will suffice for our analysis of the Atiyah-Hirzebruch spectral sequences

$$H_*(\Omega^\nu S^3X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^\nu S^3X; \mathbf{Z}/2) \quad \nu = 2, 3$$

via the groups

$$H_*\left(S^{\nu-1} \underset{\pi}{\times} (\Omega^\nu S^3X)^2; \mathbf{Z}/2\right) \quad \text{and} \quad K_*\left(S^{\nu-1} \underset{\pi}{\times} (\Omega^\nu S^3X)^2; \mathbf{Z}/2\right)$$

$S^{\nu-1}$ the sphere, ν as above, (see Section 5). Moreover, we present in Theorems 2.3 to 2.5 only the properties of the mod 2 Dyer-Lashof operations for finite loop spaces, taken from [**19**, Part III].

2.1. *Definitions*. The H_n structure of a space X is given by an equivariant map

$$\phi: S^n \times (X \times X) \to X$$

where π acts by the antipodal action on S^n and by permuting factors on $X \times X$, while X is a trivial π-space, [**14**].

Following Browder [**14**], a map

$$\phi_*: H(S^n) \otimes H_*(X, A) \otimes H_*(X, A) \to H_*(X, A)$$

is defined by composing the map induced by

$$\nabla: C(S^n) \otimes C(X) \otimes C(X) \to C(S^n \times X \times X),$$

(the natural map of normalized singular chains) and the map induced by ϕ. A is any coefficient system. The Browder operation λ_n is then defined by

$$\text{(2.2)} \quad \lambda_n(x, y) = \phi_*(\gamma \otimes x \otimes y), \quad x, y \in H_*(X, A)$$

where γ is a generator of $H_*(S^n)$.

The composite

$$C(S^n) \otimes C(X) \otimes C(X) \xrightarrow{\nabla} C(S^n \times X \times X) \xrightarrow{\phi_\#} C(X)$$

factors through the collapsed module, so as to give

$$C(S^n) \otimes C(X) \otimes C(X) \xrightarrow{\eta} C(S^n) \bigotimes_{\pi} (C(X) \otimes C(X)) \xrightarrow{\phi_\#} C(X)$$

and then the method of Steenrod, used to define cohomology operations, can be paralleled in this situation [**14**]. This consists in constructing elementary complexes $M(u, q)$ such that every chain map

$$f: M(u, q) \to C(X)$$

defines a homology class $\bar{u} \in H_q(X, \mathbf{Z}/2)$ and conversely, for every $\bar{u} \in H_q(X, \mathbf{Z}/2)$ a representative chain u of $\bar{u}$ can be chosen which gives a map

$$f: M(u, q) \to C(X).$$

Thus a map

$$f_\#: C(S^n) \otimes M \otimes M \to C(S^n) \otimes C(X) \otimes C(X)$$

is defined, which is equivariant and so induces

$$\bar{f}:C(S^n) \underset{\pi}{\otimes} (M \otimes M) \to C(S^n) \underset{\pi}{\otimes} (C(X) \otimes C(X));$$

composing $\bar{f}$ with $\phi_\#$ one obtains

$$\phi_\# \bar{f}:C(S^n) \underset{\pi}{\otimes} (M \otimes M) \to C(X)$$

and this induces

$$\Phi:H_*\left(C(S^n) \underset{\pi}{\otimes} (M \otimes M)\right) \to H_*(X).$$

By the methods of Steenrod it can be shown [**14**] that any two chain representations of the cycle u give the same homomorphism Φ.

The group

$$H_*\left(C(S^n) \underset{\pi}{\otimes} (M \otimes M)\right)$$

is the homology of RP^n, the n-dimensional projective space, with coefficients $H_*(M \otimes M)$. The mth operation of Araki and Kudo is defined by

$$(2.3) \qquad Q_m(\bar{u}) = \{\phi_\pi \bar{f}(e_m) \otimes u \otimes u\} = \Phi(\xi_m)$$

where ξ_m is the generator of $H_m(RP^n, A)$, with

$$A = u \otimes u \otimes \mathbf{Z}/2,$$

u as above.

In order to consider the homology operations as abstract elements of an algebraic structure the following change of notation is useful.

2.2. *Definition.* ([**28**], Definition 2.3). Let X be an H_n-space, $x \in H_q(X, \mathbf{Z}/2)$, and define

$$Q^s:H_q(X, \mathbf{Z}/2) \to H_{q+s}(X; \mathbf{Z}/2), \; s \leqq q + n$$

by:

$$Q^s(x) = 0 \text{ if } s < q \quad \text{and} \quad Q^s(x) = Q_{s-q}(x) \quad \text{if } s \geqq q.$$

We record the mod 2 cases of Theorems 1.1 to 1.4 of [**19**, Part III]. Through Theorems 2.3-2.6 the coefficients $\mathbf{Z}/2$ are understood.

2.3. THEOREM. ([**19**], P. III, Theorem 1.1). *There exist homomorphisms*

$$Q^s:H_q(X) \to H_{q+s}(X) \quad s \geqq 0, s - q < n,$$

natural with respect to maps of H_n-spaces, such that

a) $Q^s x = 0$ *if* $s < \deg(x)$, $x \in H_*(X)$.

b) $Q^s x = x^2$ *if* $s = \deg(x)$, $x \in H_*(X)$.

c) $Q^s \phi = 0$ *if* $s > 0$, $\phi \in H_0(X)$ *the identity element.*

d) *The following Cartan formulas hold*:

i) $$Q^s(x \otimes y) = \sum_{i+j=s} Q^i(x) \otimes Q^j(y), \quad x \otimes y \in H_*(X \times Y),$$

ii) $$Q^s(xy) = \sum_{i+j=s} Q^i(x)Q^i(y), \quad x, y \in H_*(X),$$

iii) $$\psi Q^s(x) = \sum_{i+j=s} Q^i(x') \otimes Q^j(x''),$$

if

$$\psi(x) = \sum x' \otimes x'', \quad x \in H_*(X).$$

e) *The Adem relations hold*:

$$Q^r Q^s = \sum_i \binom{i - s - 1}{2i - r} Q^{r+s-i} Q^i.$$

f) *The Nishida relations hold*: *If*

$$Sq^r_*: H_*(X) \to H_*(X)$$

is dual to Sq^r_*, *then*

$$Sq^r_* Q^s = \sum_i \binom{s - r}{r - 2i} Q^{s-r+i} Sq^i_*.$$

The next theorem states the properties of the Browder operation which are relevant to our purposes.

2.4. THEOREM. (Ibid, Theorem 1.2). *There exist homomorphisms*

$$\lambda_n: H_q(X) \otimes H_r(X) \to H_{q+r+n}(X),$$

natural with respect to maps of H_n *spaces which satisfy*:

a) *If* X *is an* H_{n+1} *space*, $\lambda_n(x, y) = 0$, *for* $x, y \in H_*(X)$.

b) $\lambda_0(x, y) = xy + yx$, *for* $x, y \in H_*(X)$.

c) $\lambda_n(x, y) = \lambda(y, x)$, $x, y \in H_*(X)$, *and* $\lambda_n(x, x) = 0$.

d) $\lambda_n(\phi, x) = 0 = \lambda_n(x, \phi)$, $\phi \in H_0(X)$ *the identity element of* $H_*(X)$ *and* $x \in H_*(X)$.

e) $$Sq^s_* \lambda_n(x, y) = \sum_{i+j=s} \lambda_n[Sq^i_* x, Sq^j_* y].$$

f) $$\lambda_n[x, Q^s y] = 0 = \lambda_n[Q^s x, y], \quad x, y \in H_*(X).$$

We next list the properties of the top operation Q_n.

2.5. THEOREM. (Ibid, Theorem 1.3). *There is a function*

$$Q_n: H_q(X) \to H_{2q+n}(X)$$

defined for all $q \geqq 0$, *natural with respect to maps of* H_n *spaces, which satisfies the following formulas, where*

$$\mathrm{ad}_n(x)(y) = \lambda_n(y, x), \ \mathrm{ad}_n^i(x)(y) = \mathrm{ad}_n(x)(\mathrm{ad}_n^{i-1}(x)(y)).$$

a) *If* X *is an* H_{n+1} *space, then*

$$Q_n(x) = Q^{n+q}(x).$$

b) *Denoting* $Q_n(x)$ *by* $Q^{n+q}(x)$, *the formulas* a)-c) *and* d), i, iii *of* 2.3 *hold, as well as*

$$Q_n(xy) = \sum_{i+j=n+|xy|} Q^i(x)Q^j(y) + x\lambda_n(x, y)y$$

where

$$|xy| = \deg x + \deg(y).$$

c) *The Nishida relations are now*

$$Sq_*^r Q_n(x) = \sum_i \binom{\frac{n+q}{2} - r}{r - 2i} Q^{m-r+i} Sq_*^i(x)$$
$$+ \sum \frac{1}{i_1} \mathrm{ad}_n(Sq_*^{i_2}(x))(Sq_*^{i_1}(x))$$

where $m = n + |x|$, *and the second sum runs over all sequences* (i_1, i_2) *such that* $i_1 + i_2 = r$, $i_1 < i_2$.

d) $\beta Q_n(x) = (|x| + n - 1)Q^{|x|+n-1}(x) + \lambda_n(\beta x, x)$.

e) $\lambda_n(x, Q_n(y)) = \mathrm{ad}_n^2(y)(x), \quad x, y \in H_*(X)$.

f) $Q_n(x + y) = Q_n(x) + Q_n(y) + \lambda_n(x, y)$.

The following formulas relate the homology operations to the homology suspension

$$\sigma: H_*(\Omega^{n+1}Y) \to H_*(\Omega^n Y).$$

2.6. THEOREM. (Ibid, Theorem 1.4). *If* $x \in H_*(\Omega^{n+1}X; \mathbf{Z}/2)$

a) $\sigma_* Q^s(x) = Q^s(\sigma_* x), \quad x \in H_*(X)$,

b) $\sigma_* Q_n(x) = Q_{n-1}(\sigma_* x), \quad x \in H_*(X)$,

c) $\sigma_* \lambda_n(x, y) = \lambda_{n-1}(\sigma_* x, \sigma_* y), \quad x, y \in H_*(X)$.

The homology with mod 2 coefficients of $\Omega^n S^n X$ was computed by Browder [**14**] using the homology operations of Definition 2.1 which by a suitable change of notation (cf. Definition 2.2) are those listed in Theorems 2.3 to 2.5.

2.7. THEOREM. ([**14**], Theorem 3).

$$H_*(\Omega^n S^n X;\ \mathbf{Z}/2) = P(QH_*(X;\ \mathbf{Z}/2)),\quad n \geqq 2,$$

where $P(M)$ *is the graded polynomial ring over* $\mathbf{Z}/2$ *generated by* M, $Q(H_*(X;\ \mathbf{Z}/2))$ *is the submodule of* $H_*(\Omega^n S^n X;\ \mathbf{Z}/2)$ *generated by all elements* $Q_1^{i_1} \dots Q_{n-1}^{i_{n-1}}(\lambda_n(x, y))$, $\lambda_n(x, y)$ *as in* (2.2), Q_m *as in* (2.3), *and* $(i_1, \dots, i_{n-1})$ *is any sequence of nonnegative integers, with* $Q_m^{i_m}$ *denoting the iteration of* Q_m, $(Q_m^0 = \textit{identity})$.

2.8. *The* mod 2 *Bockstein spectral sequence for homology.* The exact sequence

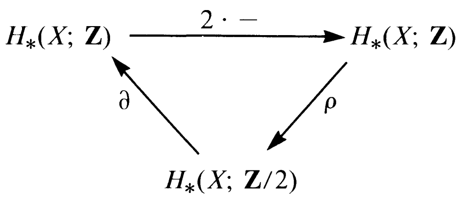
$$\dots \to H_*(X, \mathbf{Z}) \xrightarrow{2\cdot -} H_*(X, \mathbf{Z}) \xrightarrow{\rho} H_*(X, \mathbf{Z}/2) \xrightarrow{\partial} H_*(X, \mathbf{Z}) \xrightarrow{2\cdot -} \dots$$

derived from the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{2\cdot -} \mathbf{Z} \xrightarrow{r} \mathbf{Z}/2\mathbf{Z} \to 0$$

gives rise to an exact couple [**32**], which is a triangle of graded groups and graded maps

(2.4)



where $\deg(2 \cdot -) = \deg \rho = 0$ and $\deg \partial = -1$, with the triangle exact at each corner.

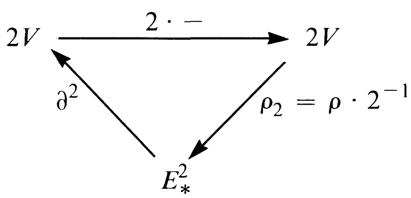
Let E_*^1 denote $H_*(X;\ \mathbf{Z}/2)$ and V denote $H_*(X;\ \mathbf{Z})$. The mod 2 Bockstein homomorphism $\beta = \rho\partial$ satisfies $\beta^2 = 0$, and allows the definition

2 = 0, and allows the definition

$$E_*^2 = \frac{\ker \beta}{\operatorname{im} \beta},$$

which fits in an exact sequence

(2.5)



called the derived couple of the exact couple [**1**, P. III]. The maps of (2.5) are induced by those in (2.4) in the obvious manner. Setting

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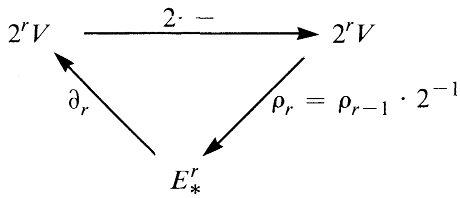
$$B_2 = \rho_2 2^{-1}\partial_2$$

one can define

$$E^3_* = \frac{\ker B_2}{\operatorname{im} B_2}.$$

Iteration of the procedure above gives the rth derived couple



with

$$B_r = \rho_{r-1}2^{-1}\partial_r : E^r_* \to E^r_*.$$

The groups E^r_* and the maps B_r constitute the terms and differentials of a spectral sequence $\{E^r_*, B_r\}$, ([7], Section 11), called the Bockstein spectral sequence. Denoting Z^r the group $\ker(B_r)$, and by B^r the group $\operatorname{im}(B_r)$ we state the following property of the Bockstein spectral sequence.

2.9. Proposition. ([Ibid, Section 11]).

$$\partial(\mathbf{Z}^r(E^1_*)) = (2^rV) \cap (\ker 2\cdot -), \quad r \geqq 1$$

$$\rho^{-1}(B^r(E^1)) = \ker(2^r: V \to V) + 2V, \quad r \geqq 1.$$

The Bockstein spectral sequence provides information on the 2-primary part of the group $H_*(X; \mathbf{Z})$ as follows.

2.10. Proposition. ([Ibid, Section 11]).

$$E^\infty_i(X, \mathbf{Z}/2) \cong \left[\frac{H_i(X; \mathbf{Z})}{\operatorname{tors} H_i(X; \mathbf{Z})}\right] \otimes \mathbf{Z}/2$$

where tors $H_*(X; \mathbf{Z})$ *denotes the torsion subgroup.*

Analysis of the effect of higher Bocksteins on the operations Q_m and λ_n, (2.1, 2.2) allowed F. R. Cohen to prove the following result.

2.11. Proposition. ([**19**], P. III, Corollary 3.13). *If X has no 2-torsion, then the 2-torsion of $H_*(\Omega^2S^2X; \mathbf{Z}/2)$ is all of order 2.*

This proposition will be useful for us in Section 5.

3. K-theories. Our method will involve the computation of π-equivariant mod 2 K-homology of certain spaces, understanding this functor as the dual of its cohomology counterpart $K_\pi^*\mathbf{Z}/2$ defined in [**41**, Section 1], (see also [**35**]). A thorough analysis of (non-equivariant) $K\mathbf{Z}/2$ cohomology is presented in the papers by Araki and Toda [**7**].

We give a brief account of the facts from [**7**], and from elsewhere, which we need.

3.1. *Definitions*. Let M be the space $S^1 \cup_2 CS^1$, and let

$$Un = \begin{cases} \mathbf{Z} \times BU_n & n \text{ even} \\ U_n & n \text{ odd} \end{cases}$$

be the spaces of the unitary spectrum. Then the space consisting of the based maps from M into the spaces U_n constitute the spaces of a $\mathbf{Z}/2$-graded Ω spectrum [**46**]. The maps of this spectrum are induced by the Bott maps

$$\{SU_n \xrightarrow{\alpha_n} U_{n+1}\}.$$

The spectrum above, $\{U_n^M, \alpha_n\}$, represents the generalized cohomology theory, [**46**], denoted $\widetilde{K}^*(\_; \mathbf{Z}/2)$, which is $\mathbf{Z}/2$ graded by virtue of Bott periodicity.

Thus

$$\widetilde{K}^0(X, \mathbf{Z}/2) = [X, U_{2n}^M], \quad \widetilde{K}^1 = [X, U_{2n+1}^M].$$

The associated homology theory, also $\mathbf{Z}/2$-graded, denoted $\widetilde{K}_*(\_; \mathbf{Z}/2)$ is given by

$$\widetilde{K}_0(X; \mathbf{Z}/2) = \lim_{\substack{\to \\ n}} [S^n, X \wedge U_n^M]$$

$$\widetilde{K}_1(X; \mathbf{Z}/2) = \lim_{\substack{\to \\ n}} [S^{n+1}, X \wedge U_n^M].$$

From the cofibration

$$S^1 \xrightarrow{i} M \xrightarrow{\pi} S^2,$$

i the inclusion and π shrinking S^1 to a point, and using Bott periodicity one gets the exact sequence

$$[S^n, X \wedge U_{n-1}] \cong [S^n, X \wedge U_n^{S^2}] \to [S^n, X \wedge U_n^M] \to [S^n, X \wedge U_n^{S^1}] \cong [S^n, X \wedge U_{n-1}]$$

which, taking direct limits, yields the exact sequence

$$\widetilde{K}_*(X) \xrightarrow{\rho} \widetilde{K}_*(X; \mathbf{Z}/2) \xrightarrow{\delta} \widetilde{K}_{*-1}(X).$$

ρ is called the reduction mod 2 and ξ the Bockstein homomorphism.

The mod 2 Bockstein homomorphism β is defined as $\rho\xi$. The exact sequence above extends to infinity on both sides to give an exact sequence

$$\dots \xrightarrow{2\cdot -} \widetilde{K}_*(X) \xrightarrow{\rho} \widetilde{K}_*(X, \mathbf{Z}/2) \xrightarrow{\partial} \widetilde{K}_{*-1}(X; \mathbf{Z}/2)$$
$$\xrightarrow{2\cdot -} \widetilde{K}_{*-1}(X) \xrightarrow{\rho} \dots$$

([**7**, Section 2]) and there holds the universal coefficient exact sequence

$$0 \to (\widetilde{K}_*(X) \otimes \mathbf{Z}/2) \xrightarrow{\bar{\rho}} \widetilde{K}_*(X; \mathbf{Z}/2) \xrightarrow{\bar{\partial}} \operatorname{Tor}(\widetilde{K}_{*-1}(X), \mathbf{Z}/2) \to 0$$

where $\bar{\rho}$ and $\bar{\partial}$ are induced by ρ and ∂, respectively, [Ibid]. Moreover $\widetilde{K}_*(X; \mathbf{Z}/2)$ is a $\mathbf{Z}/2$-vector space, a fact proved in [**7**, Section 2] for $\widetilde{K}^*(\_; \mathbf{Z}/2)$ and which holds for $K_*(\_; \mathbf{Z}/2)$ by the duality

$$\widetilde{K}_*(\_; \mathbf{Z}/2) = \operatorname{Hom}(\widetilde{K}^*(\_; \mathbf{Z}/2), \mathbf{Z}/2), \quad [\mathbf{3}].$$

This duality defines a non-singular pairing

$$K_\alpha(X; \mathbf{Z}/2) \otimes K^\alpha(X; \mathbf{Z}/2) \to \mathbf{Z}/2,$$

[**41**, Section 1], which will be useful for our purposes.

3.2. *Multiplication in K-theory.* The external product of complex vector bundles defines a multiplication in periodic, reduced K-theory

$$v: \widetilde{K}^i(X) \otimes \widetilde{K}^j(Y) \to \widetilde{K}^{i+j}(X \wedge Y), \quad [\mathbf{11}].$$

The definition of $\widetilde{K}^*(X; \mathbf{Z}/2)$ given in 3.1 is, by adjunction, clearly equivalent to set $\widetilde{K}^i(X; \mathbf{Z}/2) = \widetilde{K}^{i+2}(X \wedge M)$, by the suspension isomorphism σ of K-theory. The suspension isomorphism σ_2 of $K^*\mathbf{Z}/2$ theory can be expressed as the composite

$$\begin{array}{ccccc} \sigma_2: \widetilde{K}^{j+2}(X \wedge M) & \underset{\cong}{\xrightarrow{\sigma}} & \widetilde{K}^{j+3}(X \wedge M \wedge S^1)^{1\wedge T^*} & \underset{\cong}{\to} & K^{j+3}(X \wedge S^1 \wedge M) \\ & & & & \| \\ & & & & \widetilde{K}^{j+1}(SX; \mathbf{Z}/2) \end{array}$$

where

$$T: X \wedge S^1 \to S^1 \wedge X$$

is the switching map, [**7**, Section 2].

The product v induces the following maps v_R and v_L, [Ibid, Section 3]

$$\begin{array}{ccc} \widetilde{K}^i(X; \mathbf{Z}/2) \otimes \widetilde{K}^i(Y) & \xrightarrow{\quad v_R \quad} & \widetilde{K}^{i+j}(X \wedge Y; \mathbf{Z}/2) \\ \| & & \| \\ \widetilde{K}^{i+2}(X \wedge M) \otimes K^j(y) \to \widetilde{K}^{i+j+2}(X \wedge M \wedge Y) & \xrightarrow{(1\wedge T)^*} & \widetilde{K}^{i+j+2}(X \wedge Y \wedge M) \end{array}$$

$$\begin{array}{ccc} \widetilde{K}^i(X) \otimes \widetilde{K}^j(Y; \mathbf{Z}/2) & \xrightarrow{v_L} & \widetilde{K}^{i+j}(X \wedge Y; \mathbf{Z}/2) \\ \| & & \| \\ \widetilde{K}^i(X) \otimes \widetilde{K}^{j+2}(Y \wedge M) & \xrightarrow{v} & \widetilde{K}^{i+j+2}(X \wedge Y \wedge M). \end{array}$$

The maps v_L and v_R enjoy the following properties:

(3.1) $$v_R(\rho \otimes 1) = \rho v = v_L(1 \otimes \rho)$$
$$\delta v_R(x \otimes y) = v(\delta x \otimes y) \qquad \delta v_L(x \otimes y) = v(x \otimes \delta y)$$
$$\beta v_R(x \otimes y) = v_R(\beta x \otimes y) \quad \beta v_L(x \otimes y) = v_L(x \otimes \beta y).$$

There is a multiplication

$$v_2:\widetilde{K}^i(X; \mathbf{Z}/2) \otimes \widetilde{K}^j(Y; \mathbf{Z}/2) \to \widetilde{K}^{i+j}(X \wedge Y; \mathbf{Z}/2)$$

defined in [7] and which can be quickly described as follows: For $x \in \widetilde{K}^i(X \wedge M)$ and $y \in \widetilde{K}^j(Y \wedge M)$ the external product of vector bundles gives

$$x \cdot y \in \widetilde{K}^{i+j}(X \wedge Y \wedge M \wedge M).$$

In [7, Section 4] is defined a complex

$$N = S^2 U_g C(SM)$$

and a map

$$\alpha:N \to M \wedge M$$

with the property that for all X the cofibration sequence

$$0 \to \widetilde{K}^*(X \wedge S^2M) \to \widetilde{K}^*(X \wedge N) \to \widetilde{K}^*(X \wedge S^2) \to 0$$

is naturally split exact.

$$v_2(x \otimes y) \in \widetilde{K}^{i+j}(X \wedge Y; \mathbf{Z}/2) = \widetilde{K}^{i+j}(X \wedge Y \wedge S^2M)$$

is then defined as the component of $\alpha^*(x \cdot y)$ in this group.

The multiplication v_2 satisfies the following formulas [Ibid]

(3.2) $$v_R = v_2(1 \otimes \rho) \quad v_L = v_2(\rho \otimes 1)$$
$$\beta v_2(x \otimes y) = v_2(\beta x \otimes y) + v_2(x \otimes \beta y)$$
$$v_2(v_2(x \otimes y) \otimes z) = v_2(x \otimes v_2(y \otimes z))$$
$$v_2(\rho \otimes \rho) = \rho v$$
$$T^* v_2(x \otimes y) = v_2(y \otimes x) + v_2(\beta x \otimes \beta y),$$

where

$$T:X \wedge Y \to Y \wedge X$$

is the switching map.

3.3. Proposition. ([7, Section 6]). *For X, Y finite complexes, ν_2 is an isomorphism*

$$\nu_2 : \tilde{K}(X; \mathbf{Z}/2) \otimes \tilde{K}^*(Y; \mathbf{Z}/2) \rightarrow \tilde{K}^*(X \wedge Y; \mathbf{Z}/2).$$

The maps ν, ν_R, ν_L, and ν_2 have their K-homology counterparts when one considers the category of finite CW-complexes. This is a consequence of the fact that

$$\tilde{K}_*(X) \cong \tilde{K}^{-*}(DX),$$

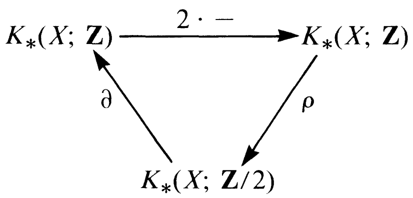
where DX denotes the Spanier-Whitehead dual of X [**18**]. We denote by μ, μ_R, μ_L and μ_2 the respective duals of the maps above, and the K-homology versions of the formulas in (3.1) and (3.2) are valid.

Dual to Proposition 3.3 we have

3.4. Proposition. *μ_2 is an isomorphism,*

$$\mu_2 : \tilde{K}_*(X; \mathbf{Z}/2) \otimes \tilde{K}_*(Y, \mathbf{Z}/2) \rightarrow \tilde{K}_*(X \wedge Y; \mathbf{Z}/2).$$

3.5. *The Bockstein Spectral Sequence in* mod 2 *K-homology*. In analogy to the homology Bockstein spectral sequence of 2.8, the exact sequence of 3.1 determines an exact couple



for K-homology, ([**1**], Part III). Properties 2.9 hold in this situation, except that, since we are in periodic K-homology, degree must be replaced by filtration. The analogous of Proposition 2.10 also holds in K-homology ([7, Section 11]). Moreover the subquotient

$$\frac{\ker \beta}{\operatorname{im} \beta} \subset \frac{K_*(X; \mathbf{Z}/2)}{\operatorname{im} \beta}$$

is selfdual under the duality

$$K_*(X; \mathbf{Z}/2) \cong \operatorname{Hom}(K^*(X; \mathbf{Z}/2), \mathbf{Z}/2), \quad ([\mathbf{41}]).$$

3.6. *Equivariant K-theory*. Definitions. We follow [**35**]. If G is a topological group, a G-space is a topological space X together with a continuous map $G \times X \rightarrow X$ denoted $(g, x) \rightarrow g \cdot x$, such that

$$(g \cdot (g' \cdot x) = (g \cdot g') \cdot x) \quad \text{and} \quad 1x = x,$$

1 the identity element of G. If X is a G-space, then a G-vector bundle on X consists of a G-space E together with a G-map $p_E : E \rightarrow X$, (i.e., p_E

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satisfies $p_E(g \cdot x) = g \cdot p_E(x)$) such that (a) p_E is a complex vector bundle on X, (b) for any $g \in G$, $x \in X$, the group action

$$g:E_x \to E_{g(x)}$$

is a homomorphism of vector spaces. Direct sum and tensor product of G-vector bundles on X are defined fibrewise, as in the non-equivariant case, and

$$\mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x)$$

gives rise to a G-vector bundle on X. Homomorphisms of G-vector bundles on X, denoted $f:E \to F$, are continuous maps such that $p_F f = p_E$. If M is a complex finite dimensional representation of G, i.e., if M is a G-module, then the G-vector bundle $p_1:X \times M \to X$ is denoted by $\mathbf{M}$. For a map $\phi:Y \to X$ of G-spaces, and E a G-vector bundle on X, the induced vector bundle $\phi^*(E)$ is defined as usual, and it is a G-vector bundle on Y. Suppose from now on that G is a compact group. Then the theory of G-vector bundles is linked to homotopy theory as follows.

3.7. PROPOSITION. ([**35**, Proposition 11]). *If ϕ_0, $\phi_1:Y \to X$ are G-homotopic G-maps, Y is compact, and E is a G-vector bundle on X, then*

$$\phi_0^*E \cong \phi_1^*(E).$$

3.8. *Definition*. Let X be a compact G-space, (G compact group). The set of isomorphism classes of G-vector bundles on X forms an abelian semigroup under direct sum $\oplus$. The Grothendieck construction on this semigroup is called $K_G(X)$, and it consists of formal differences $E_0 - E_1$ of G-vector bundles on X, modulo the equivalence relation:

$$E_0 - E_1 \sim E_0' - E_1'$$

if and only if

$$E_0 \oplus E_1' \oplus F \cong E_0' \oplus E_1 \oplus F$$

for some G-vector bundle F on X, ([**35**, Section 2]). The tensor product of G-vector bundles induces a commutative ring structure in $K_G(X)$, which is then a contravariant functor from compact G-spaces to commutative rings, via the induced G-vector bundle construction mentioned above. Moreover, if $\phi:Y \to X$ is a map from a compact H-space to a compact G-space, and $\alpha:H \to G$ is a homomorphism of compact groups, such that

$$\phi(h \cdot y) = \alpha(h) \cdot \phi(y),$$

then ϕ^* produces H-vector bundles on Y out of G-vector bundles on X, and induces

$$\phi^*:K_G(X) \to K_H(Y).$$

If $G = 1$, write $K(X)$ for $K_G(X)$.

3.9. *Example.* If X is a point, then $K_G(X) \cong R(G)$, the representation ring of G. $K_G(X)$ is an algebra over $R(G)$ via the map $X \to pt$, which induces the map $M \to [\mathbf{M}]$ form $R(G)$ to $K_G(X)$, ([Ibid]).

If $H \subset G$ is a compact subgroup, and X is a compact H-space, then $G \times X$ has the diagonal action of H, and the space

$$G \underset{H}{\times} X = G \times X/H$$

is constructed. There is an embedding

$$\phi: X \to G \underset{H}{\times} X$$

which identifies X with the H-subspace $H \times_H X$ of $G \times_H X$. $G \times_H X$ is a G-space and the induced vector bundle construction is an isomorphism between G-vector bundles on $G \times_H X$ and H-vector bundles on X, ([Ibid]).

Let X be a compact G-space. The projection $p: X \to X/G$ induces

$$p^*: K(X/G) \to K_G(X).$$

If G acts freely on X, i.e., if $g \cdot x = x$ only if $g = 1$, then the following proposition is true.

3.10. Proposition. ([Ibid, Proposition 2.1]). *Let G act freely on X. Then*

$$p^*: K(X/G) \to K_G(X)$$

is an isomorphism.

G acts trivially on X if $g \cdot x = x$, all g and x, and in this case there is the homomorphism $K(X) \to K_G(X)$ which gives a vector bundle the trivial G-action. There is also the natural map mentioned above: $R(G) \to K_G(X)$. Combine these two homomorphisms to define

$$\mu: R(G) \otimes K(X) \to K_G(X).$$

3.11. Proposition. ([Ibid, Proposition 2.2]). *If X is a trivial G-space, the natural map*

$$\mu: R(G) \otimes K(X) \to K_G(X)$$

is a ring isomorphism.

Notice that $K_G(X)$ is G-homotopy invariant as the induced G-vector bundle construction described above suggests, i.e.:

3.12. Proposition. ([Ibid, Proposition 2.3]). *If $\phi_0, \phi_1: Y \to X$ are G-homotopic G-maps then*

$$\phi_0^* = \phi_1^*: K_G(X) \to K_G(Y).$$

The next proposition is basic for the definition of reduced G-equivariant K-theory.

3.13. PROPOSITION. ([Ibid, Proposition 2.4]). *If E is a G-vector bundle then there is a G-module M and a G-vector bundle $E^{\perp}$ such that $E \oplus E^{\perp} = \mathbf{M}$.*

3.14. *Definition* ([Ibid]). Two G-vector bundles E, E' on X are called *stably equivalent* if there exist G-modules M, M' such that

$$E \oplus \mathbf{M} \cong E' \oplus \mathbf{M}'.$$

By 3.13 the stable equivalence classes of G-vector bundles on X form an abelian group under $\oplus$. This group is called $\widetilde{K}_G(X)$ and is naturally equivalent to a quotient of $K_G(X)$.

For a compact G-space X and a closed G-subspace A, both based at $x_0 \in A$, the Puppe's construction gives the following exact sequence, ([Ibid]):

$$\widetilde{K}_G(SX) \to \widetilde{K}_G(SA) \to \widetilde{K}_G\left(X \underset{A}{\cup} CA\right) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A). \tag{3.3}$$

Defining

$$\widetilde{K}_G^{-q}(X) = \widetilde{K}_G(S^q X),$$

$$S^q X = S(\dots(SX)),$$

$$\widetilde{K}_G^{-q}(X, A) = \widetilde{K}_G\left(S^q\left(X \underset{A}{\cup} CA\right)\right),$$

and

$$\widetilde{K}_G^{-q}(X, x_0) = \widetilde{K}_G^{-q}(X)$$

the exact sequence (3.3) can be prolonged to infinity to the left to get the exact sequence:

$$\begin{aligned} \dots \widetilde{K}_G^{-q}(X, A) \to \widetilde{K}_G^{-q}(X) \to \widetilde{K}_G^{-q}(A) \to \widetilde{K}_G^{-q+1}(X, A) \\ \to \widetilde{K}_G^{-q+1}(X) \to \widetilde{K}_G^{-q+1}(A) \to \dots \to \widetilde{K}_G(X) \to \widetilde{K}_G(A). \end{aligned} \tag{3.4}$$

3.15. *Remark*. In fact $\widetilde{K}_G$ satisfies the conditions of a generalized cohomology theory, (46), defined on compact G-spaces. As usual, $\widetilde{K}_G$ can be extended to non-compact locally compact spaces using the one point compactification X^+:

$$K_G^{-q}(X) = \widetilde{K}^{-q}(X^+), \quad K_G^{-q}(X, A) = \widetilde{K}_G^{-q}(X^+, A^+).$$

If X is already compact, $X^+ = X \cup x_0$, (disjoint), and $K_G^0(X) = K_G(X)$. In particular

$$K_G^{-q}(X, \emptyset) = K_G^{-q}(X), \quad ([\text{Ibid}]).$$

Using the equivariant Thom homomorphism the following proposition holds.

3.16. PROPOSITION. ([Ibid, Proposition 3.5]). *$K_G^{-q}(X)$ is naturally isomorphic to $K_G^{q-2}(X)$, by a map which is multiplication by a certain element of $K_G^{-2}(pt)$.*

This proposition allows the definition of $K_G^q(X)$ for positive q, and $K_G^*(X)$ can then be regarded as $\mathbf{Z}/2$ graded,

$$K_G^*(X) \cong K_G^0(X) \otimes K_G^1(X).$$

3.17. PROPOSITION. ([Ibid, Proposition 5.4]). *If X is a locally G-contractible compact G-space such that X/G has finite covering dimension, then $K_G^0(X)$ is a finite $R(G)$-module.*

3.18. *Completion.* The augmentation ideal I_G of the representation ring $R(G)$ induces the I_G-adic topology on $R(G)$, and if G is compact, Atiyah and Segal [**12**] showed that the completed ring character $R(G)^\wedge$ is isomorphic to $K^*(B_G)$, B_G denoting the classifying space for G. Furnishing $K_G(X)$ with the I_G-adic topology, the completion $K_G(X)^\wedge$ is defined and it is identified with

$$\lim_{\substack{\leftarrow \\ n}} K_G^*(X)/I_G^n, \quad [\text{Ibid}].$$

If X is a compact G-space, $X_G = (X \times E_G)/G$ where E_G is the universal G-space. To each G-vector bundle F on X the vector bundle $(F \times E_G)/G$ on X_G is associated and this defines a homomorphism

$$\alpha: K_G^*(X) \to K^*(X_G).$$

Filtering E_G by use of the Milnor resolution of G, $\{E_G^n\}$, [**12**, Section 2], maps

$$\alpha_n: K_G^*(pt) \to K_G^*(E_G^n)$$

are defined by

$$\mathbf{M} \to [M \times E_G^n/G],$$

(for M a G-module). The augmentation ideal I_G is the kernel of

$$R(G) \xrightarrow{\alpha_n} K_G^*(E_G^n) \cong K(B_G^n) \to \mathbf{Z}.$$

There is a homomorphism

$$\alpha_n: K_G^*(X) \to K_G^*(X \times E_G^n)$$

induced by $X \times E_G^n \to X$, and it factorizes through

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$$\alpha_n: K_G^*(X)/I_G^n \to K^*(X \times E_G^n), \quad \text{[Ibid]}.$$

Then the inverse limit map, $\varprojlim_n \alpha_n$, gives an isomorphism

$$K_G^*(X)^\wedge \xrightarrow{\alpha} \varprojlim_n K^*(X \times E_G^n) \quad \text{[Ibid]}.$$

We will consider G-spaces X with free G-action, and for them the following theorem holds, which is part of Proposition 4.3 of [**12**].

3.19. THEOREM. ([**12**]). *Let X be a compact G-space. X has free G-action if and only if $K_G^*(X)$ is complete (and Hausdorff).*

3.20. *Definition.* ([**41**, Section 1]).

$$K_\pi^*(X; \mathbf{Z}/2) = K_\pi^*(X \wedge M)$$

where M is the 2-Moore space, (definition 3.2).

3.21. *The Transfer Homomorphism.* For X and Y compact spaces and $f:X \to Y$ a finite covering, the direct image construction of bundles associates to a vector bundle E over X a vector bundle $f_!(E)$ over Y, where the fibre of $f_!(E)$ on y is

$$\bigoplus_{f(x)=y} E_x.$$

The function $E \to f_!(E)$ is functorial on vector bundles, and gives rise to a homomorphism

$$f_!: K^*(X) \to K^*(Y),$$

called the transfer homomorphism. There is a reduced version of the transfer homomorphism $f_!: \widetilde{K}^*(X) \to \widetilde{K}^*(Y)$. ([**9**], [**41**, Section 2]).

3.22. PROPOSITION. ([**9**, Section 1]). *If F is a vector bundle over Y and E is a vector bundle over X, then*

$$f_!(E \otimes f^*(F)) \cong f_!(E) \otimes F.$$

Let X be a compact G-space, and Y a closed subspace. For $H \subset G$ a subgroup of finite index define a map $f:G \times X/H \to X$ by

$$f[g, x] = x \cdot g^{-1}.$$

Then f and its restriction to $G \times Y/H$ are finite coverings. G acts on $G \times X/H$ by multiplication on the G-factor and then f is a G-map, thus defining

$$f_!: K_G^*(G \times X/H, G \times Y/H) \to K_G^*(X, Y),$$

([**41**, Section 2]). There is the isomorphism

$$\phi:K_H^*(X, Y) \to K_G^*(G \times X/H, G \times Y/H)$$

([**35**]). If $(X, Y) = (pt, \emptyset)$, $f_!$ is the induced representation construction. Moreover, if X is a free G-space, the map

$$k_!:K^*(X/H) \to K^*(X/G)$$

induced by the finite covering $k:X/H \to X/G$ coincides, via

$$K_G^*(X) \cong K^*(X/G) \quad \text{and} \quad K_H^*(X) \cong K^*(X/H),$$

with the homomorphism

$$f_!\phi:K_H^*(X) \to K_G^*(X)$$

described above, ([**41**, Section 2]).

Let $G = \pi$ be the cyclic group of order 2, and consider the class $y \in R(\pi)$ determined by the one-dimensional complex representation of π whose character is $e^{(i\pi/2)}$ on the canonical cycle.

Then

$$R(\pi) = \frac{\mathbf{Z}[y]}{(y^2 - 1)}.$$

If $\sigma = 1 + y \in R(\pi)$, then $\sigma^2 = 0$ in $R(\pi) \otimes \mathbf{Z}/2$ and $\{1, \sigma\}$ is a $\mathbf{Z}/2$-basis for $R(\pi) \otimes \mathbf{Z}/2$.

3.23. PROPOSITION. *Let $i_!:K^*(X) \to K_\pi^*(X)$ be the transfer homomorphism associated to $e \to \pi$, the inclusion of the identity element. Then*

$$i_!(1) = \sigma \in R(\pi) \otimes \mathbf{Z}/2.$$

Let

$$i^*:K_\pi^*(X, Y) \to K^*(X, Y)$$

be the forgetful homomorphism. Then the composite $i_!i^*$ is multiplication by σ, and $i^*i_! = 1 + \tau^*$, ([**41**, Section 2]).

4. The Rothenberg-Steenrod spectral sequence in K-theory. In this section we state the Rothenberg-Steenrod spectral sequence for π-equivariant, mod 2 K-cohomology and K-homology, where π is the group of order 2. We follow fairly closely the exposition of [**41**, Section 1]. The Rothenberg-Steenrod spectral sequence was adapted to K-theory by L. Hodgkin [**25**] and improved by D. W. Anderson and L. Hodgkin in [**4**]; it is modelled in the corresponding spectral sequence for ordinary theory, ([**34**]). In [**41**, Section 3], Snaith computed the spectral sequence for $K_\pi^*(X^2; \mathbf{Z}/2)$, where X^2 has the permutation action of π, the group of order 2, and then defined secondary operations in the manner of Dyer-Lashof, when X is an infinite loop space [Ibid, Section 5]. An application of these operations is given in [**43**]. A serious difficulty in applying the operations arises from the fact that they cannot be iterated at will [**41**, Section 4]. In an attempt to carry out the analogous project for finite iterated loop spaces Snaith computed in [**42**] the π-equivariant mod 2 K-theory of $S^{\vee} \times X^2$, $\nu = 1, 2$. Then using the H_1-structure of X he defined a secondary operation

$$\bar{Q}_1 : \frac{\ker \beta}{\operatorname{im} \beta} \xrightarrow{q_1} K_*^{\pi}(S^1 \times X^2; \mathbf{Z}/2) \xrightarrow{\theta} K_*(X; \mathbf{Z}/2)$$

(with

$$\frac{\ker \beta}{\operatorname{im} \beta} \subset K_*(X; \mathbf{Z}/2)/\operatorname{im} \beta,$$

q_1 defined below, and θ the structure map), which made possible the computation of the Atiyah-Hirzebruch spectral sequence for $K_*(\Omega^2 S^3 X; \mathbf{Z}/2)$, where X is a finite CW-complex with $H_*(X; \mathbf{Z}/2)$ torsion free. We parallel the method of [**42**] in order to determine the Atiyah-Hirzebruch spectral sequence for $K_*(\Omega^3 S^3 X; \mathbf{Z}/2)$, where X is as above, (see Section 5). In this section we collect the necessary results to carry out this project.

4.1. *The Spectral Sequences.* Consider the Milnor resolution of π,

$$* \subset S^0 \subset \dots S^n \subset S^{n+1} \subset \dots S^{\infty} = E\pi,$$

[**34**], [**41**, Section 1]. If X is a π-space with $Y \subset X$ a closed π-subspace, then the functors $\widetilde{K}^*(-; \mathbf{Z}/2)$ and $\widetilde{K}_*(-; \mathbf{Z}/2)$ applied to the filtered space

$$\dots \subset \left(\frac{X}{Y} \wedge S^{n^+}\right)/\pi \subset \left(\frac{X}{Y} \wedge S^{n+1^+}\right)/\pi \subset$$

$$\dots \subset \left(\frac{X}{Y} \wedge E\pi\right)/\pi$$

give spectral sequences convergent to

$$\widetilde{K}^*\left(\left(\frac{X}{Y} \wedge E\pi^+\right)/\pi; \mathbf{Z}/2\right) \cong K^*((X \times Y)_{\pi}; \mathbf{Z}/2)$$

$$\cong K_{\pi}^*(X, Y; \mathbf{Z}/2)$$

and

$$\widetilde{K}_*\left(\left(\frac{X}{Y} \wedge E\pi^+\right)/\pi; \mathbf{Z}/2\right) \cong K_*^{\pi}(X, Y; \mathbf{Z}/2),$$

[**25**], [**41**], since in this situation

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$$K^*(X, Y; \mathbf{Z}/2)^{\wedge} \cong K^*((X, Y)_\pi; \mathbf{Z}/2)$$

where

$$(X, Y)_\pi = \left(X \underset{\pi}{\times} E\pi, Y \underset{\pi}{\times} E\pi\right)$$

and one can define

$$K_*^\pi(X, Y; \mathbf{Z}/2) = K_*((X, Y)_\pi; \mathbf{Z}/2),$$

([**41**, Section 1]). We next describe the spectral sequences above.

The properties of the Milnor resolution of π are such that the complexes

$$\mathbf{Z}/2 \xrightarrow{\xi} K^*(S^0; \mathbf{Z}/2) \xrightarrow{d_I} \widetilde{K}^*(S^1, S^0; \mathbf{Z}/2) \xrightarrow{d_I} K^*(S^2, S^1; \mathbf{Z}/2) \xrightarrow{d_I} \dots$$

and

$$\mathbf{Z}/2 \xleftarrow{\eta} K_*(S^0; \mathbf{Z}/2) \xleftarrow{d_{II}} K_*(S^1, S^0; \mathbf{Z}/2) \xleftarrow{d_{II}} K_*(S^2, S^1; \mathbf{Z}/2) \xleftarrow{d_{II}}$$

are respectively free $K^*(\pi; \mathbf{Z}/2)$-comodule and $K_*(\pi; \mathbf{Z}/2) = \mathbf{Z}/2[\pi]$-module resolutions of $\mathbf{Z}/2$, and such properties also imply

$$\widetilde{K}^*\left(\left(\frac{X}{Y} \wedge E\pi_n\right)/\pi, \left(\frac{X}{Y} \wedge E\pi\right)/\pi_{n-1}\right)/\pi; \mathbf{Z}/2\right)$$

$$\cong K^*(X, Y; \mathbf{Z}/2) \underset{K^*(\pi;\mathbf{Z}/2)}{\square} K^*(S^n, S^{n-1}, \mathbf{Z}/2)$$

and

$$\widetilde{K}_*\left(\left(\frac{X}{Y} \wedge E\pi_n\right)/\pi\right), \left(\frac{X}{Y} \wedge E\pi_{n-1}\right)/\pi; \mathbf{Z}/2\right)$$

$$\cong K_*(X, Y; \mathbf{Z}/2) \underset{\mathbf{Z}/2[\pi]}{\otimes} K_*(S^n, S^{n-1}; \mathbf{Z}/2), \quad [\mathbf{25}], [\mathbf{41}].$$

The E_1-terms of the spectral sequences are then, [Ibid],

(4.1) $$K^*(X, Y; \mathbf{Z}/2) \underset{K^*(\pi;\mathbf{Z}/2)}{\square} K^*(S^0; \mathbf{Z}/2)$$

$$\xrightarrow{1 \square d_I} K^*(X, Y; \mathbf{Z}/2) \underset{K^*(\pi;\mathbf{Z}/2)}{\square} (S^1, S^0; \mathbf{Z}/2) \xrightarrow{1 \square d_I} \dots$$

and

(4.2) $$K_*(X, Y; \mathbf{Z}/2) \underset{\mathbf{Z}/2[\pi]}{\otimes} K_*(S^0; \mathbf{Z}/2)$$

$$\xleftarrow{1 \otimes d_{II}} K_*(X, Y; \mathbf{Z}/2) \underset{\mathbf{Z}/2[\pi]}{\otimes} K(S^1, S^0; \mathbf{Z}/2) \xleftarrow{1 \otimes d_{II}} \dots$$

The E_2^{**} and E^2_{**} terms are given by the homology of (4.1) and (4.2) respectively, and

(4.3) $E_2^{q,\alpha}(X, Y) = \mathrm{Cotor}^{q,\alpha}_{K^*(\pi;\mathbf{Z}/2)}(K^*(X, Y; \mathbf{Z}/2), \mathbf{Z}/2)$,

(4.4) $E^2_{q,\alpha}(X, Y) = \mathrm{Tor}^{\mathbf{Z}/2[\pi]}_{q,\alpha}(K_*(X, Y; \mathbf{Z}/2), \mathbf{Z}/2)$,

[**25**], [**26**], [**41**].

4.2.I. Theorem. ([**41**, Theorem 1.4, b]). *The Rothenberg-Steenrod spectral sequence of* X, $\{E_r, d_r\}$, *is a strongly convergent spectral sequence of* $\mathbf{Z} \times \mathbf{Z}/2$ *bigraded* $\mathbf{Z}/2$*-algebras and* $E_r^{**}(pt, \phi, \mathbf{Z}/2)$*-modules such that* a) E_2^{**} *is as in* (4.3), b) $d_r:E_r^{q,\alpha} \to E_r^{q+r,\alpha-r+1}$ *is a derivation with respect to the* $\mathbf{Z}/2$ *and* $E_r^{**}(pt, \phi; \mathbf{Z}/2)$ *actions above,* c) *if* X *is a finite complex the spectral sequence converges strongly to* $K_\pi^*(X, Y; \mathbf{Z}/2)$.

II. Dually. ([Ibid, Theorem 1.4.a]). *The Rothenberg-Steenrod spectral sequence for* (X, Y), $\{E^r, d^r\}$, *is a strongly convergent spectral sequence of* $\mathbf{Z} \times \mathbf{Z}/2$ *bigraded* $\mathbf{Z}/2$*-coalgebras and* $E^r_{**}(pt, \phi; \mathbf{Z}/2)$*-comodules such that* a) E^2_{**} *is as in* (4.4) b) $d^r:E^r_{q,\alpha} \to E^r_{q-r,\alpha+r-1}$ *is a derivation with respect to the coactions above,* c) *the spectral sequence converges strongly to* $K^\pi_*(X, Y; \mathbf{Z}/2)$.

4.2. *Remark*. The filtrations for K_π^* and K_*^π above are, as usual, [**16**], decreasing and increasing, respectively, and the spectral sequences converge to

$$\oplus \frac{F^p}{F^{p+1}} \quad \text{and} \quad \oplus \frac{F_p}{F_{p-1}}.$$

4.3. *The transfer in* π*-equivariant* mod 2 *K-theory*. In order to analyse the transfer homomorphism induced by the inclusion $e \to \pi$, (see 3.21), Snaith characterized the homomorphism

$$i_!:K^*(X, Y; \mathbf{Z}/2) \to K_\pi^*(X, Y; \mathbf{Z}/2)$$

in a useful way. We record some results from [**41**] which we will apply in this section.

Let $D^n_\pi = CS^{n-1}_\pi$, the cone on S^{n-1}_π, with π acting on D^n_π by the con-wise action. Recall from 4.1 the space $E\pi$ and the notation we use.

4.4. Proposition. ([**41**, Proposition 2.3]). *Let* X *be a compact* π*-space. For* $m > 0$ *there are isomorphisms*

$$K_\pi^*(X, Y; \mathbf{Z}/2) \cong K_\pi^*((X, Y) \times (D^{2m}, S^{2m-1}); \mathbf{Z}/2)$$
$$\cong K_\pi^*((X, Y) \times (E\pi, S^{2m-1}); \mathbf{Z}/2).$$

4.5. Proposition. ([Ibid, Proposition 2.4]). *Let* $i:e \to \pi$ *be as above, and* X *a compact* π*-space. Under the isomorphism in* 4.4 *and the isomorphism*

$$K^{\alpha}_{\pi}((X, Y) \times (S^1_{\pi}, S^0_{\pi}); \mathbf{Z}/2) \to K^{\alpha+1}(X, Y; \mathbf{Z}/2)$$

the coboundary

$$\delta: K^{\alpha}_{\pi}((X, Y) \times (S^1_{\pi}, S^0_{\pi}); \mathbf{Z}/2) \to K^{\alpha+1}_{\pi}((X, Y) \times (E\pi, S^1_{\pi}); \mathbf{Z}/2)$$

corresponds to the transfer $i_!$.

A $\mathbf{Z}/2[\pi]$-free resolution of $\mathbf{Z}/2$ is given by

$$\mathbf{Z}/2 \overset{\epsilon}{\leftarrow} D_0 \leftarrow D_1 \overset{d_1}{\leftarrow} D_2 \overset{d_2}{\leftarrow} \dots$$

where D_q is the free, left $\mathbf{Z}/2[\pi]$-module on one generator e_q, $q \geqq 0$, $\epsilon(e_0) = 1$, and

$$d_{2k}(e_{2k+1}) = (1 + \tau) \cdot e_{2k},$$

$$d_{2k+1}(e_{2k+2}) = (1 + \tau) \cdot e_{2k+1},$$

with $\tau \in \pi$ the non-identity element, [**40**].

Let π act on $(X, Y)^2$ by factor permutation. The isomorphisms, ([**41**, Section 1]),

$$K_*((X, Y)^2; \mathbf{Z}/2) \cong K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}$$
$$\cong K_*(X, Y; \mathbf{Z}/2)^{\otimes 2} \underset{\mathbf{Z}/2[\pi]}{\otimes} D_i$$

imply that

$$\mathrm{Tor}^{\mathbf{Z}/2[\pi]}_{*,*}(K_*((X, Y)^2; \mathbf{Z}/2), \mathbf{Z}/2)$$

is the homology of the complex

$$0 \leftarrow K_*(X, Y; \mathbf{Z}/2)^{\otimes 2} \underset{(1+\tau_*)}{\longleftarrow} K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}$$
$$\underset{(1+\tau_*)}{\longleftarrow} K_*(X, Y; \mathbf{Z}/2)^{\otimes 2} \leftarrow \dots$$

and dually

$$\mathrm{Cotor}^{**}_{K^*(\pi;\mathbf{Z}/2)}(K^*((X, Y)^2; \mathbf{Z}/2), \mathbf{Z}/2)$$

is the homology of

$$0 \to K^*(X, Y; \mathbf{Z}/2)^{\otimes 2} \underset{1+\tau^*}{\longrightarrow} K^*(X, Y; \mathbf{Z}/2)^{\otimes 2}$$
$$\underset{1+\tau^*}{\longrightarrow} K^*(X, Y; \mathbf{Z}/2)^{\otimes 2} \to \dots$$

4.6. *Remark*. The π-action on $K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}$ is

$$\tau_*(x \otimes y) = y \otimes x + \beta y \otimes \beta x,$$

([7], [41]), and there is a canonical choice of a basis for $K_*(X, Y; \mathbf{Z}/2)$ which makes the π-module $K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}$ expressible as a direct sum of two dimensional submodules of the form

$$\{u_{\alpha_1} \otimes u_{\alpha_2}, u_{\alpha_2} \otimes u_{\alpha_1} + \beta u_{\alpha_2} \otimes \beta u_{\alpha_1}\}$$

and one dimensional submodules of the form

$$\{u_\alpha | \beta u_\alpha = 0, u_\alpha \notin \text{im } \beta\},$$

[8], [41]. $\text{Tor}_{q,0}^{\mathbf{Z}/2[\pi]}$ is zero on the two-dimensional submodules if $q > 0$, and $\text{Tor}_{0,*}^{\mathbf{Z}/2[\pi]}$ is the module of coinvariants; finally

$$\text{Tor}_{q,0}^{\mathbf{Z}/2[\pi]} \cong \mathbf{Z}/2$$

for the one-dimensional submodules, generated by the class of $u_\alpha^{\otimes 2} \otimes e_q$, and

$$\text{Tor}_{q,1}^{\mathbf{Z}/2[\pi]} = 0$$

[41, Section 1]. The situation is expressed in the following:

4.7. PROPOSITION. ([41, Proposition 1.7]). *There are natural isomorphisms*, $q > 0$:

$$\phi:\text{Tor}_{q,0}^{\mathbf{Z}/2[\pi]}(K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}, \mathbf{Z}/2) \xrightarrow{\cong} \frac{\ker \beta}{\text{im } \beta} \subset \frac{K_*(X, Y; \mathbf{Z}/2)}{\text{im } \beta}$$

given by

$$\phi(x^{\otimes 2} \otimes e_q) = x + \text{im } \beta,$$

$\text{Tor}_{q,1}$ *is zero, and*

$$\text{Tor}_{0,\alpha}^{\mathbf{Z}/2[\pi]}(K_*(X, Y; \mathbf{Z}/2)^{\otimes 2}, \mathbf{Z}/2)$$

is isomorphic to the coinvariants of the π-action. Dually, there are isomorphisms, $q > 0$:

$$\phi:\text{Cotor}_{K^*(\pi;\mathbf{Z}/2)}^{q,0}(K^*(X, Y; \mathbf{Z}/2)^{\otimes 2}, \mathbf{Z}/2)$$

$$\xrightarrow{\cong} \frac{\ker \beta}{\text{im } \beta} \subset \frac{K^*(X, Y; \mathbf{Z}/2)}{\text{im } \beta}$$

such that

$$\phi_2(x^{\otimes 2} \otimes w_q) = x + \text{im } \beta,$$

and $\text{Cotor}^{q,1}$ *is zero.*

$$\text{Cotor}_{K^*(\pi,\mathbf{Z}/2)}^{0,\alpha}(K^*(X, Y; \mathbf{Z}/2)^{\otimes 2}, \mathbf{Z}/2)$$

is isomorphic to the module of invariants of the π-action on $K^*(X, Y; \mathbf{Z}/2)^{\otimes 2}$.

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The computation of $\mathrm{Tor}_{**}^{\mathbf{Z}/2[\pi]}$ and $\mathrm{Cotor}_{K^*(\pi,\mathbf{Z}/2)}^{**}$ described in 4.6 and Proposition 4.7 can be carried out for $S^v \times X^2$ furnished with the diagonal action, where S^v is given the antipodal action, $v = 1, 2$. Let

$$A^v = K^*(S^v; \mathbf{Z}/2)$$

as $\mathbf{Z}/2$-vector space, generated by 1 and γ_v.

4.8. PROPOSITION. ([**42**], Section 1).

$$\mathrm{Cotor}_{K^*(\pi,\mathbf{Z}/2)}^{q*}(K^*(S^v \times X^2; \mathbf{Z}/2), \mathbf{Z}/2)$$
$$\cong \begin{cases} \pi\text{-invariants in } A^v \otimes K^*(X, \mathbf{Z}/2)^{\otimes 2} & \text{if } q = 0 \\ A^v \otimes \dfrac{\ker \beta}{\operatorname{im} \beta} & \text{if } q > 0. \end{cases}$$

Moreover

$$E_1^{q*} \cong A^v \otimes K^*(X; \mathbf{Z}/2)^{\otimes 2} \otimes e_q, \quad \text{and} \quad E_2^{0,*} \subset E_1^{0,*}$$

is the inclusion of the π-invariants. If $q > 0$,

$$E_2^{q,*} \underset{\delta}{\overset{\cong}{\leftarrow}} A^v \otimes \frac{\ker \beta}{\operatorname{im} \beta} \subset A^v \otimes \frac{K^*(X, \mathbf{Z}/2)}{\operatorname{im} \beta}$$

is defined by

$$\delta(a \otimes [x + \operatorname{im} \beta]) = a \otimes x^{\otimes 2} \otimes e_q.$$

In ([**41**], Section 3) Snaith determined the Rothenberg-Steenrod spectral sequence of π-spaces $(X, Y)^2$, permutation action. His method uses bundle theoretic constructions called Massey triple products as well as the quadratic construction ([**41**, Section 3 and Appendix II]). The result is stated in the following theorem.

4.9.I. THEOREM. ([**41**], Theorem 3.8). *In the spectral sequence*

$$\{E^r((X, Y)^2, d^r)\}$$

the only non-trivial differential is d_3, and if

$$x \in \operatorname{Ker} \beta - \operatorname{im} \beta \subset K_\alpha(X, Y; \mathbf{Z}/2)$$

then

$$d_3(x^{\otimes 2} \otimes e_{2q}) = x^{\otimes 2} \otimes e_{2q-3}, \qquad \alpha \equiv 0 \bmod 2$$
$$d_3(x^{\otimes 2} \otimes e_{2q+1}) = x^{\otimes 2} \otimes e_{2(q-1)}, \qquad \alpha \equiv 1 \bmod 2;$$

d_3 is zero otherwise. Dual to I *is the following statement*:

II. *In the spectral sequence $\{E_r(X, Y)^2, dr\}$ the only non-trivial differential is d_3, whose action on*

$$x \in \ker \beta - \operatorname{im} \beta \subset K^*(X, Y, \mathbf{Z}/2)$$

is

$$d_3(x^{\otimes 2} \otimes e_{2q+1}) = x^{\otimes 2} \otimes e_{2q+4}, \quad \alpha \equiv 0 \bmod 2,$$
$$d_3(x^{\otimes 2} \otimes e_{2q}) = x^{\otimes 2} \otimes e_{2q+3}, \quad \alpha \equiv 1 \bmod 2,$$

d_3 *is zero otherwise.*

4.9.I and II together with the properties of the spectral sequence listed in Theorem 4.2 allow one to compute the Rothenberg-Steenrod spectral sequence

$$\{E_r(S^\nu \times X^2, d_r)\} \quad \text{and} \quad \{E^r(S^\nu \times X^2, d^r)\}, \quad \nu = 1, 2,$$

whose E_2 and E^2 terms are given in Proposition 4.8. More specifically the Rothenberg-Steenrod spectral sequence for S^ν, $\{E_r(S^\nu, d_r)\}$, can be easily computed from the fact that

$$K_\pi^*(S^\nu; \mathbf{Z}/2) \cong K^*(RP^\nu; \mathbf{Z}/2)$$

(see Proposition 3.10) and one obtains that ([**42**, Section 1])

$$d_2(\gamma_1 \otimes e_0) = 1 \otimes e_2,$$

which by the derivation property of d_2 implies that

$$d_2(\gamma_1 \otimes e_q) = 1 \otimes e_{q+2}$$

if $q > 0$, and that higher differentials are all trivial. Similarly,

$$d_3(\gamma_2 \otimes e_0) = \gamma_2 \otimes e_2 + 1 \otimes e_2,$$

so that if $\lambda, \mu \in \{0, 1\}$, then

$$d_3((\lambda + \mu\gamma_2) \otimes e_{2q}) = \mu(1 + \gamma_2) \otimes e_{2q+3},$$
$$d_3((\lambda + \mu\gamma_2) \otimes e_{2q+1}) = (\lambda + \mu) \otimes e_{2q+4},$$

and higher differentials are trivial (notice that d_2 is trivial in this case, by dimension argument).

The preceding remarks, and use of the derivation property of the differentials with respect to the $E_r(pt, \phi; \mathbf{Z}/2)$ action and to the factors of $S^\nu \times X^2$, have as a consequence the following proposition.

4.10. Proposition. ([**42**, Lemma 1.1]). *In the spectral sequence*

$$E_1^{**}(S^\nu \times X^2, \phi) \Rightarrow K_\pi^*(S^\nu \times X^2; \mathbf{Z}/2), \quad \nu = 1, 2,$$

the only non-trivial differential is $d_{\nu+1}$ *and it acts as follows*:

a) $d_2(\gamma_1 \otimes x^{\otimes 2} \otimes e_j) = 1 \otimes x^{\otimes 2} \otimes e_{j+2}$

b) $d_3(\lambda + \mu\gamma_2) \otimes x^{\otimes 2} \otimes e_j)$

$$= \begin{cases} \mu(1 + \gamma_2) \otimes x^{\otimes 2} \otimes e_{j+3}, \ j \equiv \deg x \bmod 2 \\ (\lambda + \mu) \otimes x^{\otimes 2} \otimes e_{j+3}, \ \textit{otherwise} \end{cases}$$

where $\beta x = 0$, $x \notin \operatorname{im} \beta$ *in both* a *and* b.

4.11. PROPOSITION. ([**42**, Lemma 1.2]). *Dually, in*

$$E^2_{**}(S^v \times X^2, \phi) \Rightarrow K^\pi_*(S^v \times X^2; \mathbf{Z}/2), \quad v = 1, 2,$$

the only non-trivial differential is d_{v+1} *and*

a) $d_2(1 \otimes x^{\otimes 2} \otimes e_{j+2}) = \gamma_1 \otimes x^{\otimes 2} \otimes e_j$

b) $d_3(\lambda + \mu\gamma_2) \otimes x^{\otimes 2} \otimes e_{j+3})$

$$= \begin{cases} (\lambda + \mu)\gamma_2 \otimes x^{\otimes 2} \otimes e_j, j \not\equiv \deg x \bmod 2 \\ \lambda(1 + \gamma_2) \otimes x^{\otimes 2} \otimes e_j, \ \textit{otherwise} \end{cases}$$

where $x \in \ker \beta - \operatorname{im} \beta$.

c) $E^\infty_{j,*}(S^v \times X^2; \phi) = 0 \quad \textit{if } j \geqq v + 1$

d) *If* $1 \leqq j \leqq v$,

$$\frac{\ker \beta}{\operatorname{im} \beta} \underset{\delta}{\overset{\cong}{\rightarrow}} E^\infty_{j,0},$$

with

$$\delta(x) = 1 \otimes x^{\otimes 2} \otimes e_q$$

e) $\rho_*: K^\pi_*(S^2 \times X^2; \mathbf{Z}/2) \rightarrow K^\pi_*(X^2; \mathbf{Z}/2)$

is onto, where ρ *collapses* S^2 *to a point.*

The following proposition, proved in [**41**] using bundle theoretic constructions, will be essential for our computation in the rest of this section.

4.12. PROPOSITION. ([**41**, Proposition 4.10]).

i) *Let* $z_1^{\otimes 2} \otimes e_1 \in K^1_\pi((X, Y)^2; \mathbf{Z}/2)$

be the element represented by this class in $E^{1,0}_\infty((X, Y)^2$. *Then*

$$\beta(z_1^{\otimes 2} \otimes e_1) = i_!(B_2(z_1)^{\otimes 2}) \in K^0_\pi((X, Y)^2; \mathbf{Z}/2).$$

ii) *Let*

$$z_0^{\otimes 2} \otimes e_0 \in i^*(z_0^{\otimes 2}) \subset K^0_\pi((X, Y)^2; \mathbf{Z}/2),$$

where i^* *is the forgetful map. Then*

$$\beta(z_0^{\otimes 2} \otimes e_0) = B_2(z_0)^{\otimes 2} \otimes e_1 \in K^1_\pi((X, Y)^2; \mathbf{Z}/2).$$

4.13. *Remark*. One can use the proposition above to investigate ker β in $K_\pi^*(S^2 \times X^2; \mathbf{Z}/2)$ and in $K_*^\pi(S^2 \times X^2; \mathbf{Z}/2)$ through the maps induced by

$$\rho: S^2 \times X^2 \to X^2,$$

(see Proposition 4.11). This is indeed what we do in Proposition 4.24.

In what follows we apply the results in the Rothenberg-Steenrod spectral sequences of 4.10 and 4.11 to define certain classes $q_1(x)$ in $K_1^\pi(S^1 \times Y^2; \mathbf{Z}/2)$ and $q_2(x)$ in $K_0^\pi(S^2 \times Y^2; \mathbf{Z}/2)$ where q_1 and q_2 are functions to be defined. q_1 and q_2 will play a major role in the computation of the Atiyah-Hirzebruch spectral sequences for $\Omega^2 S^3 X$ and $\Omega^3 S^3 X$, X a finite torsion free CW-complex. There will actually be an indeterminancy in defining the classes $q_2(x)$ of $K_0(S^2 \times Y^2; \mathbf{Z}/2)$, which nevertheless will turn out to be harmless when we arrive to the analysis of the Atiyah-Hirzebruch spectral sequence in Section 5. The indeterminancy arising is a reflect of the properties of the Rothenberg-Steenrod spectral sequence, as will be discussed in 4.20.

Both definitions of q_1 and q_2 can be thought of as part of the program introduced by L. Hodgkin [**26**] of defining Dyer-Lashof operations in K-theory mod p. In case of infinite loop spaces QX, $p = 2$, the project of constructing such operations has been accomplished by Snaith ([**41**, Section 5]).

The technique used by Snaith involves the study of the Rothenberg-Steenrod spectral sequence for $K_*^\pi(X^2; \mathbf{Z}/2)$ whose properties are contained in [**38**], [**39**], [**40**], [**41**]. The last one of the sequel of papers above focuses on $K_*^\pi(X^2; \mathbf{Z}/2)$. The content of the first 4 sections of [**41**] will be used in our analysis of the Rothenberg-Steenrod spectral sequence for $K(S^2 \times X^2; \mathbf{Z}/2)$ in the rest of this section, and the auxiliary results we need to define q_2 are modelled, and rely on the corresponding ones of the sections of [**41**] mentioned above. Moreover, the application of q_2 to the Atiyah-Hirzebruch spectral sequence for $\Omega^3 S^3 X$ is done by imitating the procedure followed in [**42**] to determine the graded group $K_*(\Omega^2 S^3 X; \mathbf{Z}/2)$.

We define functions q_1 and q_2 on

$$\frac{\ker \beta}{\operatorname{im} \beta} \subset \frac{K_*(Y; \mathbf{Z}/2)}{\operatorname{im} \beta},$$

for Y a compact space, such that

$$q_1: \frac{\ker \beta}{\operatorname{im} \beta} \to K_1(S^1 \times Y^2; \mathbf{Z}/2)$$

$$q_2: \frac{\ker \beta}{\operatorname{im} \beta} \to K_0(S^2 \times Y^2; \mathbf{Z}/2)/\mathrm{Ind}$$

with Ind a certain subspace to be determined, (see Proposition 4.22).

These functions will be crucial in the determination of the Atiyah-Hirzebruch spectral sequence for Ω^3S^3X.

4.14. *The function* q_1. q_1 was defined by Snaith ([**42**, Section 2]) in the following way. First, the Rothenberg-Steenrod spectral sequence for $S^1 \times Y^2$ (Proposition 4.10) implies the existence of a natural map

$$\phi:\frac{\ker \beta}{\text{im } \beta} \to K^1(S^1 \times Y^2; \mathbf{Z}/2)$$

given by

$$\phi(x + \text{im } \beta) = 1 \otimes x^{\otimes 2} \otimes e_1 \in E^{\infty}_{1,0} \subset K^1(S^1 \times Y^2; \mathbf{Z}/2).$$

Next, the transfer homomorphism (see 3.21)

$$i_!:K^1(S^1 \times Y^2; \mathbf{Z}/2) \to K^1_{\pi}(S^1 \times Y^2; \mathbf{Z}/2)$$

has kernel generated by

$$\{ (1 + \tau^*)(w), 1 \otimes x^{\otimes 2} | \beta x = 0\} \subset [K^*(S^1; \mathbf{Z}/2) \otimes K^*(Y; \mathbf{Z}/2)^{\otimes 2}]^1,$$

[**42**, Section 1], and denoting by J such a kernel one defines the monomorphism

$$\bar{i}_!:\frac{K^1(S^1 \times Y^2; \mathbf{Z}/2)}{J} \to K^1_{\pi}(S^1 \times Y^2; \mathbf{Z}/2).$$

The images of ϕ and $i_!$ generate $K^1(S^1 \times Y^2; \mathbf{Z}/2)$, [Ibid], so that

$$\Phi = \bar{i}_! \oplus \phi:K^1(S^1 \times Y^2; \mathbf{Z}/2)/J \oplus \frac{\ker \beta}{\text{im } \beta} \to K^1_{\pi}(S^1 \times Y^2; \mathbf{Z}/2)$$

is an isomorphism, and the duality of 3.5 allows one to define $q_1(x)$, $x \in \ker \beta - \text{im } \beta \subset K_*(Y, \mathbf{Z}/2)$, as

$$(0 \oplus f)\Phi^{-1}:K^1_{\pi}(S^1 \times Y^2; \mathbf{Z}/2) \to \mathbf{Z}/2$$

where

$$f:\frac{\ker \beta}{\text{im } \beta} \to \mathbf{Z}/2$$

is dual to x.

In order to iterate the function q_1 it must be checked that $\beta q_1(x)$ is zero, and an analysis of this situation is carried out in Section 2 of [**42**], the result being

4.15. PROPOSITION. ([**42**, Proposition 2.7]). *If* $x \in K_{\alpha}(Y; \mathbf{Z}/2)$, *then*

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$$\beta(q_1(x)) = \begin{cases} i_*(B_2(x)^{\otimes 2}) & \text{if } \alpha \equiv 0 \bmod 2 \\ i_*(B_2(x)^{\otimes 2} + x^{\otimes 2}) & \text{if } \alpha \equiv 1 \bmod 2, \end{cases}$$

where B_2 *is the second Bockstein,* $x \in \ker \beta$.

A property of q_1 important for our objectives is

4.16. PROPOSITION. ([**42**, Proposition 2.8]). *Let*

$$\Delta_*:(K_*^{\pi}(S^1 \times (Y \times Y)^2; \mathbf{Z}/2) \to K_*^{\pi}(S^1 \times Y^2; \mathbf{Z}/2) \otimes K_*^{\pi}(S^1 \times Y^2; \mathbf{Z}/2)$$

be the diagonal homomorphism. If $x, y \in \ker \beta \subset K_*(Y; \mathbf{Z}/2)$ *then*

$$\Delta_*(q_1(x \otimes y)) = q_1(x) \otimes i_*(y^{\otimes 2}) + i_*(x^{\otimes 2}) \otimes q_1(y).$$

4.17. *Definition.* ([**42**, Definition 2.13]). Let Y be an H_1-space with structure map $\theta:S^1 \times Y^2 \to Y$, (Definition 2.1). The composite

$$\theta_* q_1 : \frac{\ker \beta}{\operatorname{im} \beta} \to K_1(Y; \mathbf{Z}/2)$$

defines classes denoted

$$\bar{Q}_1(x) = \theta_* q_1 x, \quad x \in \ker \beta - \operatorname{im} \beta \subset K_*(Y; \mathbf{Z}/2).$$

It results from Proposition 4.15 the following:

4.18. PROPOSITION. ([**42**, Theorem 2.15 iv]). *With the notation of Definition* 4.17,

$$\beta \bar{Q}_1(x) = \begin{cases} (B_2(x))^2 & \text{if } \deg x \equiv 0 \bmod 2 \\ (B_2(x))^2 + x^2 & \text{if } \deg x \equiv 1 \bmod 2. \end{cases}$$

As a consequence of Proposition 4.16 it holds:

4.19. PROPOSITION. ([**42**, Theorem 2.15 ii]). *Let*

$$x, y \in \ker \beta \subset K_*(Y; \mathbf{Z}/2),$$

where Y *is an* H_1*-space. Then*

$$\bar{Q}_1(x \cdot y) = \bar{Q}_1(x) \cdot y^2 + x^2 \cdot \bar{Q}_1(y).$$

4.20. *The function* q_2. Consider the Rothenberg-Steenrod spectral sequence for $S^2 \times Y^2$ determined in 4.10 and 4.11. Its K-homology version implies the short exact sequence

$$0 \to F_{0,\alpha} \to F_{2,\alpha} \to \frac{F_{2,\alpha}}{F_{0,\alpha}} \to 0, \quad \alpha \in \mathbf{Z}/2,$$

and since the filtration F_r is zero for $r < 0$, we have in the usual way

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$$0 \to E^{\infty}_{0,\alpha} \to F_{2,\alpha} \to E^{\infty}_{2,\alpha} \oplus E^{\infty}_{1,\alpha-1} \to 0,$$

after moding up by F_{-1} and writing

$$\frac{F_{2,\alpha}}{F_{0,\alpha}} \cong \frac{F_{2,\alpha}}{F_{1,\alpha-1}} \oplus \frac{F_{1,\alpha-1}}{F_{0,\alpha}}.$$

Now, from 4.11 b), $E^{\infty}_{0,*}$, $E^{\infty}_{1,*}$ and $E^{\infty}_{2,*}$ constitute the whole of

$$K^{\pi}_{*}(S^2 \times Y^2; \mathbf{Z}/2),$$

and as in [**16**, Chapter XV] one concludes that $F_2 = F_3 = \dots$. Moreover $E^{\infty}_{0,*}$ is contained in the π-coinvariants of

$$K_*(S^2 \times Y^2; \mathbf{Z}/2),$$

and we obtain the exact sequence

$$(4.5) \qquad 0 \to \frac{[K_*(S^2 \times Y^2; \mathbf{Z}/2)]_{\pi}}{M} \xrightarrow{i_*} K^{\pi}_{*}(S^2 \times Y^2; \mathbf{Z}/2) \xrightarrow{\Delta} B \to 0$$

where

$$B = L_1 \oplus L_2 \subset \left(A_2 \otimes \frac{K_*(Y; \mathbf{Z}/2)}{\operatorname{im} \beta}\right) \oplus \left(A_2 \otimes \frac{K_*(Y; \mathbf{Z}/2)}{\operatorname{im} \beta}\right),$$

L_i the subspace isomorphic to the non-bounding subspace of

$$K_*(S^2 \times Y^2; \mathbf{Z}/2) \otimes e_i$$

in 4.11 $i = 1, 2$, and where

$$\begin{aligned} M &= \langle \{\pi\text{-coinvariants which bound}\} \rangle \\ &= \langle \{\gamma_2 \otimes x^{\otimes 2} | \deg x \equiv 1, \beta x = 0\} \cup \{ (1 + \gamma_2) \otimes x^{\otimes 2} | \\ &\qquad\qquad \deg x \equiv 0, \beta x = 0\} \rangle. \end{aligned}$$

Notice that from 4.11.d,

$$L_i \cong \frac{\ker \beta}{\operatorname{im} \beta} \subset \frac{K_*(Y; \mathbf{Z}/2)}{\operatorname{im} \beta},$$

and that the map Δ is the direct sum of the restrictions on δ of 4.8 to L_i.

Dual to (4.5) there is the following exact sequence

$$(4.6) \qquad 0 \leftarrow \frac{\{K^*(S^2 \times Y^2; \mathbf{Z}/2)\}^{\pi}}{M'} \xleftarrow{i^*} K^{*}_{\pi}(S^2 \times Y^2; \mathbf{Z}/2) \xleftarrow{\Delta'} B' \leftarrow 0$$

with groups and maps correspondingly defined.

We will make considerable use of the exact sequences (4.5) and (4.6) in the rest of this section. Some remarks are necessary in order to express the way in which the exact sequences (4.5) and (4.6) determine the groups

$$K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2)$$

and its dual

$$K_{\pi}^*(S^2 \times Y^2; \mathbf{Z}/2).$$

First, notice that in (4.6) the classes of B' determine, through the monomorphism Δ', corresponding classes in

$$K_{\pi}^*(S^2 \times Y^2; \mathbf{Z}/2),$$

while a π-invariant

$$z \in \frac{\{K^*(S^2 \times Y^2; \mathbf{Z}/2)\}^{\pi}}{M'}$$

is such that a whole coset of

$$\frac{K_{\pi}^*(S^2 \times Y^2; \mathbf{Z}/2)}{B'}$$

goes to it under i^*. The situation is interchanged in the exact sequence (4.5) for

$$K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2),$$

where now i_*, is a monomorphism on

$$\frac{[K_*(S^2 \times Y^2; \mathbf{Z}/2)]_{\pi}}{M}$$

while a whole coset of

$$K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2)$$

hits one element under Δ. Later in 4.23 we will define a class

$$w \in K_{\pi}^*(S^2 \times Y^2; \mathbf{Z}/2)$$

determined by an element z of B', and we will then consider the dual class

$$w^0 \in K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2)$$

which we will identify as being determined by the dual to z in B. Thus there will possibly be a summand of the form $i_*(y)$ in the element

$$w^0 \in K_*^{\pi}(S^2 \times Y^2; \mathbf{Z}/2)$$

so defined, which introduces an indeterminancy in the classes we will consider. This indeterminancy is an essential feature in the approach to equivariant K-$\mathbf{Z}/2$ homology as derived from the Rothenberg-Steenrod spectral sequence, (**[26]**, **[41]**, and 4.1 above). In connection with the problem of the indeterminancy, we will have to prove some technical results in 4.21 and 4.22, which are necessary in the sequel.

In order to study the properties of the function q_2 we are aiming to define, we require knowledge of the kernel and the image of the transfer homomorphism

$$i_!:K^0(S^2 \times Y^2; \mathbf{Z}/2) \to K^0_\pi(S^2 \times Y^2; \mathbf{Z}/2).$$

4.21. PROPOSITION. *Let* $Q \subset K^0(S^2 \times Y^2; \mathbf{Z}/2)$ *be the subspace generated by*

$$\{1 \otimes x^{\otimes 2} | \deg x \equiv 1, \beta x = 0\}$$
$$\cup \{ (1 + \gamma_2) \otimes x^{\otimes 2} | \deg x \equiv 0, \beta x = 0\}$$
$$\cup \{ (1 + \tau^*)(w) \}.$$

Then

$$\frac{K^0(S^2 \times Y^2; \mathbf{Z}/2)}{Q} \cong \operatorname{im} i_! \subset \ker(\sigma \cdot -) \subset K^0_\pi(S^2 \times Y^2; \mathbf{Z}/2).$$

Proof. If $x \in \ker \beta$,

$$i^*i_!(a \otimes x^{\otimes 2}) = (1 + \tau^*)(a \otimes x^{\otimes 2}) = 0$$

by the action of τ^*, so that

$$i_!(a \otimes x^{\otimes 2}) = \Delta'(a \otimes [x + \operatorname{im} \beta]).$$

Moreover

$$\sigma(i_!(a \otimes x^{\otimes 2})) = i_!i^*i_!(a \otimes x^{\otimes 2})$$
$$= i_!i^*(\Delta'(a \otimes [x + \operatorname{im} \beta])) = 0,$$

by (4.6). Then, for general $i_!(a \otimes x_1 \otimes x_2)$, we have

$$\sigma i_!(a \otimes x_1 \otimes x_2) = i_!((a \otimes x_1 \otimes x_2)i^*(\sigma)) = \Delta'(x + [\operatorname{im} \beta])$$

for some x, by (4.6), thus $x_1 = x_2 = x \in \ker \beta$ and hence

$$\sigma i_!(a \otimes x_1 \otimes x_2) = 0$$

by the previous case, and we have shown

$$\operatorname{im} i_! \subset \ker(\sigma \cdot -).$$

To prove that Q is the kernel of $i_!$, consider the classes $(1 + \gamma_2) \otimes x^{\otimes 2}$ if $\deg(x) \equiv 0$ and $1 \otimes x^{\otimes 2}$ if $\deg x \equiv 1$, ($\beta x = 0$ for both types). From Proposition 4.10 we have that

$$(1 + \gamma_2) \otimes x^{\otimes 2} \otimes e_1 \quad \text{and} \quad 1 \otimes x^{\otimes 2} \otimes e_1,$$

with deg $x \equiv 0$ and deg $x \equiv 1$, respectively, are permanent cycles in the Rothenberg-Steenrod spectral sequence. Use of the isomorphisms in Proposition 4.4 and of the characterization of the transfer in Proposition 4.5 gives that both

$$(1 + \gamma_2) \otimes x^{\otimes 2} \otimes e_1 \quad \text{if deg } x \equiv 0, \quad \text{and}$$
$$1 \otimes x^{\otimes 2} \otimes e_1 \quad \text{if deg } x \equiv 0,$$

are in the image of j in the exact sequence

$$K^1_\pi((S^2 \times Y^2) \times (E\pi, S^0), \mathbf{Z}/2) \xrightarrow{j} K^1_\pi((S^2 \times Y^2) \times (S^1, S^0); \mathbf{Z}/2) \xrightarrow{\delta = i_!} K^0_\pi((S^2 \times Y^2) \times (E\pi, S^1); \mathbf{Z}/2)$$

thus implying that

$$i_!((1 + \gamma_2) \otimes x^{\otimes 2}) = 0 \quad \text{if deg } x \equiv 0 \quad \text{and}$$
$$i_!(1 \otimes x^{\otimes 2}) = 0 \quad \text{if deg } x \equiv 1,$$

($\beta x = 0$ in both cases). Similarly, since $\gamma_2 \otimes x^{\otimes 2} \otimes e_1$ is not in im(j) for both deg(x) $\equiv 0$ and deg(x) $\equiv 1$ (by Theorem 4.10) we have that

$$i_!(\gamma_2 \otimes x^{\otimes 2}) \neq 0, \quad \text{deg } x \equiv 0 \text{ or } 1.$$

That $\{ (1 + \tau^*)(w) \} \subset \ker(i_!)$ is seen as follows:

$$i_!((1 + \tau^*)(w)) = i_! i^* i_!(w) = \sigma(i_!(w)) = 0$$

as shown before. Thus $Q \subset \ker(i_!)$ and to prove the converse contention suppose

$$i_!(\sum a_i \otimes x_i' \otimes x_i'') = 0$$

so that

$$0 = i^* i_!(\sum a_i \otimes x_i' \otimes x_i'') = (1 + \tau^*)(\sum a_i \otimes x_i' \otimes x_i''),$$

which means that $\sum a_i \otimes x_i' \otimes x_i''$ is π-invariant, i.e.,

$$\sum a_i \otimes x_i' \otimes x_i'' = (1 + \tau^*)(w) + \sum a_i \otimes x_i^{\otimes 2},$$

with $x_i \in \ker \beta$. Hence

$$0 = i_!(\sum a_i \otimes x_i' \otimes x_i'') = i_!(\sum a_i \otimes x_i^{\otimes 2})$$

which by the arguments above is possible only if $a = 1 + \gamma_2$ for deg $x_i \equiv 0$ and if $a = 1$ for deg $x_i \equiv 1$, thus proving $\ker i_! \subset Q$.

Use of the exact sequences (4.5) and (4.6) gives the following characterization of the dual of the even degree component of the image of $i_!$.

4.22. PROPOSITION. *Let Y be a compact space. Then*

$$\operatorname{im} i_! \subset K^0_\pi(S^2 \times Y_2; \mathbf{Z}/2)$$

is dual to

$$K^\pi_0(S^2 \times Y^2; \mathbf{Z}/2)/\mathrm{Ind},$$

where Ind *is the subspace generated by*

$$\{i_*(a \otimes x^{\otimes 2}) \mid \beta x = 0,\ a = 1 \text{ if } \deg x \equiv 0,\ a = 1 + \gamma_2 \text{ if } \deg x \equiv 1\}$$
$$= \{i_*(1 \otimes x^{\otimes 2}) \mid \beta x = 0\}.$$

Proof. Let

$$\langle , \rangle : K_0(-; \mathbf{Z}/2) \otimes K^0(-; \mathbf{Z}/2) \to \mathbf{Z}/2$$

denote the nonsingular pairing (3.1). Suppose

$$z \in K^\pi_0(S^2 \times Y^2; \mathbf{Z}/2)$$

is such that

$$0 = \langle z, i_!(w) \rangle \quad \text{for all } w \in K^0_\pi(S^2 \times Y^2; \mathbf{Z}/2),$$

so that in particular

$$0 = \langle z, i_!(a \otimes x^{\otimes 2}) \rangle, \quad x \in \ker \beta,$$

hence (4.6) and Proposition 4.5 imply

$$i_!(a \otimes x^{\otimes 2}) = \Delta' x,$$

and so

$$0 = \langle z, i_!(a \otimes x^{\otimes 2}) \rangle = \langle z, \Delta'(x) \rangle = \langle \Delta z, x \rangle$$

for all x, which by (4.5) gives $z = i_*(z')$ for some z'. Now, by assumption,

$$0 = \langle z, i_!(a \otimes x_1 \otimes x_2) \rangle = \langle i_* z', i_!(a \otimes x_1 \otimes x_2) \rangle$$
$$= \langle z', i^* i_!(a \otimes x_1 \otimes x_2) \rangle = \langle z', (1 + \tau^*)(a \otimes x_1 \otimes x_2) \rangle,$$

so that z' does not pair with any $(1 + \tau^*)(w)$, forcing

$$z' = \sum a_i \otimes z_i''^{\otimes 2}.$$

By assumption on z, for any $b \otimes x_1 \otimes x_2$ we have

$$0 = \sum \langle i_* a_i \otimes z_1''^{\otimes 2}, i_!(b \otimes x_1 \otimes x_2) \rangle$$
$$= \sum \langle a_i \otimes z_1''^{\otimes 2}, (1 + \tau^*)(b \otimes x_1 \otimes x_2) \rangle$$
$$= \sum \langle a_i \otimes z_i''^{\otimes 2}, b \otimes \beta x_2 \otimes \beta x_1 \rangle$$
$$= \sum \langle a_i, b \rangle \langle z_i'', \beta x_2 \rangle \langle z_i'', \beta x_1 \rangle$$
$$= \sum \langle a_i, b \rangle \langle \beta z_i'', x_1 \rangle \langle \beta z_i'', x_2 \rangle$$

which, taking $b = a_i$, $x_1 = x_2$, implies that $\beta z_i'' = 0$. Then

$$\mathrm{im}(i_!) \subset K^0_\pi(S^2 \times Y^2; \mathbf{Z}/2)$$

has the dual in the statement of the proposition since from the spectral sequence 4.11.b), the generators of Ind determine non-trivial elements in

$$K_0^\pi(S^2 \times Y^2; \mathbf{Z}/2).$$

The other description of Ind comes from d) in Proposition 4.11, as $a \otimes x^{\otimes 2}$ is equivalent to $1 \otimes x^{\otimes 2}$ in the Rothenberg-Steenrod spectral sequence for

$$K_*^\pi(S^2 \times Y^2; \mathbf{Z}/2).$$

4.23. *Definition.* Let

$$x \in \ker \beta \subset K_0(Y; \mathbf{Z}/2) \cong \mathrm{Hom}(K^0(Y; \mathbf{Z}/2), \mathbf{Z}/2)$$

denote the class

$$x + [\mathrm{im}\, \beta] \in \frac{\ker \beta}{\mathrm{im}\, \beta}$$

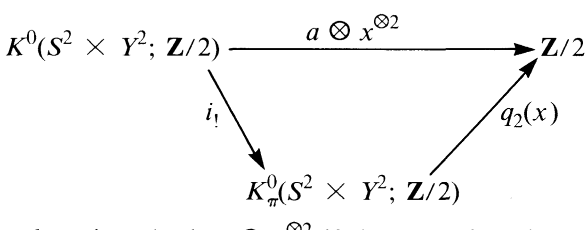
and consider the functionals

$$1 \otimes x^{\otimes 2} \in \mathrm{Hom}(K^0(S^2 \times Y^2; \mathbf{Z}/2), \mathbf{Z}/2)$$

if $\deg x \equiv 0 \bmod 2$, and

$$(1 + \gamma_2) \otimes x^{\otimes 2} \in \mathrm{Hom}(K^0(S^2 \times Y^2; \mathbf{Z}/2), \mathbf{Z}/2)$$

if $\deg x \equiv 1 \bmod 2$. For x as above, define $q_2(x)$ as the functional making the following diagram commutative, in which a depends on $\deg x$ as above:



Notice that since both $1 \otimes x^{\otimes 2}$ if $\deg x \equiv 0$ and $(1 + \gamma_2) \otimes x^{\otimes 2}$ if $\deg x \equiv 1$ are zero on $Q = \ker i_!$, the map $q_2(x)$ exists. Thus $q_2(x)$ is in the dual of im $i_!$, and from the description of the transfer $i_!$ given in Proposition 4.5 we see that, in terms of the spectral sequence of Propositions 4.11 and of 4.5, it has the form

$$q_2(x) = 1 \otimes x^{\otimes 2} \otimes e_2 + i_*(y^{\otimes 2}),$$

for some $y \in \ker \beta$, $\deg y \equiv 0$.

We will be interested in the composite $q_1 q_2(x)$, deg $x \equiv 1$, which to be defineable requires that $\beta q_2(x) = 0$. We now determine $\beta q_2(x)$ and later we will see that for the space $\Omega^3 S^3 X$, with suitable X and $x \in \Omega^3 S^3 X$, we will have $\beta q_2(x) = 0$.

4.24. PROPOSITION. $\beta q_2(x) = q_1(B_2 x)$, *where* deg $x \equiv 1$ mod 2.

Proof. Consider the odd dimensional class $\beta q_2(x)$, for x as above, and suppose it pairs with a class

$$i_!(w) + 1 \otimes y^{\otimes 2} \otimes e_1 + (1 + \gamma_2) \otimes z^{\otimes 2} \otimes e_1$$

in K^1_π-cohomology; we then have

$$\begin{aligned} 1 &= \langle \beta q_2(x), i_!(w) + 1 \otimes y^{\otimes 2} \otimes e_1 + (1 + \gamma_2) \otimes z^{\otimes 2} \otimes e_1 \rangle \\ &= \langle q_2(x), i_!(\beta w) + i_!(B_2(y)^{\otimes 2}) \\ &\qquad + \beta[((1 + \gamma_2) \otimes e_1) \cdot (z^{\otimes 2} \otimes e_0)] \rangle \\ &= \langle q_2(x), (1 + \gamma_2) \otimes B_2 z^{\otimes 2} \otimes e_2 \rangle = \langle x, B_2 z \rangle = \langle B_2 x, z \rangle. \end{aligned}$$

Thus $\beta q_2(x)$ is dual to

$$(1 + \gamma_2) \otimes B_2 x^{\otimes 2} \otimes e_1$$

and by 4.11.b we conclude that

$$\beta q_2(x) = 1 \otimes B_2 x^{\otimes 2} \otimes e_1.$$

(In the equations above we made use of Proposition 4.12 and Remark 4.13, as well as of the fact that neither $i_!(\beta w)$, deg$(w) \equiv 1$, nor $i_!(B_2(y)^{\otimes 2})$, deg $y \equiv 1$, pair with $q_2(x)$, deg$(x) \equiv 1$).

5. The Atiyah-Hirzebruch spectral sequence for $\Omega^v S^3 X$, $V = 1, 2$.

5.1. *Definitions.* The *Atiyah-Hirzebruch spectral sequence for K-theory* was set up in the paper [**11**] by M. Atiyah and F. Hirzebruch; it arises from the filtration

$$\widetilde{K}^n_p(X) = \ker[\widetilde{K}^n(X) \to \widetilde{K}^n(X^{p-1})]$$

of $\widetilde{K}^n(X)$ and it has the following properties. Let X be a finite CW-complex. Then

(5.1) $E_1^{p,q} \cong C^p(X; K^q(pt))$, d_1 the ordinary coboundary.

(5.2) $E_2^{p,q} \cong H^p(X; K^q(pt))$.

(5.3) $E_\alpha^{p,q} \cong Gr\widetilde{K}^{p+q}(X) = \dfrac{\widetilde{K}_p^{p+q}(X)}{\widetilde{K}_{p+1}^{p+q}(X)}$.

(5.4) $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanishes for r even.

(5.5) It is natural on X.

(5.6) The spectral sequence is compatible with Bott periodicity so that the grading q can be disregarded, [Ibid].

5.2. *The Multiplicativity of the Atiyah-Hirzebruch Spectral Sequence.* Concerning the multiplicative structure of $\widetilde{K}^*(X)$, the following is satisfied. Consider the spectral sequence above, $E_r^p(X)$, $r \geqq 2$, with differentials d_r. The cup-product

$$E_2^p(X) \otimes E_2^q(X) \to E_2^{p+q}(X)$$

induces pairings

$$E_r^p(X) \otimes E_r^q(X) \to E_r^{p+q}(X)$$

which are maps of spectral sequences if $E_r^p(X) \otimes E_r^q(X)$ is endowed with the usual differential. Moreover the pairing

$$E_\infty^*(X) \otimes E_\infty^*(X) \to E_\infty^*(X)$$

so obtained coincides with the product induced by the ring structure of $\widetilde{K}^*(X)$, [Ibid]. Notice that this means that the spectral sequence is multiplicative modulo lower filtration.

5.3. *K-homology*. The filtration

$$K_n^p(X) = \operatorname{Im}[K_n(X^p) \to K_n(X)]$$

defines the Atiyah-Hirzebruch spectral sequence $\{E_*^q(X)\}$ for $K_*(X)$, with properties analogous to those for $K^*(X)$, though now the multiplicativity of the spectral sequence refers to the external product, giving

$$E_*^\infty(X) \otimes E_*^\infty(X) \to E_*^\infty(X \wedge X).$$

[**1**, Part III]

We will make use of the following well known result (see e.g. [**11**]).

5.4. PROPOSITION. *Let X be a finite CW-complex for which $H_*(X; \mathbf{Z})$ is torsion free. Then the Atiyah-Hirzebruch spectral sequence collapses, i.e., $E_*^2 = E_*^\infty$, so that $H_*(X) \cong K_*(X)$.*

Suppose a complex X is the direct limit of a certain family of subcomplexes $\{X_m\}$. Then:

5.5. PROPOSITION. ([**1**, Part III]). *$K_*(X)$ is canonically isomorphic to*

$$\varinjlim_m K_*(X_m).$$

5.6. *Remark.* The results above on the Atiyah-Hirzebruch spectral sequence hold if coefficients are introduced, and we will be mainly concerned with the case of mod 2 coefficients.

5.7. *Example*. We will apply Proposition 5.5 to the space Ω^nS^nX in Theorems 5.10 and 5.12. A filtration satisfying the condition of Proposition 5.5 has been given for Ω^nS^nX by J. P. May in [**30**]. The subspaces of this filtration are denoted by F_kC_nX, ([Ibid]). We will pursue this subject in Section 6.

5.8. *Remark*. The first differential in the Atiyah-Hirzebruch spectral sequence

$$H_*(X; \mathbf{Z}/2) \Rightarrow K_*(X; \mathbf{Z}/2)$$

is known to be

$$d_3 = Sq^1_*Sq^2_* + Sq^3_*,$$

where Sq^m_* is dual to the Steenrod square Sq^m, ([**42**], Section 3).

We present the theorem of Snaith on the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^2S^3X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^2S^3X; \mathbf{Z}/2),$$

for X a finite, torsion free CW-complex, a result we will use and whose proof we imitate in order to determine

$$K_*(\Omega^3S^3X; \mathbf{Z}/2),$$

X as above.

First we state the following consequence of the Nishida relations, although we use the lower notation for the homology operations, (see Definition 2.2).

5.9. Proposition. ([**42**], Lemma 3.4). *Let*

$$x \in H_s(\Omega^2S^3X; \mathbf{Z}/2),$$

with X such that $H_*(X; \mathbf{Z})$ is torsion free. Then, for $p \geqq 0$

$$(Sq^3_* + Sq^1_*Sq^2_*)(Q_1^{p+2}(x)) = \begin{cases} (Q_1^p(x))^4 \text{ if } p > 0 \text{ or } s \text{ odd} \\ 0 \text{ otherwise.} \end{cases}$$

We notice that in proving the proposition the summands of the Nishida relations involving only $\lambda_1(-, -)$ play no role, by the assumption on $H_*(X, \mathbf{Z})$, (see Theorem 2.5.c).

5.10. Theorem. ([**42**], Theorem 3.6). *Let X be a finite, torsion free CW-complex. Then, in the Atiyah-Hirzebruch spectral sequence*

$$H_*(\Omega^2S^3X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^2S^3X; \mathbf{Z}/2),$$

$E^4_* \cong E^\infty_* \cong A \otimes B$, *where A and B are*

$$A = \bigotimes_{\lambda_i} \frac{P(\lambda_i, Q_1\lambda_i)}{(\lambda_i^4, (Q_1\lambda_i)^4)} \otimes \left[\bigotimes_{t \geqq 2} E((Q_1^t(\lambda_i))^2)\right] \text{ if } \deg(\lambda_i) \equiv 1 \bmod 2;\ \lambda_i$$

denotes the Browder generators $\lambda_1(x, y)$, (2.2);

$$B = \bigotimes_{\lambda_i} \frac{P(\lambda_i, Q_1\lambda_i, Q_1^2\lambda_i)}{((Q_1\lambda_i)^4, (Q_1^2\lambda_i)^4)} \otimes \left[\bigotimes_{t \geqq 3} E((Q^t\lambda_i)^2)\right]$$

if $\deg(\lambda_i) \equiv 0 \bmod 2$; λ_i *as in* A.

In A *and* B, P *means a polynomial algebra and* E *an exterior algebra.*

Proof. Notice first that by properties 2.4.f and 2.5.1 of the Browder operations, each λ_i equals

$$\lambda_1(X_1, \lambda_1(x_1, \lambda_1(\dots \lambda_1(x_k, x_1))))$$

with

$$x_i \in H_*(SX; \mathbf{Z}/2) \subset H_*(\Omega^2 S^3 X; \mathbf{Z}/2);$$

since X is finite, torsion free, this observation and 5.4 implies that the λ_i's are infinite cycles. Now, $E_*^4 \cong A \otimes B$ by Proposition 5.9; we must prove that any higher differential is trivial. Observe that the algebra

$$H_*(\Omega^2 S^3 X; \mathbf{Z}/2)$$

is primitively generated, which passes to all the E_r-terms, $r \geqq 2$ and that the multiplicative odd dimensional (primitive) generators are only λ_i (odd degree), $Q_1(\lambda_i)$, and $Q_1^2(\lambda_i)$. If we show that these classes are infinite cycles which are not hit by any differential, we are done. For if there were a d_r acting non-trivially (let us take the smallest $r > 3$ with this property) then d_r should be non-trivial on some $(Q_1^t(\lambda_i))^2$, which is primitive, and hence $d_r[(Q_1^t(\lambda_i))^2]$ would be primitive, odd dimensional, hit by a differential, which is not possible under our assumption on these classes. It remains to show that the odd primitive multiplicative generators are infinite cycles which do not bound. Let

$$\bar{\lambda}_i \in K_\alpha(F_k C_2 SX; \mathbf{Z}/2)$$

be a Browder generator, where $F_k C_2 X$ is a subspace of $\Omega^2 S^3 X$ in its filtration given in 5.7. Then

$$i_{l_*}(x) = \bar{\lambda}_i,$$

for $x \in (F_k C_2 SX)^l$, the l-skeleton. Since $\bar{\lambda}_i$ is in the image of ρ, (Definition 3.1), so is x, and moreover $\rho(\gamma_i) = \bar{\lambda}_i$ with γ_i of infinite order. Now let

$$\lambda_i \in H_l(F_k C_2 SX; \mathbf{Z}/2)$$

represent $\bar{\lambda}_i$ in the Atiyah-Hirzebruch spectral sequence

$$H_*(F_k C_2 SX; \mathbf{Z}/2) \Rightarrow K_* H_*(F_k C_2 SX; \mathbf{Z}/2).$$

Since x is in $\mathrm{im}(\rho)$ the class

$$q_1(x) \in K_1^\pi(S^1 \times (F_k C_2 SX)^l \times (F_k C_2 SX)^l; \mathbf{Z}/2)$$

is defined, (see 4.14), and

$$(j_{2l+1})_*(q_1(x)) \in K_\alpha\left((S^1, S^0) \underset{\pi}{\times} [(F_k C_2 SX)^l, (F_k C_2 SX)^{l-1}]^2; \mathbf{Z}/2\right)$$
$$\cong C_{2l+1}\left(S^1 \underset{\pi}{\times} (F_k C_2 SX)^2; \mathbf{Z}/2\right),$$

(j_{2l+1} induced by the obvious projection), is a homology class determining the element

$$Q_1(\lambda_i) \in H_*(\Omega^2 S^3 X; \mathbf{Z}/2).$$

Moreover, if

$$\bar{\lambda}_i \in K_0(\Omega^2 S^3 X; \mathbf{Z}/2)$$

then

$$\beta(q_1(\bar{\lambda}_i)) = i_* B_2(\bar{\lambda}_i)^{\otimes 2}$$

(by Proposition 4.15), is zero as $\bar{\lambda}_i$ is in $\mathrm{im}(\rho)$, whence

$$q_1 q_1(\bar{\lambda}_i) \in K_1\left(S^1 \underset{\pi}{\times} \left(S^1 \underset{\pi}{\times} [(F_k C_2 SX)^l]^2\right)^2; \mathbf{Z}/2\right)$$

is defined and the canonical projection of it to the cycles

$$C_{4l+3}\left(S^1 \underset{\pi}{\times} S^1 \underset{\pi}{\times} \left([F_k C_2 SX]^2\right)^2; \mathbf{Z}/2\right)$$

determines the homology class $Q_1 Q_1(\lambda_i)$. The proof of the odd primitive multiplicative generators being infinite cycles is then complete. That they are not the target of any differential is seen by noting that if $d_r(z) = y$ is one of the classes in question, then the naturality of the Atiyah-Hirzebruch spectral sequence implies that

$$0 \neq d_r(\sigma_* x) = \sigma_* y \in H_*(\Omega S^3 X; \mathbf{Z}/2)$$

which is impossible, as the stable splitting of $\Omega S^3 X$ involves only smashed copies of suspensions of X, (**[37]**).

We now determine the Atiyah-Hirzebruch spectral sequence for $\Omega^3 S^3 X$. The following proposition is established using the Nishida relations, (Theorems 2.3 to 2.5).

5.11. PROPOSITION. *Let X be a finite torsion free CW-complex and consider the Atiyah-Hirzebruch spectral sequence*

$$H_*(\Omega^3 S^3 X; \mathbf{Z}/2) \Rightarrow K_*(\Omega^3 S^3 X; \mathbf{Z}/2).$$

Then $E_*^4 \cong A \otimes B$, *where*

$$A \cong \bigotimes_{\lambda_i} \frac{\mathbf{Z}/2[\lambda_i, Q_1\lambda_i, Q_1Q_2\lambda_i, Q_2(\lambda_i), Q_2^2\lambda_i]}{(\lambda_i^4, (Q_1\lambda_i)^4, (Q_1Q_2\lambda_i)^2, (Q_2\lambda_i)^4, (Q_2^2\lambda_i)^4}$$

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$$\otimes\left[\bigotimes_{\alpha_1=\alpha_2} E((Q_{\alpha_1}Q_{\alpha_2}\ldots(\lambda_i))^2)\right]$$

if deg $\lambda_i \equiv 1 \bmod 2$, *and*

$$B \cong \bigotimes_{\lambda_i}\frac{\mathbf{Z}/2[\lambda_i, Q_1\lambda_i, Q_2\lambda_i]}{((Q_1\lambda_i)^2, (Q_2\lambda_i)^4)} \otimes \left[\bigotimes_{\alpha_1=\alpha_2} E((Q_{\alpha_1}Q_{\alpha_2}\ldots(\lambda_i))^2)\right]$$

if deg $\lambda_i \equiv 0 \bmod 2$. *In A and B, E denotes exterior algebra,* λ_i *is as in Proposition* 5.10.

Proof. We analyze the effect of

$$d_3 = Sq^1_*Sq^2_* + Sq^3_*$$

on the generators of

$$H_*(\Omega^3S^3X; \mathbf{Z}/2).$$

This has been stated in case of $Q^t_1(\lambda_i)$ in Proposition 5.1. Consider now $Q^t_2(\lambda_i)$ and let $m = \deg Q_2^{t-1}(\lambda_i)$.

Then by the Nishida relations

$$(5.6)\quad Sq^2_*Q^t_2(\lambda_i) = \binom{m}{2}Q^mQ_2^{t-1}(\lambda_i) + Q^{m+1}Sq^1_*Q_2^{t-1}(\lambda_i) + \sum\frac{1}{i_1}\mathrm{ad}^2(Sq^{i_2}_*(x))(Sq^{i_1}_*(x)),$$

$$i_1 + i_2 = 3,\ i_1 < i_2,$$

where whenever a subindex appears we are using the lower notation for the operations, while if no subindex is present, Q^k denotes the Dyer-Lashof operation in upper notation, (Definition 2.2). Notice that the λ_i's are all torsion free by our assumption on X, so that the last summation in (5.6) is zero, a fact which holds for all the rest of the paper. We proceed to our computations, considering now

$$(5.7)\quad Sq^1_*(Q^t_2(\lambda_i)) = \binom{m}{1}Q^{m+1}Q_2^{t-1}(\lambda_i)$$

$$(5.8)\quad Sq^3_*(Q^t_2(\lambda_i)) = \binom{m-1}{3}Q^{m-1}Q_2^{t-1}(\lambda_i) + \binom{m-1}{1}Q^mSq^1_*Q_2^{t-1}(\lambda_i).$$

Combining (5.6) and (5.7) we get

(5.9) $$Sq^1_*Sq^2_*(Q^t_2(\lambda_i)) = \binom{m-1}{1}\binom{m}{2}Q^{m-1}Q_2^{t-1}(\lambda_i) + \binom{m}{1}Q^m Sq^1_* Q_2^{t-1}(\lambda_i)$$

If m is even, (5.9) gives

$$Sq^1_*Sq^2_*Q^t_2(\lambda_i) = 0.$$

In (5.8)

$$Q^{m-1}Q_2^{t-1}(\lambda_i) = 0,$$

by degree (Theorems 2.3 and 2.5) while

(5.10) $$Sq^1_*Q_2^{t-1}(\lambda_i) = \binom{\frac{m-2}{2}+1}{1} Q^{(m-2/2)+1}Q_2^{t-2}(\lambda_i).$$

Together (5.9) and (5.10) imply

(5.11) $$(Sq^3_* + Sq^1_*Sq^2_*)Q^t_2(\lambda_i)$$
$$= \binom{\frac{m-2}{2}+1}{1} Q^m Q^{(m-2/2)+1}Q_2^{t-2}(\lambda_i)$$
$$= \binom{\frac{m-2}{2}+1}{1} Q_1 Q^{(m-2/2)+1}Q_2^{t-2}(\lambda_i)$$
$$= \binom{\frac{m-2}{2}+1}{1} Q_1^2 Q_2^{t-2}(\lambda_i).$$

Clearly (5.11) is zero if and only if

$$\frac{m-2}{2} \equiv 1, \bmod 2.$$

Now,

$$\deg Q_2(\lambda_i) = 2\deg(\lambda_i) + 2 \quad \text{and}$$
$$\deg Q_2^2(\lambda_i) = 2(2\deg(\lambda_i) + 2) + 2,$$

(Section 2.1), and so

$$\frac{m-2}{2} \equiv 1 \bmod 2$$

is satisfied only if

$$\deg(\lambda_i) \equiv 1 \bmod 2 \quad \text{and} \quad t = 2.$$

Observe from (5.11) that $d_3Q_2(\lambda_i) = 0$, all λ_i.

Thus we have established

(5.12) $$d_3(Q_2^t(\lambda_i)) = \begin{cases} Q_1^2Q_2^{t-2}(\lambda_i) & \text{if } t > 2, \text{ or if } t = 2 \text{ and } \deg(\lambda_i) \equiv 0 \bmod 2 \\ 0 & \text{if } t = 1, \text{ or if } t = 2 \text{ and } \deg(\lambda_i) \equiv 1 \bmod 2. \end{cases}$$

Similarly, analysis of the action of d_3 on $Q_1Q_2^t(\lambda_i)$ gives

(5.13) $$d_3(Q_1Q_2^t(\lambda_i)) = \begin{cases} Q_1Q_2^{t-1}(\lambda_i) & \text{if } t \geqq 2 \\ 0 & \text{if } t < 2. \end{cases}$$

Clearly squares are all d_3-cycles, and the λ_i's are all infinite cycles by the torsion-freeness of X, (see proof of Theorem 5.10).

Thus the primitive generators of A and B are determined.

One then checks that the powers in the denominators of A and B in the statement of the proposition are d_3-boundaries. For example, the exterior classes of A and B in the proposition are so since $(Q^t(\lambda_i))^4$ is a d_3-boundary for $t > 1$ if $\deg(\lambda_i) \equiv 1 \bmod 2$ and for $t > 2$ and $\deg(\lambda_i) \equiv 0 \bmod 2$; this follows from Proposition 5.9. Thus E_*^4 has the asserted form, since

$$H_*(\Omega^3S^3X; \mathbf{Z}/2) = E_*^2,$$

and then all E_*^r, are primitively generated Hopf algebras, (**[5]**, **[8]**).

In analogy to Snaith's result, Theorem 5.10, on the Atiyah-Hirzebruch spectral sequence for Ω^2S^3X we prove

5.12. THEOREM. *For a finite, torsion free CW-complex X, the Atiyah-Hirzebruch spectral sequence for $K_*(\Omega^3S^3X; \mathbf{Z}/2)$ is such that $E_*^4 \cong E_*^\infty$.*

Proof. As in the proof of Theorem 5.10 it suffices to show that the odd degree primitive generators are infinite cycles which do not bound. λ_i and $Q_1(\lambda_i)$ are infinite cycles, as seen in Theorem 5.10, and to check that they do not bound we apply the homology suspension to them, getting classes $\sigma_*\lambda_i$ and $(\sigma_*\lambda_i)^2$ in

$$H_*(\Omega^2S^3X; \mathbf{Z}/2),$$

which we know not to be boundaries from that theorem. It remains to consider only the classes of type $Q_1Q_2\lambda_i$, $\deg \lambda_i \equiv 1$. Proceeding as in Definition 4.23 we construct classes

$$q_1q_2(\bar{\lambda}_i) \in K_*^\pi(S^1 \times (S^2 \times [(F_kC_3SX)^l]^2)^2; \mathbf{Z}/2).$$

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Since

$$\beta q_2(\bar{\lambda}_i) = 1 \otimes B_2\bar{\lambda}_i^{\otimes 2} \otimes e_1 = 0,$$

(Proposition 4.24), by torsion freeness, the composite $q_1q_2(\bar{\lambda}_i)$ is defined. Projecting $q_1q_2(\bar{\lambda}_i)$ to the chains

$$C_{4l+5}\left(S^1 \underset{\pi}{\times} \left(S^2 \underset{\pi}{\times} [F_kC_3SX]^2\right)^2; \mathbf{Z}/2\right)$$

we obtain a cycle

$$(\lambda_i \otimes \lambda_i \otimes e_2) \otimes (\lambda_i \otimes \lambda_i \otimes e_2) \otimes e_1 + (y \otimes y) \otimes (y \otimes y) \otimes e_1,$$

(Definition 4.23), which gives rise to the homology class

$$Q_1Q_2\lambda_i + Q_1(y^2) = Q_1Q_2(\lambda_i),$$

since $Q_1(y^2) = 0$ by Theorem 2.3.d. Now

$$\theta_*q_1q_2(\lambda_i) = \theta_*([(\bar{\lambda}_i \otimes \bar{\lambda}_i \otimes e_2) \otimes (\bar{\lambda}_i \otimes \bar{\lambda}_i \otimes e_2)] \otimes e_1)$$
$$+ \theta_*([(y \otimes y) \otimes (y \otimes y)] \otimes e_1)$$

in $K_*^{\pi}(\Omega^3S^3X; \mathbf{Z}/2)$ and

$$\theta_*([(y \otimes y) \otimes (y \otimes y)] \otimes e_1) = \bar{Q}_1(y^2) = 0$$

by Proposition 4.19, since $\beta(y^2) = 0$, and so $Q_1Q_2\lambda_i$ is an infinite cycle. Finally, $Q_1Q_2\lambda_i$ is not a boundary since

$$\sigma_*Q_1Q_2\lambda_i = (Q_1(\lambda_i))^2 \in H(\Omega^2S^3X; \mathbf{Z}/2)$$

is not a boundary by Theorem 5.10 and the proof of the theorem is complete.

6. $K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$ as an algebra. Araki and Toda [7] defined an admissible multiplication for mod 2 K-theory

$$\nu_2:K^*(X; \mathbf{Z}/2) \otimes K^*(Y; \mathbf{Z}/2) \to K^*(X \wedge Y); \mathbf{Z}/2),$$

and the dual notion is a multiplication

$$\mu_2:K_*(X; \mathbf{Z}/2) \otimes K_*(Y; \mathbf{Z}/2) \to K_*(X \wedge Y; \mathbf{Z}/2).$$

ν_2 and μ_2 are not commutative, the effect of

$$T:X \wedge Y \to Y \wedge X$$

being given by

$$\nu_2T^*(x \otimes y) = \nu_2(y \otimes x) + \nu_2(\beta y \otimes \beta x) \quad \text{and}$$

$$\mu_2T_*(x \otimes y) = \mu_2(y \otimes x) + \mu_2(\beta y \otimes \beta x),$$

([7], [**41**]). All the expected relations between the multiplication ν in integral K-theory and ν_2 are satisfied. ν becomes ν_2 after reduction mod 2, i.e.,

$$\rho\nu(x \otimes y) = \nu_2(\rho x \otimes \rho y),$$

and the Bockstein homomorphism β acts as a derivation,

$$\beta\nu_2(x \otimes y) = \nu_2(\beta x \otimes y) + \nu_2(x \otimes \beta y).$$

Details on these facts are given in [7] and we recorded some results on this subject in Section 3.

If X is a CW-complex which is an associative H-space with unit e let

$$i:\{e\} \to X \quad \text{and} \quad p:X \to e$$

denote the inclusion and constant maps, and

$$h:X \times X \to X, \quad \Delta:X \to X \times X$$

the h-space product and the diagonal maps. Define a product and a coproduct in $K_*(X; \mathbf{Z}/2)$ as the composites, ([**5**]),:

$$\phi = h_*\mu_2:K_*(X; \mathbf{Z}/2) \otimes K_*(X; \mathbf{Z}/2) \to K_*(X \wedge X; \mathbf{Z}/2) \to K_*(X; \mathbf{Z}/2)$$

$$\Psi = \mu_2^{-1}\Delta_*:K_*(X; \mathbf{Z}/2) \to K_*(X \wedge X; \mathbf{Z}/2) \to K_*(X; \mathbf{Z}/2) \otimes K_*(X; \mathbf{Z}/2).$$

From [**5**] we have the following result.

6.1. PROPOSITION. *Suppose*

$$\mu_2:K_*(X; \mathbf{Z}/2) \otimes K_*(X; \mathbf{Z}/2) \to K_*(X \wedge X; \mathbf{Z}/2)$$

is commutative. Then $K_*(X; \mathbf{Z}/2)$ *is a Hopf algebra with multiplication* ϕ, *comultiplication* Ψ, *unit* $\eta = i_*$ *and counit* $\epsilon = p_*$. (*We will show later that if* $X = \Omega^2 S^{2n+1}$, *then* μ_2 *is commutative and so Proposition* 6.1 *holds for* $K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2$.)

Roughly speaking we proceed as follows. The stable splitting of $\Omega^2 S^{2n+1}$, [**37**], will allow us to express both the homology and K-homology of $\Omega^2 S^{2n+1}$ as direct sums of the homology and K-homology of the pieces of the splitting. Moreover the naturality of the Atiyah-Hirzebruch spectral sequence and the stability of its differentials imply that the homology of a piece of the splitting determines, in the E^∞ term, the K-homology of that piece. These observations and F. R. Cohen's result, [Proposition 2.11], on the torsion of

$$H_*(\Omega^2 S^3 X; \mathbf{Z}/2)$$

will give the commutativity of

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$$K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$$

as an algebra, (Sections 6.11, 6.12). Although the spectral sequence above is multiplicative, we observe that it is so only modulo lower filtration and that this fact reflects itself in the formula

$$h_*T_*\mu_2(x \otimes y) = h_*(\mu_2(y \otimes x) + \mu_2(\beta y \otimes \beta x)),$$

so that we do need the proof of commutativity in order to know that we are dealing with a commutative $\mathbf{Z}/2$-graded Hopf-algebra, ([**5**], [**8**]). Granted this, a possible method of determining the algebra relations in

$$K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$$

consists in showing that the primitive generators for it exhibited in Theorem 5.10 have the height suggested by the expression of A in this theorem. One is encouraged to expect this when simple inspection of filtration, plus the properties of the stable splitting of Ω^2S^{2n+1}, (see Theorem 6.4), show that the classes ι and $Q_1(\iota)$ have height 4, (Theorem 6.13). This conjecture turns out to hold for all the primitive generators of $K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$.

6.2. *The Stable Splitting of* Ω^nS^nX. An important result we will require in order to determine the algebra

$$K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$$

is the stable splitting of finite iterated loop spaces due to V. P. Snaith and further studied in [**21**]. In order to state the properties of the splitting of Ω^2S^{2n+1} we shall use, the following notions are necessary.

6.3. *Definitions*. Let $\mathscr{C}_k(q)$ denote the space of ordered q-tuples of little cubes disjointly embedded in I^k, ([**13**], [**30**]), on which the symmetric group Σ_q acts freely. For a based space X, C_kX is defined by

$$C_kX = \bigvee_{q \geqq 1} \mathscr{C}_k(q) \underset{\Sigma_q}{\times} X^q/\sim$$

where

$$[(c_1, \dots, c_q), (x_1, \dots, x_{q-1}, *)]$$
$$\sim [(c_1, \dots, c_{q-1}), (x_1, \dots, x_{q-1})]$$

determines an equivalence relation on

$$\bigvee_{q \geqq 1} \mathscr{C}_k(q) \underset{\Sigma_q}{\times} X^q,$$

[**30**]. The spaces C_kX are approximations to Ω^kS^kX in the sense that there are natural maps

$$\alpha_k: C_kX \to \Omega^kS^kX$$

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which are homotopy equivalences when X is connected [Ibid]. C_kX is filtered by closed subspaces

$$F_nC_kX \subset F_{n+1}C_kX, \quad F_nC_kX = \bigvee_{q=0} \mathscr{C}_k(q) \underset{\Sigma q}{\times} X^q/\sim$$

and there are maps

$$F_nC_kX \times F_mC_kX \rightarrow F_{n+m}C_kX$$

obtained from the operad action defined by May [Ibid]. These maps define a product on C_kX and the approximation α_k sends products in C_kX to loop products in Ω^kS^kX. The quotients of successive filtrations in C_kX,

$$F_qC_kX/F_{q-1}C_kX = D_{k,q}X$$

are called the reduced extended power spaces, and

$$D_{k,q} = \mathscr{C}_k(q)^+ \underset{\Sigma_q}{\wedge} \mathrm{X}^{[q]},$$

where Y^+ is the union of Y with a disjoint base point and $X^{[q]}$ is the q-fold smash product, [**30**]. With the notation above, the stable splitting of Ω^kS^kX can now be stated.

6.4. THEOREM. ([**37**], Theorem 1.1). *Let $\Sigma^\infty Y$ denote the suspension spectrum of a space Y. Then there is a weak homotopy equivalence for X a connected space*:

$$\Sigma^\infty\Omega^kS^kX \underset{w}{\cong} \bigvee_{q\geqq 1} \Sigma^\infty D_{k,q}X.$$

The maps

$$F_nC_kX \times F_mC_kX \rightarrow F_{n+m}C_kX$$

are such that the composite

$$F_nC_kX \times F_mC_kX \rightarrow F_{n+m}C_kX \rightarrow D_{n+m}X$$

factors through the projection

$$F_nC_kX \times F_mC_kX \rightarrow D_{k,n}X \wedge D_{k,m}X$$

thus giving maps

$$\eta: D_{k,n}X \wedge D_{k,m}X \rightarrow D_{k,n+m}X,$$

[**30**]. Projection on each component $\Sigma^\infty D_{k,q}X$ of $\bigvee_q \Sigma^\infty D_{k,q}X$ gives the components of the stable splitting of Theorem 6.4, which we denote j_q. The following refined version of Snaith's stable splitting of Ω^kS^kX will be useful for our purposes.

6.5. THEOREM. ([**17**], Theorem H). *For $n \geqq 1$ and connected spaces X, the following is a natural commutative diagram in the stable category, in which the horizontal arrows are equivalences*:

$$\begin{array}{ccc}
\Sigma^\infty(\Omega^k S^k X \times \Omega^k S^k X) & \xrightarrow{\sum_{r \geqq 1} \sum_{p+q=r} j_p \wedge j_q} & \bigvee_r \bigvee_{p+q=r} \Sigma^\infty(D_{k,p}X \wedge D_{k,q}X) \\
\downarrow & & \downarrow \\
\Sigma^\infty \Omega^k S^k X & \xrightarrow{\sum_{r \geqq 1} j_r} & \bigvee_{r \geqq 1} \Sigma^\infty D_{k,r}X
\end{array}$$

Here the map on the left is loop addition and the one on the right is induced by the maps η mentioned above. The stable equivalence $\sum j_r$ is said to be *exponential*, in the sense that it sends sums in $\Sigma^\infty \Omega^k S^k X$ to products in

$$\bigvee_{r \geqq 1} \Sigma^\infty(D_{k,r}X).$$

We now specialize to the case $X = S^r$, r odd, in the discussion above, so that we are dealing with $\Omega^2 S^2 S^r$, r odd. We suppress the index 2 in the symbols $D_{2,q}S^r$, and denote this last space simply by D_q^r. With these conventions we state a result of F. Cohen, Mahowald, and Milgram on D_q^r, $q > 1$.

6.6. THEOREM. ([**20**]).

$$D_q^r = S^{q(r-1)} D_q^1.$$

Thus Snaith's splitting, Theorem 6.4, becomes:

6.7. PROPOSITION.

$$\Sigma^\infty \Omega^2 S^{r+2} \cong \bigvee_{q=1}^{\infty} \Sigma^\infty S^{q(r-1)} D_q^1, \quad r \textit{ odd.}$$

Let

$$\iota \in H_*(\Omega^2 S^3; \mathbf{Z}/2)$$

be the fundamental class, and $Q_1^{j-1}(\iota)$ the $j - 1$ iteration of the homology operation Q_1 (Section 2). Define a weight function wt on $H_*(\Omega^2 S^3; \mathbf{Z}/2)$ by

$$\mathrm{wt}(Q_1^{j-1}(\iota)) = 2^{j-1},$$

and extend it to decomposables by

$$\mathrm{wt}(x \cdot y) = \mathrm{wt}(x) + \mathrm{wt}(y).$$

The image of $H_*(D_1, \mathbf{Z}/2)$ in $H_*(\Omega^2 S^3; \mathbf{Z}/2)$ under the map induced by the stable splitting of 6.4 has been characterized in terms of the function wt above. We quote the result, stated in [**15**].

6.8. Proposition. ([**15**]). *$H_*(D_q^1; \mathbf{Z}/2) \subset H_*(\Omega^2 S^3; \mathbf{Z}/2)$ is generated by all monomials of weight q. Due to Proposition* 6.7 *the proposition holds also for*

$$H_*(D_q^r; \mathbf{Z}/2) \subset H_*(\Omega^2 S^{r+2}, \mathbf{Z}/2), \quad r \text{ odd},$$

if wt *is defined on indecomposables by*

$$\mathrm{wt}(Q_1^{j-1}(\iota)) = 2^{j-1}$$

and then extended to decomposables as above.

We are now prepared to compute the algebra

$$K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2);$$

first we show that it is commutative, and to do so we need the following considerations.

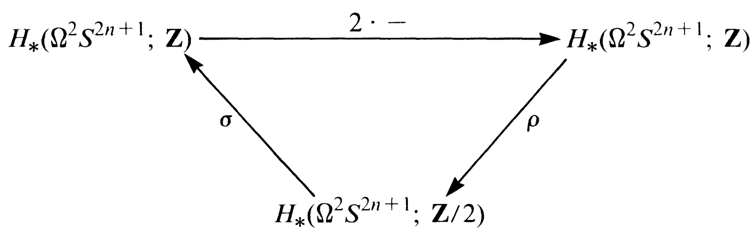
6.9. *Notation.* Denote

$$G = \{ (Q_1^t(\iota))^2 | t \geqq 2\} \subset H_{\mathrm{even}}(\Omega^2 S^{2n+1}; \mathbf{Z}/2).$$

Since $\beta Q_1^{t+1}(\iota) = (Q_1(\iota))^2$ we have that $g \in \operatorname{im} \rho$ for all $g \in G$, where

$$\rho : H_*(\Omega^2 S^{2n+1}; \mathbf{Z}) \rightarrow H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$$

is the mod 2 reduction. So from the exact couple for mod 2 homology (see (2.4)):



we conclude that $G \subset \ker \delta$.

Consider now the Atiyah-Hirzebruch spectral sequences for both integral and mod 2 K-homology. They are multiplicative and thanks to the H-space map there is a pairing (for $\mathbf{Z}$ and $\mathbf{Z}/2$)

$$E_*^{\infty}(\Omega^2 S^{2n+1}) \otimes E_*^{\infty}(\Omega^2 S^{2n+1}) \rightarrow E_*^{\infty}(\Omega^2 S^{2n+1}).$$

The naturality of these spectral sequences implies the following "diagram convergence" modulo lower filtration shown on the next page. The top and bottom triangles are respectively the mod 2 homology and mod 2 K-homology exact couples [**32**], [**1**, P. 3]. Recall the set G defined in 6.9, and notice from Theorem 5.10 that G consists of infinite cycles of the spectral sequence for $K_*\mathbf{Z}/2$.

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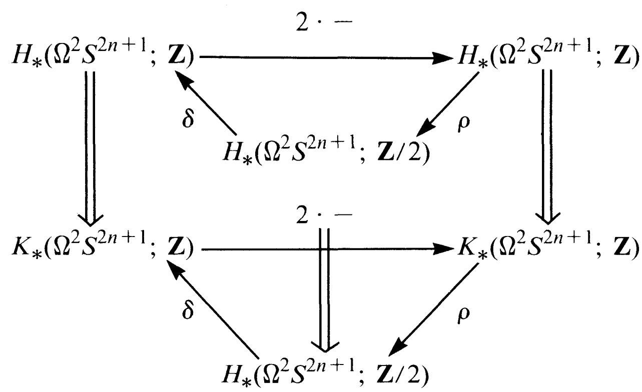
(6.1)



6.10. THEOREM. *$\bar{G} = \{K_*\mathbf{Z}/2$-classes determined by $G\}$. Then $\bar{G} \subset \operatorname{im} \rho$, (in $K_*\mathbf{Z}/2$).*

Proof. Recall from Proposition 2.9 that

$$\rho^{-1}(\operatorname{im} \beta) = \{\text{order 2 elements}\} + \{\text{2-divisible elements}\} \subset H_*(\Omega^2S^{2n+1}; \mathbf{Z}).$$

Choose for each $g \in G$ a 2-torsion element y such that $\rho(y) = g$. We claim that y is an infinite cycle in the integral spectral sequence

$$H_*(\Omega^2S^{2n+1}; \mathbf{Z}) \Rightarrow K_*(\Omega^2S^{2n+1}; \mathbf{Z}).$$

For suppose there is a differential d_{2r+1} in this spectral sequence for which $d_{2r+1}(y) \neq 0$. Then the naturality of the spectral sequence implies that

$$\rho(d_{2r+1}(y)) = d_{2r+1}(\rho(y)) + \{\text{terms in } (\operatorname{im} d_3)\} \subset H_*(\Omega^2S^{2n+1}; \mathbf{Z}/2).$$

This is so by Theorem 5.10, which also implies that

$$d_{2r+1}(\rho(y)) = 0 \quad \text{if } r > 1,$$

thus giving

$$d_{2r+1}(\rho(y)) \in (\operatorname{im} d_3).$$

We next show that this forces $\rho(d_{2r+1}(y)) = 0$. For if

$$0 \neq \rho d_{2r+1}(y) = d_3(z)$$

for some z, then Theorem 5.10 implies that

$$z = \sum_k \left[\bigotimes_{j_k} Q_1^{j_k}(\iota)\right] \otimes w_k,$$

where the j_k's are distinct and bigger than 1, and w_k is a square. Moreover, notice that there must be an even number of factors $Q_1^{j_k}(\iota)$, $j_k > 1$ in each summand, at least two of them as comes from the fact that deg(z) is even and

$$d_3 z \in D^{2n-1}_{2^r+1},$$

the component of g, (by Proposition 6.8). Then

$$0 \neq d_3(z) = \sum_k \left[\bigotimes_{l_k} Q_1^{l_k}(\iota)\right] \otimes w_k,$$

since d_3 is a derivation, where now there are an odd number of factors $Q_1^{l_k}(\iota)$ in each summand, and w_k is a square. Since

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$$

is a polynomial algebra, one can show by a Borel basis argument that

$$\beta d_3(z) = \beta \sum_k \left(\bigotimes_k Q_1^{l_k}(\iota) \otimes w_k\right)$$

is non-zero, which contradicts that $d_3(z) = \rho d_{2r+1}(y)$, thus proving that

$$\rho d_{2r+1}(y) = 0.$$

This implies that $d_{2r+1}(y)$ is 2-divisible, say $2x$, and $d_{2r+1}(y)$ is also 2-torsion by the linearity of the differentials (recall the choice of y). Then

$$0 = 2d_{2r+1}(y) = 2(2x),$$

which, however, contradicts F. R. Cohen's result, (Proposition 2.11), on the torsion of

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2).$$

So we have that $d_{2r+1}(y) = 0$ for all $r \geqq 1$, and y is a permanent cycle in

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}) \Rightarrow K_*(\Omega^2 S^{2n+1}; \mathbf{Z}).$$

Looking at diagram (6.1) we see that the naturality of the spectral sequence implies that $\bar{G} \subset \operatorname{im} \rho$, which proves the theorem.

Theorem 6.11 together with

$$\beta \bar{Q}_1(\iota) = \iota^2 \quad \text{in } K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2),$$

(see 4.18) have the following consequence.

6.12. COROLLARY. *$K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$ is commutative.*

As a further step in our computation of $K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$ as an algebra we will now determine the height of the multiplicative generators for this algebra, which are exhibited in the computation of $K^*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$ as a vector space in Snaith's Theorem 5.2. This theorem takes the following form for $X = S^{2n-1}$:

$$G_rK_*(\Omega^2S^{2n+1}; \mathbf{Z}/2) = \frac{\mathbf{Z}/2[\iota, Q(\iota)]}{(\iota^4, (Q_1(\iota))^4)} \otimes \left(\bigotimes_{t \geqq 2} E(Q_1^t(\iota))^2\right),$$

the fundamental class, deg $(\iota) \equiv 1 \bmod 2$.

6.13. THEOREM. *In the algebra* $K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2)$, *the classes* ι, $Q_1(\iota)$ *have height* 4, *while classes* $(Q_1^t(\iota))^2$, $t \geqq 2$, *have height* 2.

Proof. ι^4 is a d_3-boundary by Proposition 5.9, so if $\iota^4 \neq 0$ in $K\mathbf{Z}/2$ multiplication, then it is a combination of classes in filtration lower than that of ι^4. Inspection of filtrations shows that

$$\iota^4 = \lambda\iota \otimes Q_1(\iota) + \mu\iota^2, \quad \lambda, \mu \in \{0, 1\}.$$

However, we see from Proposition 6.8 that the right members of the above equality do not fall in the component determined by D_4^{2n-1} according to Theorem 6.5, while $\iota^4 \in D_4^{2n-1}$. Thus $\lambda = \mu = 0$ and $\iota^4 = 0$.

Similarly, using $\iota^4 = 0$, we have as the only possibility the following equation

$$(6.2) \qquad (Q_1(\iota))^4 = \lambda_1\iota(Q_1(\iota))^3 + \lambda_2\iota^2(Q_1(\iota))^2 + \lambda_3\iota^3Q_1(\iota) + \lambda_4(Q_1(\iota))^2 + \lambda_5\iota Q_1(\iota) + \lambda_6\iota^2, \quad \lambda_j \in \{0, 1\}.$$

Once again we see by Proposition 6.8 and Theorem 6.5 that $(Q_1(\iota))^4$ lies in the component determined by D_8^{2n-1} in Proposition 6.7, while none of the right member summands of (6.2) does so. Thus $(Q_1(\iota))^4 = 0$.

We consider now $(Q_1^t(\iota))^4$, $t \geqq 2$. If this class is non-trivial in

$$K_*(\Omega^2S^{2n+1}; \mathbf{Z}/2),$$

then

$$(6.3) \qquad (Q_1^t(\iota))^4 = \sum_i \iota^{k_i} \otimes (Q_1(\iota))^{l_i} \otimes \left(\bigotimes_{j_i}(Q_1^{t_{j_i}}(\iota))^2\right)$$

where

$$0 \leqq k_i, \quad l_i < 4, \quad k_i + l_i \equiv 0 \bmod 2, \quad t_{j_i} \geqq 2$$

and with each ith summand at the right of filtration lower than that of $(Q_1^t(\iota))^4$. We prove by induction on $t \geqq 2$ that (6.3) is impossible. For $t = 2$,

$$(Q_1^2(\iota))^4 \in D_{2^4}^{2n-1}$$

by Proposition 6.8, and one checks that no values of k_i, l_i and t_{j_i} satisfying the conditions above are such that any summand at the right of (6.3) is in the component determined by $D_{2^4}^{2n-1}$. Suppose that

$$(Q_1^s(\iota))^4 = 0 \quad \text{for } s < t - 1, t > 2.$$

Then in the expression (6.3) for $(Q_1^t(\iota))^4$ we have that if $2 \leqq t_{j_i} < t$ then t_{j_i} appears at most once in each right summand. Also, by filtration, $(Q_1^t(\iota))^2$ appears at most once in each right term. Again use of Proposition 6.8 allows us to see that no values of k_i, l_i and t_{j_i} are such that the monomial

$$\iota^{k_i} \otimes (Q_1(\iota))^{l_i} \otimes \left(\bigotimes_{j_i} (Q_1^{t_{j_i}}(\iota))^2\right)$$

lies in the component of $K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$

determined by $D_{2^t+2}^{2n-1}$, which is the component of $(Q_1^t(\iota))^4$. Thus

$$(Q_1^t(\iota))^4 = 0 \quad \text{in } K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$$

and the proof of the theorem is complete.

We are ready to prove our result on the algebra structure of

$$K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2).$$

6.14. THEOREM. *As an algebra*

$$K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2) \cong \frac{\mathbf{Z}/2[\iota, Q_1(\iota)]}{(\iota^4, (Q_1(\iota))^4)} \otimes \left(\bigotimes_{t \geqq 2} E(Q_1^t(\iota))^2\right)$$

where ι *is a* $K_*\mathbf{Z}/2$ *representative for the fundamental class of*

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2),$$

and similarly for the other generators at the right.

Proof. Recall from 5.2 that the Atiyah-Hirzebruch spectral sequence is multiplicative, but care must be taken of the fact that it converges to the graded group defined by the quotients of successive filtrations. Due to this last fact, we have that, in $E_*^4 \cong E_*^\infty$ of the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2) \Rightarrow K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2),$$

there are two possible sources of algebra relations, namely:

a) those arising from the identity in $K_*\mathbf{Z}/2$-theory

$$x \cdot y + y \cdot x = \beta x \cdot \beta y,$$

(see (3.2)), and

b) those given by d_3-boundaries which are non-trivial as elements of

$$K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2).$$

From Theorem 6.10 we have that if x is a multiplicative generator, $x \neq Q_1(\iota)$, then $\beta x = 0$, while by Theorem 6.13

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$$\iota^4 = (Q_1(\iota))^4 = 0,$$

so that $x \cdot y + y \cdot x = 0$ for all x and y. Moreover Theorem 6.13 also shows that

$$(Q_1^t(\iota))^4 = 0 \quad \text{for } t \geqq 2.$$

Thus neither a) nor b) produce new relations among the multiplicative generators of $K_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2)$, other than those derived from the Atiyah-Hirzebruch spectral sequence, which proves the theorem.

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