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SYMMETRIC ITINERARY SETS

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Abstract

We consider a one-parameter family of dynamical systems $W : [0, 1] \to [0, 1]$ constructed from a pair of monotone increasing diffeomorphisms W_i such that $W_i^{-1} : [0, 1] \to [0, 1]$ $(i = 0, 1)$. We characterise the set of symbolic itineraries of W using an attractor $\overline{\Omega}$ of an iterated closed relation, in the terminology of McGehee, and prove that there is a member of the family for which $\overline{\Omega}$ is symmetrical.

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1. Introduction

Let $W_0 : [0, a] \to [0, 1]$ and $W_1 : [1 - b, 1] \to [0, 1]$ be continuous and differentiable and such that $a + b > 1$, $W_0(0) = W_1(1 - b) = 0$ and $W_0(a) = W_1(1) = 1$. Let the derivatives $W_i'(x)$ $(i = 0, 1)$ be uniformly bounded below by $d > 1$.

As illustrated in Figure 1, for $\rho \in [1 - b, a]$ we define $W : [0, 1] \to [0, 1]$ by

$$[0, 1] \ni x \mapsto \begin{cases} W_0(x) & \text{if } x \in [0, \rho], \\ W_1(x) & \text{otherwise.} \end{cases}$$

Similarly, we define $W_+ : [0, 1] \to [0, 1]$ by replacing $[0, \rho]$ by $[0, \rho)$.

Let $I = \{0, 1\}$. Let $I^\infty = \{0, 1\} \times \{0, 1\} \times \cdots$ have the product topology induced from the discrete topology on I. For $\sigma \in I^\infty$, write $\sigma = \sigma_0\sigma_1\sigma_2 \ldots$, where $\sigma_k \in I$ for all $k \in \mathbb{N}$. The product topology on I^∞ is the same as the topology induced by the metric $d(\omega, \sigma) = 2^{-k}$, where k is the least index such that $\omega_k \neq \sigma_k$. It is well known that (I^∞, d) is a compact metric space. We define a total order relation $\preceq$ on I^∞, and on I^n for any $n \in \mathbb{N}$, by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_k < \omega_k$, where k is the least index such that $\sigma_k \neq \omega_k$. For $\sigma \in I^\infty$ and $n \in \mathbb{N}$, we write $\sigma|_n = \sigma_0\sigma_1\sigma_2 \ldots \sigma_n$. The space I^∞ is the appropriate one in which to embed and study the itineraries of the family of discontinuous dynamical systems $W : [0, 1] \to [0, 1]$.

For $k \in \mathbb{N}$ and $W_{(+)} \in \{W, W_+\}$, let $W_{(+)}^k$ denote $W_{(+)}$ composed with itself k times and let $W_{(+)}^{-k} = (W_{(+)}^k)^{-1}$. We define a map $\tau : [0, 1] \to I^\infty$, using all of the orbits of W, by

$$\tau(x) = \sigma_0\sigma_1\sigma_2 \ldots,$$

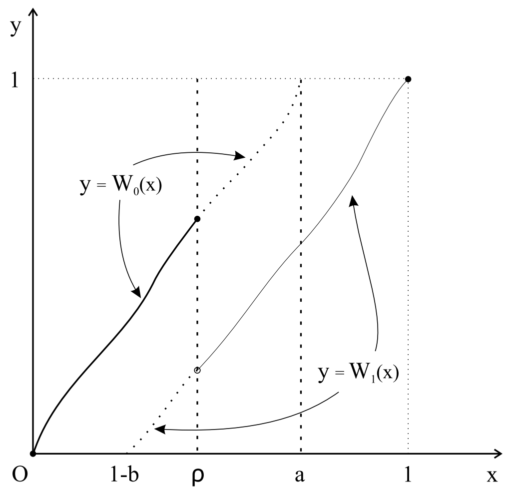

FIGURE 1. The piecewise-continuous dynamical system $W : [0, 1] \to [0, 1]$ is defined in terms of two monotone strictly increasing differentiable functions $W_0(x)$ and $W_1(x)$ and a real parameter ρ.

where σ_k equals 0 or 1, according as $W^k(x) \in [0, \rho]$ or $(\rho, 1]$, respectively. We call $\tau(x)$ the *itinerary* of x under W, or an *address* of x, and we call $\Omega = \tau([0, 1])$ an *address space* for $[0, 1]$. Similarly, we define $\tau^+ : [0, 1] \to I^\infty$ so that $\tau^+(x)_k$ equals 0 or 1, according as $W_+^k(x) \in [0, \rho)$ or $[\rho, 1]$, respectively, and we define $\Omega_+ = \tau^+([0, 1])$. Note that W, W_+, Ω, Ω_+, τ and τ^+ all depend on ρ.

The main goals of this paper are to characterise $\overline{\Omega}$ and to show that there exists a value of ρ such that $\overline{\Omega}$ is symmetric.

THEOREM 1.1. *Define an iterated closed relation $r \subset I^\infty \times I^\infty$ by*

$$r := \{(\sigma, 0\sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(\sigma, 1\sigma) \in I^\infty \times I^\infty : \sigma \succeq \beta\},$$

where $\alpha = \tau(W_0(\rho))$ and $\beta = \tau^+(W_1(\rho))$. The only attractors of r are $\{\overline{0}\}$, $\{\overline{1}\}$, $\{\overline{0}, \overline{1}\}$ and $\overline{\Omega}$. The corresponding dual repellers are the sets $\{\sigma \in I^\infty : \beta \preceq \sigma\}, \{\sigma \in I^\infty : \sigma \preceq \alpha\}$, $\{\sigma \in I^\infty : \beta \preceq \sigma\} \cup \{\sigma \in I^\infty : \sigma \preceq \alpha\}$ and the empty set, respectively. The chain recurrent set for r is $\{\overline{0}, \overline{1}\} \cup \{\sigma \in \overline{\Omega} : \beta \preceq \sigma \preceq \alpha\}$.

We write $\overline{E}$ to denote the closure of a set E, but we write $\overline{0} = 000\ldots$ and $\overline{1} = 111\ldots \in I^\infty$. For $\sigma = \sigma_0\sigma_1\sigma_2\ldots \in I^\infty$, we write 0σ to mean $0\sigma_0\sigma_1\sigma_2\ldots \in I^\infty$ and $1\sigma = 1\sigma_0\sigma_1\sigma_2\ldots \in I^\infty$.

Define a symmetry function $^* : I^\infty \to I^\infty$ by $\sigma^* = \omega$, where $\omega_k = 1 - \sigma_k$ for all k.

THEOREM 1.2. *There exists a unique $\rho \in [1 - b, a]$ such that $\overline{\Omega}^* = \overline{\Omega}$.*

Theorem 1.1 tells us that $\overline{\Omega}$ is fixed by itineraries of two inverse images of the critical point ρ and provides the basis for a stable algorithm to determine $\overline{\Omega}$. It relates the address spaces of dynamical systems of the form of W to the beautiful theory of

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iterated closed relations on compact Hausdorff spaces [4] and hence to the work of Charles Conley.

Theorem 1.2 is interesting in its own right and also because it has applications in digital imaging, as explained and demonstrated, in the special case of affine maps in [1], and in the case of nonlinear maps in [2]. It enables the construction of parameterised families of nondifferentiable homeomorphisms on $[0, 1]$, using pairs of overlapping iterated function systems (see Proposition 4.6). Theorem 1.2 generalises results in [1] to nonlinear W_i's. The proof uses symbolic dynamics in place of the geometrical construction outlined in [1]. The approach and results open up the mathematics underlying [1] and [3].

To tie the present work to [1], note that τ is a section, as defined in [1], for the hyperbolic iterated function system

$$\mathcal{F} := ([0, 1]; W_0^{-1}, W_1^{-1}).$$

Our observations interrelate to, but are more specialised than, the work of Parry [5]. Our point of view is topological rather than measure-theoretic and our main results appear to be new.

2. Basic properties of τ

The following list of properties is relatively easy to check. Below the list we elaborate on points (1), (2) and (3).

(1) W^n is piecewise differentiable and its derivative is uniformly bounded below by d^n; each branch of W^n, except for the leftmost branch, is defined on an interval of the form $(r, s]$; W_+^n is piecewise differentiable and its derivative is uniformly bounded below by d^n; each branch of W_+^n, except for the rightmost branch, is defined on an interval of the form $[r, s)$.

(2) If (r, s) is the interior of the definition domain of a branch of W^n (and of W_+^n), then $\tau(x)|_n$ is constant on $(r, s]$, $\tau^+(x)|_n$ is constant on $[r, s)$ and $\tau(x)|_n = \tau^+(x)|_n$ for all $x \in (r, s)$.

(3) The boundary of the definition domain of a branch of W^n is contained in $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho)$; by (1), the length of such a domain is at most d^{-n}.

(4) The set $\bigcup_{k\in\mathbb{N}} W^{-k}(\rho)$ is dense in $[0, 1]$. This follows from (3).

(5) $\tau(x) = \tau^+(x)$ unless $x \in \bigcup_{k\in\mathbb{N}} W^{-k}(\rho)$, in which case $\tau(x) \prec \tau^+(x)$.

(6) $\tau(x)$ and $\tau^+(x)$ are strictly increasing functions of $x \in [0, 1]$ and $\tau(x) \preceq \tau^+(x)$. This follows from (4) and (5).

(7) For all $x \in [0, 1]$, $\tau(x)$ is continuous from the left and $\tau^+(x)$ is continuous from the right. Moreover, for all $x \in (0, 1)$,

$$\tau(x) = \lim_{\varepsilon\to 0^+} \tau^+(x - \varepsilon) \quad \text{and} \quad \tau^+(x) = \lim_{\varepsilon\to 0^+} \tau(x + \varepsilon).$$

These assertions follow from (2), (3) and (4).

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(8) Each $x \in W^{-n}(\rho)$, such that $\tau(x)|_n$ is constant, moves continuously with respect to ρ with positive velocity bounded above by d^{-n}. This follows from (1).

(9) For $x \in (0,1)\setminus\bigcup_{k=0}^{n} W^{-k}(\rho)$, $\tau(x)|_n = \tau^+(x)|_n$ is locally constant with respect to ρ; moreover, this holds if x depends continuously on ρ. This follows from (2), (3) and (6).

(10) The symmetry function $^* : I^\infty \to I^\infty$ is strictly decreasing and continuous.

(11) For any $\sigma|_n \in I^n$, $n \in \mathbb{N}$, the set

$$\mathcal{I}(\sigma|_n) := \{x \in [0,1] : \tau(x)|_n = \sigma|_n \text{ or } \tau^+(x)|_n = \sigma|_n\}$$

is either empty or a nondegenerate compact interval of length at most d^{-n}. This follows from (2), (3) and (6).

(12) The projection $\hat{\pi} : I^\infty \to [0,1]$ is well defined by

$$\hat{\pi}(\sigma) = \sup\{x \in [0,1] : \tau^+(x) \le \sigma\} = \inf\{x \in [0,1] : \tau(x) \ge \sigma\}.$$

This follows from (6).

(13) The projection $\hat{\pi} : I^\infty \to [0,1]$ is increasing, by (6); continuous, by (11); and, by (7),

$$\hat{\pi}(\tau(x)) = \hat{\pi}(\tau^+(x)) = x \quad \text{for all } x \in [0,1],$$
$$\tau(\hat{\pi}(\sigma)) \le \sigma \le \tau^+(\hat{\pi}(\sigma)) \quad \text{for all } \sigma \in I^\infty.$$

(14) Let $S : I^\infty \to I^\infty$ denote the left-shift map $\sigma_0\sigma_1\sigma_2\ldots \mapsto \sigma_1\sigma_2\sigma_3\ldots$. For all $\sigma \in I^\infty$ such that $\sigma \le \tau(\rho)$ or $\sigma \ge \tau^+(\rho)$,

$$\hat{\pi}(S(\sigma)) = W(\hat{\pi}(\sigma)).$$

Also, $\hat{\pi}(\tau^+(\rho)) = \rho$ and $\hat{\pi}(S(\tau^+(\rho))) = W_1(\rho)$. These statements follow from (7).

Here we elaborate on points (1)–(3). Consider the piecewise-continuous function $W^k(x)$ for $k \in \{1,2,\ldots\}$. Its discontinuities are at ρ and, for $k > 1$, other points in $(0,1)$, each of which can be written in the form $W_{\sigma_0}^{-1} \circ W_{\sigma_2}^{-1} \circ \cdots W_{\sigma_{l-1}}^{-1}(\rho)$ for some $\sigma_0\sigma_1\ldots\sigma_{l-1} \in \{0,1\}^l$ and some $l \in \{1,2,\ldots,k-1\}$. We denote these discontinuities, together with the points 0 and 1, by

$$D_{k,0} := 0 < D_{k,1} < D_{k,2} < \cdots < D_{k,D(k)-1} < 1 =: D_{k,D(k)},$$

where $D(1) = 3, D(2) = 5 < D(3) < D(4) < \cdots$. For each $k \ge 1$, one of the $D_{k,j}$'s is equal to ρ. For $k \ge 1$, we have $W^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1}]$ and $W_+^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1})$. Similarly, $W^k(x) = W_1^k(x)$ for all $x \in (D_{k,D(k)-1}, D_{k,D(k)}]$ and $W_+^k(x) = W_1^k(x)$ for $x \in [D_{D(k)-1}, D_{D(k)}]$.

Observe that, for all $x \in (D_{k,l}, D_{k,l+1})$ $(l = 0,1,\ldots,D(k)-1)$,

$$W^k(x) = W_+^k(x) = W_{\theta_k} \circ W_{\theta_{k-1}} \circ \cdots W_{\theta_1}(x)$$

for some fixed $\theta_1\theta_2\ldots\theta_k \in \{0,1\}^k$. We refer to $\theta_1\theta_2\ldots\theta_k$ as the *address of the interval* $(D_{k,l}, D_{k,l+1})$, we say that $(D_{k,l}, D_{k,l+1})$ '*has address* $\theta_1\theta_2\ldots\theta_k$' and we write, by slight abuse of notation, $\tau((D_{k,l}, D_{k,l+1})) = \theta_1\theta_2\ldots\theta_k$.

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Let $k > 1$. Consider two adjacent intervals, $(D_{k,m-1}, D_{k,m}]$ and $(D_{k,m}, D_{k,m+1}]$, for $m \in \{1, 2, \ldots, D(k) - 1\}$ and $k > 1$. Let the one on the right have address $\theta_0\theta_1 \ldots \theta_{k-1}$ and the one on the left have address $\eta_0\eta_1 \ldots \eta_{k-1}$. Then $\eta_0\eta_1 \ldots \eta_{k-1} \prec \theta_0\theta_1 \ldots \theta_{k-1}$ and

$$\begin{aligned}
\tau(x)|_{k-1} &= \eta_0\eta_1 \ldots \eta_{k-1} \quad \text{for all } x \in (D_{k,m-1}, D_{k,m}],\\
\tau^+(x)|_{k-1} &= \eta_0\eta_1 \ldots \eta_{k-1} \quad \text{for all } x \in [D_{k,m-1}, D_{k,m}),\\
\tau(x)|_{k-1} &= \theta_0\theta_1 \ldots \theta_{k-1} \quad \text{for all } x \in (D_{k,m}, D_{k,m+1}],\\
\tau^+(x)|_{k-1} &= \theta_0\theta_1 \ldots \theta_{k-1} \quad \text{for all } x \in [D_{k,m}, D_{k,m+1}).
\end{aligned}$$

In particular, $\tau(x)|_{k-1}$ and $\tau^+(x)|_{k-1}$ are constant and equal on each of the open intervals $(D_{k,m-1}, D_{k,m})$ and have distinct values at the discontinuity points $\{D_{k,m}\}_{m=1}^{D(k)-1}$.

3. The structures of Ω, Ω_+ and $\overline{\Omega}$

In this section we characterise Ω and Ω_+ as certain inverse limits and we characterise $\overline{\Omega}$ as an attractor of an iterated closed relation on I^∞. These inverse limits are natural and they clarify the structures of Ω and Ω_+. They are implied by the shift invariance of Ω and Ω_+. Recall that $S : I^\infty \to I^\infty$ denotes the left-shift map $\sigma_0\sigma_1\sigma_2 \ldots \mapsto \sigma_1\sigma_2\sigma_3 \ldots$.

Proposition 3.1.

(i) $\tau(W(x)) = S(\tau(x))$ *and* $\tau^+(W_+(x)) = S(\tau^+(x))$ *for all* $x \in [0, 1]$.

(ii) $S(\Omega) = \Omega$ *and* $S(\Omega_+) = \Omega_+$.

Proof. Part (i) follows at once from the definitions of τ and τ^+. Part (ii) follows from (i) together with $W([0, 1]) = W_+([0, 1]) = [0, 1]$. □

We say that $\Lambda \subset I^\infty$ is *closed from the left* if $\lim x_n \in \Lambda$ whenever $\{x_n\}_{n=0}^\infty$ is a nondecreasing sequence of points in Λ. We say that $\Lambda \subset I^\infty$ is *closed from the right* if $\lim x_n \in \Lambda$ whenever $\{x_n\}_{n=0}^\infty$ is a nonincreasing sequence in Λ. For $S \subset X$, where $X = I^\infty$ or $[0, 1]$, we write

$$L(S) = \{\sigma \in X : \text{there is a nondecreasing sequence } \{z_n\}_{n=0}^\infty \subset S \text{ with } \sigma = \lim z_n\}$$

to denote the closure of S from the left. Analogously, we define $R(S)$ for the closure of S from the right.

Proposition 3.2.

(i) Ω *is closed from the left and* Ω_+ *is closed from the right;*

(ii) $\overline{\Omega} = \overline{\Omega_+} = \Omega \cup \Omega_+ = \overline{\Omega \cap \Omega_+}$.

Proof. *Proof of (i).* By (6), $\tau : [0, 1] \to I^\infty$ is monotone strictly increasing. By (7), τ is continuous from the left. Let $\{z_n\}_{n=0}^\infty$ be a nondecreasing sequence of points in Ω, $y_n = \tau^{-1}(z_n)$ and $y = \lim y_n \in [0, 1]$. Since τ is continuous from the left, $\Omega \ni \tau(y) = \tau(\lim y_n) = \lim \tau(y_n) = \lim z_n$. It follows that Ω is closed from the left. Similarly, Ω_+ is closed from the right.

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Proof of (ii). Let $Q = \{x \in [0,1] : \tau(x) = \tau^+(x)\}$. By (4), $\overline{Q} = [0,1]$ and, by (5),

$$\Omega \cap \Omega_+ = \tau([0,1]) \cap \tau^+([0,1]) = \tau(Q) = \tau^+(Q).$$

Hence,

$$\overline{\Omega \cap \Omega_+} = \overline{\tau(Q)} = \overline{\tau^+(Q)} = \overline{\Omega} = \overline{\Omega_+}.$$

Finally, $\Omega \cup \Omega_+ = L(\tau(Q)) \cup R(\tau^+(Q)) = L(\tau(Q)) \cup R(\tau(Q)) = \overline{\tau(Q)} = \overline{\Omega}$. □

We define $s_i : I^\infty \to I^\infty$ by $s_i(\sigma) = i\sigma$ $(i = 0, 1)$. Note that both s_0 and s_1 are contractions with contractivity $1/2$. We write 2^{I^∞} to denote the set of all subsets of I^∞. For $\sigma, \omega \in I^\infty$, we define

$$[\sigma, \omega] := \{\zeta \in I^\infty : \sigma \preceq \zeta \preceq \omega\},$$
$$(\sigma, \omega) := \{\zeta \in I^\infty : \sigma \prec \zeta \prec \omega\},$$
$$(\sigma, \omega] := \{\zeta \in I^\infty : \sigma \prec \zeta \preceq \omega\},$$
$$[\sigma, \omega) := \{\zeta \in I^\infty : \sigma \preceq \zeta \prec \omega\}.$$

Proposition 3.3. *Let $\alpha = S(\tau(\rho))$ and $\beta = S(\tau^+(\rho))$.*

(i) $\Omega = \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$, *where $\Psi : 2^{I^\infty} \to 2^{I^\infty}$ is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \overline{1}]).$$

(ii) $\Omega_+ = \bigcap_{k\in\mathbb{N}} \Psi_+^k([\overline{0}, \overline{1}])$, *where $\Psi_+ : 2^{I^\infty} \to 2^{I^\infty}$ is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha)) \cup s_1(\Lambda \cap [\beta, \overline{1}]).$$

(iii) $\overline{\Omega} = \overline{\Omega_+} = \bigcap_{k\in\mathbb{N}} \overline{\Psi}^k([\overline{0}, \overline{1}])$, *where $\overline{\Psi} : 2^{I^\infty} \to 2^{I^\infty}$ is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap [\beta, \overline{1}]).$$

Proof. *Proof of (i).* Let $S|_\Omega : \Omega \to \Omega$ denote the domain and range restricted shift map. It is readily found that the branches of $S|_\Omega^{-1} : \Omega \to \Omega$ are $s_0|_\Omega : [\overline{0}, \alpha] \cap \Omega \to \Omega$, where

$$s_0|_\Omega(\sigma) = s_0(\sigma) = 0\sigma \quad \text{for all } \sigma \in [\overline{0}, \alpha] \cap \Omega,$$

and $s_1|_\Omega : (\beta, \overline{1}] \cap \Omega \to \Omega$, where

$$s_1|_\Omega(\sigma) = s_1(\sigma) = 1\sigma \quad \text{for all } \sigma \in (\beta, \overline{1}] \cap \Omega.$$

(Note that $\alpha_0 = 1, \beta_0 = 0$ and $\beta \prec \alpha$.) It follows that

$$S|_\Omega^{-1}(\Lambda) = s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \overline{1}]) = \Psi(\Lambda)$$

for all $\Lambda \subset \Omega$. Since $\Omega \subset [\overline{0}, \overline{1}]$,

$$\Omega = S|_\Omega^{-1}(\Omega) = \Psi(\Omega) \subset \Psi([\overline{0}, \overline{1}]).$$

Also, since $\Psi([\overline{0}, \overline{1}]) \subset [\overline{0}, \overline{1}]$, it follows that $\{\Psi^k([\overline{0}, \overline{1}])\}$ is a decreasing (nested) sequence of sets, each of which contains Ω; hence,

$$\Omega \subset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}]).$$

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It remains to prove that $\Omega \supset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$. We note that $s_0([\overline{0}, \alpha]) = [\overline{0}, \tau(\rho)]$ and $s_1((\beta, \overline{1}]) = (\tau^+(\rho), \overline{1}]$, from which it follows that

$$\bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}]) = \bigcap_{k\in\mathbb{N}} \{\sigma \in I^\infty : S^k(\sigma) \in [\overline{0}, \tau(\rho)] \cup (\tau^+(\rho), \overline{1}]\}. \tag{3.1}$$

Let $\omega \in \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$ and suppose that $\omega \notin \Omega$. Let

$$\omega_- = \sup\{\sigma \in \Omega : \sigma \preceq \omega\} \quad \text{and} \quad \omega_+ = \inf\{\sigma \in \Omega : \omega \preceq \sigma\},$$

so that

$$\omega_- \preceq \omega \preceq \omega_+.$$

But $\omega_- \in \Omega$ (since Ω is closed from the left), so

$$\omega_- \prec \omega \preceq \omega_+.$$

Since $\inf\{\sigma \in \Omega : \omega \preceq \sigma\} = \inf\{\sigma \in \Omega_+ : \omega \preceq \sigma\}$ and Ω_+ is closed from the right, we have $\omega_+ \in \Omega_+$. Let $K = \min\{k \in \mathbb{N} : (\omega_-)_k \neq (\omega_+)_k\}$. Then $S^K(\omega_-) \prec S^K(\omega) \preceq S^K(\omega_+)$ and we must have $S^K(\omega_-) = \tau(\rho)$ and $S^K(\omega_+) = \tau^+(\rho)$. So,

$$\tau(\rho) \prec S^K(\omega) \preceq \tau^+(\rho);$$

therefore, $\omega \notin \{\sigma \in I^\infty : S^K(\sigma) \in [\overline{0}, \tau(\rho)] \cup (\tau^+(\rho), \overline{1}]\}$, which, because of (3.1), contradicts our assumption that $\omega \in \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$. Hence, $\omega \in \Omega$ and

$$\Omega \supset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}]).$$

This completes the proof of (i).

Proof of (ii). This is similar to the proof of (i), with the role of $[\overline{0}, \tau(\rho)]$ played by $[\overline{0}, \tau(\rho))$ and the role of $(\tau^+(\rho), \overline{1}]$ played by $[\tau^+(\rho), \overline{1}]$.

Proof of (iii). This is similar to the proofs of (i) and (ii). □

It is helpful to note that the addresses α and β in Proposition 3.3 obey

$$\alpha = \tau(W_0(\rho)), \quad \beta = \tau(W_1(\rho)),$$
$$\tau(\rho) = 0\alpha = 01\alpha_1\alpha_2 \ldots \quad \text{and} \quad \tau^+(\rho) = 0\beta = 10\beta_1\beta_2 \ldots.$$

Let $M > 0$ be such that $D_{k,M+1} = \rho$. It follows from the discussion at the end of Section 2 that $\tau((D_{k,M}, \rho)) = \tau^+((D_{k,M}, \rho)) = 01\alpha_1\alpha_2 \ldots \alpha_{k-2}$ and $\tau((\rho, D_{k,M+2})) = \tau^+((\rho, D_{k,M+2})) = 10\beta_1\beta_2 \ldots \beta_{k-2}$.

Corollary 3.4. *Let $k \geq 1$, $\alpha = \tau(W_0(\rho))$, $\beta = \tau(W_1(\rho))$ and let $M > 0$ be such that $D_{k,M+1} = \rho$. The set of addresses $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$ is uniquely determined by $\alpha|_{k-1}$ and $\beta|_{k-1}$. For some n_1, n_2 such that $0 \leq n_1 < M < n_2 \leq D(k) - 1$, $\tau((D_{k,n_1}, D_{k,n_1+1})) = \beta_0\beta_1 \ldots \beta_{k-2}\beta_{k-1}$ and $\tau((D_{k,n_2}, D_{k,n_2+1})) = \alpha_0\alpha_1 \ldots \alpha_{k-2}\alpha_{k-1}$. The set of addresses $\{\tau((D_{k,l}, D_{k,l+1})) : l \in \{0, 1, \ldots, D(k) - 1\}, l \neq n_1, l \neq n\}$ is uniquely determined by $\alpha|_{k-2}$ and $\beta|_{k-2}$; for example, $\tau((D_{k,M}, \rho)) = 0\alpha_0\alpha_1 \ldots \alpha_{k-2}$ and $\tau((\rho, D_{k,M+2})) = 1\beta_0\beta_1 \ldots \beta_{k-2}$.*

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Proof. It follows from Proposition 3.3 that the set of addresses at level k, namely $\{\tau((D_{k,l}, D_{k,l+1}))\}_{l=0}^{D(k)-1}$, is invariant under the following operation: first, put a '0' in front of each address that is less than or equal to α and then truncate back to length k; second, put a '1' in front of each address that is greater than or equal to β and drop the last digit; finally, take the union of the two resulting sets of addresses. □

4. Symmetry of $\overline{\Omega}$ and a consequent homeomorphism of [0, 1]

Lemma 4.1. *The attractor* $\overline{\Omega} = \{\sigma \in I^\infty : \text{ for all } k \in \mathbb{N},\ \sigma_k = 0 \Rightarrow S^k(\sigma) \leq \tau(\rho)$ *and* $\sigma_0 = 1 \Rightarrow \tau^+(\rho) \leq S^k(\sigma)\}$.

Proof. This is an immediate consequence of Proposition 3.3. □

Corollary 4.2. $\overline{\Omega}$ *is symmetric if and only if* $\alpha = \beta^*$ *(or equivalently* $\tau(\rho) = (\tau^+(\rho))^*$*).*

Lemma 4.3. *The maps* $\tau(\rho)$ *and* $\tau^+(\rho)$ *are strictly increasing as functions of* $\rho \in [a, b]$ *to* I^∞.

Proof. Note that $\tau(\rho)$ depends both implicitly and explicitly on ρ. Let $1 - b \leq \rho < \rho' \leq a$ be such that $\tau(\rho) \geq \tau(\rho')$. Observe that $\tau(\rho)|_0 = \tau(\rho')|_0$.

Assume first that there is a largest $n > 0$ such that $\tau(\rho)|_n = \tau(\rho')|_n := \theta_0\theta_1 \ldots \theta_n$. Then $\tau(\rho) = \theta_0\theta_1 \ldots \theta_n 1 \ldots$ and $\tau^+(\rho) = \theta_0\theta_1 \ldots \theta_n 0 \ldots$, which imply that

$$W_\rho^{n+1}(\rho) \geq \rho \quad \text{and} \quad W_{\rho'}^{n+1}(\rho') \leq \rho'. \tag{4.1}$$

(We write $W = W_\rho$ when we want to note the dependence on ρ.) We may assume that $\tau(\rho)|_n$ is constant on $[\rho, \rho']$ for otherwise we can restrict to a smaller interval with a strictly smaller value of n. As a consequence, at every iteration, we apply the same branch W_0 or W_1 to W_ξ to compute $g(\xi) := W_\xi^n(\xi)$ for all $\xi \in [\rho, \rho']$. Therefore, g is continuous with derivative at least $d^n > 1$, which contradicts (4.1).

The only remaining possibility is that $\tau(\rho) = \tau(\rho')$. We may assume that $\tau(\rho)$ is constant on $[\rho, \rho']$, otherwise we can reduce the problem to the previous case. This would mean that for arbitrarily large n, the image of the interval $[\rho, \rho']$ under g is at an interval of size at least $d^n(\rho' - \rho)$, which is a contradiction.

Essentially the same argument, with the role of τ played by τ^+ and the role of W played by W_+, proves that $\tau^+(\rho)$ is strictly increasing as a function of $\rho \in [1 - b, a]$ to I^∞. □

Corollary 4.4. *The map* $\rho \mapsto \tau(\rho)$ *is left continuous and the map* $\rho \mapsto \tau^+(\rho)$ *is right continuous.*

Proof. Fix a parameter ρ_0 and let $\varepsilon > 0$. Then by (7) there is $x < \rho_0$ which is not a preimage of ρ_0 for any order and such that

$$d(\tau_{\rho_0}^+(x), \tau_{\rho_0}(\rho_0)) < \frac{\varepsilon}{2}.$$

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By (9), for any $n \in \mathbb{N}$ there exists $\delta > 0$ such that the prefix $\tau_\rho^+(x)|_n$ is constant when $\rho \in (\rho_0 - \delta, \rho_0 + \delta)$. Let n be such that $2^{-n} < \varepsilon$ and let $\rho > x$ and $\rho \in (\rho_0 - \delta, \rho_0)$. Then $\tau_\rho^+(x) \prec \tau_\rho^+(\rho)$ and

$$d(\tau_\rho^+(x), \tau_{\rho_0}^+(x)) < \frac{\varepsilon}{2}.$$

Combining the two inequalities,

$$d(\tau_\rho^+(x), \tau_{\rho_0}(\rho_0)) < \varepsilon$$

and, by Lemma 4.3,

$$\tau_\rho^+(x) \prec \tau_\rho(\rho) \prec \tau_{\rho_0}(\rho_0).$$

The distance d has the property that if $\sigma \prec \zeta \prec \sigma'$, then $d(\sigma, \zeta) \le d(\sigma, \sigma')$ and $d(\zeta, \sigma') \le d(\sigma, \sigma')$. This shows that $\rho \mapsto \tau(\rho)$ is left continuous. The right continuity of $\rho \mapsto \tau^+(\rho)$ admits an analogous proof. □

As a consequence of Corollary 4.2, Lemma 4.3 and (10), we obtain the unicity of ρ for which $\overline{\Omega}$ is symmetric.

Corollary 4.5. *There is at most one $\rho \in [1 - b, a]$ such that $\overline{\Omega} = \overline{\Omega}^*$.*

Proof of Theorem 1.2. By Lemma 4.3 and (10), we may define

$$\rho_0 := \sup\{\rho \in [1 - b, a] : \tau(\rho) \preceq \tau^+(\rho)^*\} = \inf\{\rho \in [1 - b, a] : \tau(\rho)^* \preceq \tau^+(\rho)\}.$$

Assume that $\tau(\rho_0) \prec \tau^+(\rho_0)^*$. It is straightforward to check that $1 - b < \rho_0 < a$.

There is a largest $n \ge 2$ such that $\tau(\rho_0)|_n = \tau^+(\rho_0)^*|_n =: \eta = 01\ldots$. Observe that $\tau(\rho_0) = 0\tau(W_0(\rho_0))$ and $\tau^+(\rho_0) = 1\tau^+(W_1(\rho_0))$. If neither $W_0(\rho_0)$ nor $W_1(\rho_0)$ belongs to $\{0, 1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho_0)$, then by (9) both $\tau(\rho)|_{n+1}$ and $\tau^+(\rho)|_{n+1}$ are constant on a neighbourhood of ρ_0, which contradicts the definition of ρ_0.

Let us consider the projection $\hat{\pi}(\tau^+(W_1(\rho_0))^*)$. If $\hat{\pi}(\tau^+(W_1(\rho_0))^*) > W_0(\rho_0)$, then by the continuity of W_0, of $\hat{\pi}$ (by (13)) and of $\rho \mapsto \tau_\rho^+(\rho)$ (by Corollary 4.4) there is a $\rho > \rho_0$ such that $\hat{\pi}_\rho(\tau_\rho^+(W_1(\rho_0))^*) > W_0(\rho)$. By (6) and (13), $\tau_\rho(\rho) \prec \tau_\rho^+(\rho)^*$, which again contradicts the definition of ρ_0.

As $\hat{\pi}$ is increasing, (13) and $\tau(W_0(\rho_0)) \prec \tau^+(W_1(\rho_0))^*$ imply that $\hat{\pi}(\tau^+(W_1(\rho_0))^*) = W_0(\rho_0)$. Let $0 < m < n$ be minimal such that $W^m \circ W_0(\rho_0) = \rho_0$ or $W^m \circ W_1(\rho_0) = \rho_0$. By applying (14) m times,

$$W^m \circ W_0(\rho_0) = \hat{\pi}(S^m(\tau^+(W_1(\rho_0))^*)) = \hat{\pi}(\tau^+(W^m \circ W_1(\rho_0))^*). \tag{4.2}$$

As $\tau^+(\rho_0) = 1\ldots$, if $W^m \circ W_1(\rho_0) = \rho_0$, then

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* = \tau^+(W^m \circ W_1(\rho_0))^* \prec \tau^+(\rho_0),$$

which by (6) and (4.2) implies that $W^m \circ W_0(\rho_0) = \rho_0$. Therefore, $\tau(\rho_0) = \tau^+(\rho_0)^*$ as both are periodic of period $m + 1$ and have the same prefix of length $n > m$, which is a contradiction.

If $W^m \circ W_1(\rho_0) \neq \rho_0$, then $W^m \circ W_0(\rho_0) = \rho_0$ and, by (13), (10) and (4.2),

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* \preceq \tau^+(W^m \circ W_1(\rho_0)) := \sigma'.$$

By (6), this means that $\rho_0 \leq W^m \circ W_1(\rho_0)$, so in fact

$$\rho_0 < W^m \circ W_1(\rho_0).$$

Since $W^{m+1}(\rho_0) = \rho_0, \tau(\rho_0) = \kappa\kappa\kappa \ldots := \kappa^\infty$, where $\kappa = \tau(\rho_0)|_{m+1} = \tau^+(\rho_0)^*|_{m+1}$ as $m + 1 \leq n$. We can write $\tau^+(\rho_0) = \kappa^*\sigma'$; therefore, $\kappa^*\sigma' \prec \sigma'$ by (6) and the previous inequality. By induction, $\kappa^{*\infty} \prec \sigma'$, so

$$\tau^+(\rho_0)^* = \kappa(\sigma'^*) \prec \kappa^\infty = \tau(\rho_0),$$

which is a contradiction.

The case $\tau(\rho_0) \succ \tau^+(\rho_0)^*$ is analogous by the symmetric definition of ρ_0; therefore, $\overline{\Omega}_{\rho_0}$ is symmetric. □

Proposition 4.6. *If $\overline{\Omega} = \overline{\Omega}^*$, then the map $h : [0, 1] \to [0, 1]$ defined by $h(x) = \hat{\pi}(\tau(x)^*)$ is a homeomorphism and $h \circ \hat{\pi} = \hat{\pi} \circ^*$ on I^∞.*

Proof. First, by Corollary 4.2, $\tau(\rho) = \tau^+(\rho)^*$ and points x for which $\tau(x) \neq \tau^+(x)$ are exactly preimages of ρ. In this case, there is $n \geq 0$ such that $\tau(x)$ and $\tau^+(x)$ have the same initial prefix $\kappa := \tau(x)|_n = \tau^+(x)|_n$, and $\tau(x) = \kappa\tau(\rho), \tau^+(x) = \kappa\tau^+(\rho)$. Therefore, by (13), for all $x \in [0, 1]$,

$$\tau(h(x)) = \tau^+(x)^* \quad \text{and} \quad \tau^+(h(x)) = \tau(x)^*.$$

Thus, $h \circ h(x) = x$. By (6), (10) and (13), h is also decreasing and so $h : [0, 1] \to [0, 1]$ is a homeomorphism.

Let $\sigma \in I^\infty$ and $x = \hat{\pi}(\sigma)$. By (13), $\tau(x) \preceq \sigma \preceq \tau^+(x)$. As $\overline{\Omega} = \overline{\Omega}^*$, by Proposition 3.2, there exists $y \in [0, 1]$ such that $\tau(x)^* = \tau^+(y)$. Also, by Lemma 4.1 and Corollary 4.2, $\tau^+(x)^* = \tau(y)$. We may compute $h \circ \hat{\pi}(\sigma) = h(x) = \hat{\pi}(\tau(x)^*) = \hat{\pi}(\tau(y)) = y$, which is also equal to $\hat{\pi}(\sigma^*)$ as $\tau(y) \preceq \sigma^* \preceq \tau^+(y)$. □

5. Iterated closed relations and Conley decomposition for itineraries of *W*

Theorem 1.1 follows from Proposition 3.3, but some extra language is needed. In explaining this language we describe the Conley–McGehee–Wiandt decomposition theorem [4, Theorem 13.1].

For X a compact Hausdorff space, let 2^X be the subsets of X. A *relation* r on X is simply a subset of $X \times X$. A relation r on X is called a *closed relation* if r is a closed subset of $X \times X$. For example, the set $r \subset I^\infty \times I^\infty$ defined in Theorem 1.1,

$$r = \{(0\sigma, \sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(1\sigma, \sigma) \in I^\infty \times I^\infty : \beta \preceq \sigma\},$$

is a closed relation. Following [4], a relation $r \in 2^X$ provides a mapping $r : 2^X \to 2^X$ defined by

$$r(C) = \{y \in X : (x, y) \in r \text{ for some } x \in C\}.$$

Notice that the image of a nonempty set may be empty. Iterated relations are defined by $r^0 = X \times X$ and, for all $k \in \mathbb{N}$,

$$r^{k+1} = r \circ r^k = \{(x, z) : (x, y) \in r, (y, z) \in r^k \text{ for some } y \in X\}.$$

The *omega limit set* of $C \subset X$ under a closed relation $r \subset X \times X$ is

$$\omega(C) = \cap \Re(C),$$

where

$$\Re(C) = \{D \text{ is a closed subset of } X : r(D) \cup r^n(C) \subset D \text{ for some } n \in \mathbb{N}\}.$$

By definition, an *attractor* of a closed relation r is a closed set A such that the following two conditions hold:

(i) $r(A) = A$;
(ii) there is a closed neighbourhood $\overline{\mathcal{N}}(A)$ of A such that $\omega(C) \subset A$ for all $C \subset \overline{\mathcal{N}}(A)$.

The basin $\mathcal{B}(A)$ of an attractor A for a closed relation r on a compact Hausdorff space X is the union of all open sets $O \subset X$ such that $\omega(C) \subset A$ for all $C \subset O$.

Given an attractor A for a closed relation r on a compact Hausdorff space X, there exists a corresponding *attractor block*, namely a closed set $E \subset X$ such that E contains both A and $r(E)$ in its interior and $A = \omega(E)$. Also, there exists a unique *dual repeller* $A^* = X \backslash \mathcal{B}(A)$ which is an attractor for the transpose relation $r^* = \{(y, x) : (x, y) \in r\}$. The set of connecting orbits associated with the attractor/repeller pair A, A^* is given by $C(A) = X \backslash (A \cup A^*)$.

If r is a closed relation on a compact Hausdorff space X, then $x \in X$ is called *chain recurrent* for r if for every closed neighbourhood f of r, x is periodic for f (that is, there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset X$ such that $x_0 = x$, $(x_{p-1}, x_0) \in f$ and $(x_{n-1}, x_n) \in f$ for $n = 1, 2, \ldots, p-1$). The chain recurrent set $\mathcal{R}$ for r is the union of all the points that are chain recurrent for r. A transitive component of $\mathcal{R}$ is a member of the equivalence class on $\mathcal{R}$ defined by $x \sim y$ when for every closed neighbourhood f of r there is an orbit from x to y under f (that is, there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset \mathcal{R}$ such that $x_0 = x$, $x_{p-1} = y$ and $(x_n, x_{n+1}) \in f$ for all $n \in \{0, 1, \ldots, p-1\}$).

THEOREM 5.1 (Conley–McGehee–Wiandt). *If r is a closed relation on a compact Hausdorff space X, then*

$$\mathcal{R} = \bigcup_{A \in \mathcal{U}} C(A),$$

where $\mathcal{R}$ is the chain recurrent set and $\mathcal{U}$ is the set of attractors.

PROOF OF THEOREM 1.1. This follows at once from Proposition 3.3 together with Theorem 5.1, but see [4]. □

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We note that $\overline{\Omega}$ can be embedded in $[0, 1] \subset \mathbb{R}$ using the (continuous and surjective) coding map $\pi : I^\infty \to [0, 1]$ which is associated with the iterated function system $([0, 1]; x \mapsto x/2, x \mapsto (1 + x)/2)$. This coding map π is defined, for all σ, by

$$\pi(\sigma) = \sum_{k \in \mathbb{N}} \frac{\sigma_k}{2^{k+1}}$$

and provides a homeomorphism between $\overline{\Omega}$ and $\pi(\overline{\Omega})$. The point $\sigma \in \overline{\Omega}$ is uniquely and unambiguously represented by the binary real number $0.\sigma$. In the representation provided by π, the map $\overline{\Psi} : 2^{I^\infty} \to 2^{I^\infty}$ becomes the action of the iterated closed relation $\widetilde{r} \subset [0, 1] \times [0, 1] \subset \mathbb{R}^2$ defined by

$$\widetilde{r} := \{(x, x/2) : x \in [0, \pi(\alpha)]\} \cup \{(x, (x + 1)/2) : x \in [\pi(\beta), 1]\}$$

on subsets of $[0, 1]$. It follows from Proposition 3.3(iii) that $\pi(\overline{\Omega})$ is the maximal attractor, as defined in [4], of $\widetilde{r}$. The corresponding dual repeller is the empty set. It is also easy to see that $\{0\}$ and $\{1\}$ are the only other attractors, with corresponding dual repellers $[\pi(\alpha), 1]$ and $[0, \pi(\beta)]$, respectively. It follows from Theorem 5.1 that the chain recurrent set of $\widetilde{r}$ is $\{0, 1\} \cup (\pi(\overline{\Omega}) \cap (\pi(\beta), \pi(\alpha)))$.

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