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Laminar forced convection at low Péclet number. II

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In this paper, the problem of heat transfer to laminar Poiseuille flow in a circular tube is discussed for the case of an insulated tube with a ring source of heat on the boundary. The solution is developed analytically for low values of the Péclet number, and formulae for calculating the eigenvalues and coefficients have been obtained. The temperature distributions in the neighbourhood of the source have been calculated for two values of the Péclet number. The extension to the case of arbitrary wall flux has also been discussed.

1. Introduction

The problem of heat transfer to fully developed laminar flow with prescribed wall heat flux was first considered by Sellars, Tribus and Klein [5] in 1956. Their method was to build up solutions from the known solutions of the classical Graetz problem. Siegel, Sparrow and Hallman [6] tackled the same problem for circular tubes in a direct manner, as did Cess and Shaffer [1], [2] for the case of a flat duct. In each case the basic problem studied was a wall flux uniform on a semi-infinite section of the duct, axial heat conduction was ignored and the incoming fluid was at a uniform temperature. Chia-Jung Hsu [3] extended the solution for the circular tube by including the effects of axial conduction but still ignoring pre-heating of the fluid.

In this paper the wall heat flux is taken to be a delta function, and the solution is obtained including the effects of both axial conduction and

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pre-heating of the fluid. The extension to an arbitrary heat flux follows from the linearity of the problem and the well-known properties of the delta function.

2. Governing equations and their solution

For the case of Poiseuille flow in a circular tube of radius a , the axi-symmetric conduction-convection equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} = \frac{2u_m \rho^* c_v}{\kappa}\left(1 - \frac{r^2}{a^2}\right)\frac{\partial T}{\partial x} , \tag{2.1}$$

where

T is the fluid temperature,

u_m is the mean fluid velocity,

ρ^* is the fluid density,

c_v is the specific heat of the fluid, and

κ is the thermal conductivity of the fluid.

The variables r, x are the usual radial and axial variables in cylindrical polar co-ordinates, and the angular variable disappears because of the symmetry of the problem.

The boundary conditions imposed on equation 2.1 are

$$T \to T_0 \ ; \quad x \to -\infty ,$$

$$\frac{\partial T}{\partial r} = Q\delta(x) \ ; \quad r = a . \tag{2.2}$$

The equation and boundary conditions are made non-dimensional by putting

$$\rho = r/a \ , \quad \xi = x/a \ , \quad \theta = (T-T_0)Q.a \ ,$$

giving

$$\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho} + \frac{\partial^2 \theta}{\partial \xi^2} = 2P\acute{e}(1-\rho^2)\frac{\partial \theta}{\partial \xi} \tag{2.3}$$

with boundary conditions

$$\theta \to 0 \ ; \quad \xi \to -\infty ,$$

$$\frac{\partial \theta}{\partial \rho} = \delta(\xi) \ ; \quad \rho = 1 , \tag{2.4}$$

where $P\acute{e}$ is the non-dimensional Péclet number $u_m \rho^* c_v a/\kappa$.

As in the author's earlier paper [4], equation 2.3 is solved formally by using the double-sided Laplace transform, giving

$$\text{(2.5)} \qquad \theta(\xi, \rho) = \sum_{n=0}^{\infty} e^{\beta_n \xi} \frac{f(\rho;\beta_n,P\acute{e})}{\frac{\partial^2 f}{\partial p \partial \rho}(1;\beta_n,P\acute{e})}, \qquad \xi > 0$$

$$= -\sum_{n=0}^{\infty} e^{\alpha_n \xi} \frac{f(\rho;\alpha_n,P\acute{e})}{\frac{\partial^2 f}{\partial p \partial \rho}(1;\alpha_n,P\acute{e})}, \qquad \xi < 0 .$$

Here $f(\rho;\, p,\, P\acute{e})$ is the solution of

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \left(p^2 - 2pP\acute{e}(1-\rho^2)\right)f = 0 ,$$

$$\text{(2.6)} \qquad f(0;\, p,\, P\acute{e}) = 1 ,$$

and $0 < \alpha_0 < \alpha_1 < \ldots$ are the positive zeros of $\frac{\partial f}{\partial \rho}(1;\, p,\, P\acute{e})$ and $0 = \beta_0 > \beta_1 > \ldots$ are the non-positive zeros.

3. Determination of the eigenvalues and coefficients

In [4], the author derived the form

$$\text{(3.1)} \quad f(\rho;\, p,\, P\acute{e}) = \sum_{n=0}^{\infty} (P\acute{e})^n \sum_{m=0}^{n} J_{n+m}(p\rho)\, \frac{2^m}{3^m p^m} \sum_{s=0}^{m} \left(-\tfrac{1}{3}\right)^s \gamma_{ms}\, \rho^{2m+3s} \phi_{n-m-s}(\rho) ,$$

where $\phi_t(\rho) = (\rho - \rho^3/3)^t/t!$.

Differentiating with respect to ρ and collecting terms, we obtain

$$\text{(3.2)} \qquad \frac{\partial f}{\partial \rho}(\rho;\, p,\, P\acute{e}) = \sum_{n=0}^{\infty} (P\acute{e})^n F_n(\rho,\, p) ,$$

where

$$\text{(3.3)} \quad F_n(\rho,\, p) = p \sum_{m=0}^{n} J_{n+m-1}(\rho p) \cdot \frac{2^m}{3^m p^m} \sum_{s=0}^{m} \left(-\tfrac{1}{3}\right)^s \delta_{ms}\, \rho^{2m+3s} \phi_{n-m-s}(\rho)$$

and

(3.4) $$\delta_{m,s} = \gamma_{m,s} - \gamma_{m-1,s} - (s+1)\gamma_{m-1,s+1} .$$

Direct calculation gives

(3.5) $$\begin{aligned} &\delta_{00} = 1 \\ &\delta_{10} = -\tfrac{1}{2} , \quad \delta_{11} = \tfrac{1}{5} \\ &\delta_{20} = -\tfrac{21}{40} , \quad \delta_{21} = -\tfrac{3}{70} , \quad \delta_{22} = \tfrac{1}{50} \\ &\delta_{30} = -\tfrac{31}{112} , \quad \delta_{31} = -\tfrac{163}{1400} , \quad \delta_{32} = \tfrac{1}{700} , \quad \delta_{33} = \tfrac{1}{750} , \end{aligned}$$

while in general

$$\delta_{r+t,r} = \frac{1}{5^r r!} \left(b_{tt} r^t + \ldots + b_{t0}\right)$$

with

(3.6) $$b_{tt} = \left(\tfrac{2}{7}\right)^t / t! .$$

Rearranging the order of summation in 3.2 and 3.3, and using Lommel's expansion

(3.7) $$(z+h)^{-\frac{1}{2}v} J_v\left((z+h)^{\frac{1}{2}}\right) = \sum_{m=0}^{\infty} \left(-\tfrac{1}{2}h\right)^m z^{-\frac{1}{2}(v+m)} J_{v+m}\left(z^{\frac{1}{2}}\right) / m! ,$$

we obtain

(3.8) $$\frac{\partial f}{\partial \rho}(\rho;\, p,\, P\acute{e}) = \frac{1}{\rho} \sum_{m=0}^{\infty} 2^m \sum_{s=0}^{m} (-1)^s \left(pP\acute{e}\rho^4/3\right)^{m+s} \delta_{ms} J_{2m+s-1}(y)/y^{2m+s-1} ,$$

where

(3.9) $$y^2 = p^2\rho^2 - 2pP\acute{e}(\rho^2-\rho^4/3) .$$

Since $\left|J_n(y)/y^n\right| < \left(\frac{1}{2}\right)^n/n!$ and is $O\left(y^{-n-\frac{1}{2}}\right)$ as $y \to \infty$, this series converges rapidly. Also

(3.10) $$\frac{\partial^2 f}{\partial p \partial \rho}(1;\, p,\, P\acute{e}) = \sum_{m=0}^{\infty} 2^m \sum_{s=0}^{m} (-1)^s \left(P\acute{e}\rho/3\right)^{m+s} \delta_{ms} \left(\frac{m+s}{p} \frac{J_{2m+s-1}(y)}{y^{2m+s-1}} - \left(p - \tfrac{2}{3} P\acute{e}\right) \frac{J_{2m+s}(y)}{y^{2m+s}}\right)$$

with $y^2 = p^2 - \frac{4}{3} pP\acute{e}$. The dominant term in 3.8 is $\frac{1}{\rho} yJ_{-1}(y)$, which takes the value $(p^2 - \frac{4}{3}\,pP\acute{e})^{\frac{1}{2}}J_{-1}\left((p^2 - \frac{4}{3}\,pP\acute{e})^{\frac{1}{2}}\right)$ when $\rho = 1$. Writing d_n for the n-th positive zero of $J_1(x)$, we see that

$$(3.11) \qquad \alpha_n,\ \beta_n \sim \pm\, d_n\left(1+(2P\acute{e}/3d_n)^2\right)^{\frac{1}{2}} + 2P\acute{e}/3 \ ,$$

and this value can be used as a starting value to determine α_n, β_n by Newton's method using the expansion 3.8 and 3.10.

For smaller values of n it is also possible to substitute $\alpha_n = \alpha_{n_0} + \alpha_{n_1}P\acute{e} + \alpha_{n_2}P\acute{e}^2 + \ldots$ in 3.2, 3.3 to obtain

$$(3.12) \qquad \begin{aligned} \alpha_{n_0} &= d_n \ , \\ \alpha_{n_1} &= \tfrac{2}{3} \ , \\ \alpha_{n_2} &= \left[4d_n^2+14\right]/\left[15d_n^3\right] \ , \end{aligned}$$

with similar but more complicated expressions for the following terms. In Table 1, the values of α_{n_2} and α_{n_3} for $n = 1, \ldots, 8$ have been listed. The equivalent expansion for β_n is

$$(3.13) \qquad \beta_n = -\,d_n + \tfrac{2}{3}\,P\acute{e} - \alpha_{n_2}P\acute{e}^2 + \alpha_{n_3}P\acute{e}^3 - \ldots \ .$$

The case $n = 0$ is slightly more complicated. Reverting to equation 2.6, we can expand f in a power series in ρ . Rearranging this series in powers of p , we obtain

$$(3.14) \quad f(\rho;\, p,\, P\acute{e}) = 1 + p\left[\frac{P\acute{e}}{2}\left(\rho^2 - \frac{\rho^4}{4}\right)\right] + p^2\left[\frac{P\acute{e}^2}{4}\left(\frac{\rho^4}{4} - \frac{5\rho^6}{36} + \frac{\rho^8}{64}\right) - \frac{\rho^2}{4}\right] + O(p^3) \ .$$

Hence

$$(3.15) \qquad \left.\frac{\partial f}{\partial \rho}\right|_{\rho=1} = \frac{P\acute{e}}{2}\,p + \left[\frac{7}{96}\,P\acute{e}^2 - \tfrac{1}{2}\right]p^2 + \left[\frac{83P\acute{e}^3}{23040} - \frac{P\acute{e}}{48}\right]p^3 + O(p^4) \ .$$

One root is obviously $p = 0$ for which $\dfrac{\partial^2 f}{\partial p\,\partial \rho} = \dfrac{P\acute{e}}{2}$, while the other root is obtained by substituting $p = \sum_1^\infty a_n P\acute{e}^n$ to obtain

$$(3.16) \qquad \alpha_0 = P\acute{e} - \frac{P\acute{e}^3}{48} + O(P\acute{e}^5) \ ,$$

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for which

$$\frac{\partial^2 f}{\partial p \partial \rho} = -\frac{P\acute{e}}{2} + \frac{17}{48} P\acute{e}^3 + O(P\acute{e}^5) \quad (3.17)$$

These values can also be improved numerically by using 3.8 and 3.10 with $J_n(y)/y^n$ replaced by $I_n(z)/z^n$, with $z^2 = \frac{4}{3} pP\acute{e} - p^2$. Once the eigenvalues have been calculated, it is a simple procedure to evaluate the eigenfunctions and $\int_0^1 \rho(1-\rho^2) f d\rho$ numerically by solving the integral equation

$$f(\rho) = 1 - \int_0^\rho s \log(\rho/s)\{p^2 - 2pP\acute{e}(1-s^2)\} f(s) ds \,, \quad (3.18)$$

which is equivalent to the equation 2.6.

The eigenvalues and some of the coefficients thus calculated for $P\acute{e} = 1$ and $P\acute{e} = .5$ are listed in Tables 2 and 3.

The temperature field in the neighbourhood of the origin is then obtained from 2.5, and some of these values are listed in Tables 4 and 5. The values given for $\xi = 0$ were obtained by using Fejér summation because of the slowness of convergence at this point.

4. Generalizations

If the heat flux at $\rho = 1$ is changed from $\delta(\xi)$ to $f(\xi)$, we can determine the resulting temperature field by using the formula

$$f(\xi) = \int_{-\infty}^{\infty} \delta(t) f(\xi - t) dt \,. \quad (4.1)$$

The temperature field $\psi(\xi, \rho)$ resulting from f is therefore given by

(4.2) $\psi(\xi, \rho)$

$$= \int_{-\infty}^{\infty} \theta(t, \rho) f(\xi - t) dt$$

$$= \sum_0^\infty \frac{f(\rho;\beta_n,P\acute{e})}{\frac{\partial^2 f}{\partial p \partial \rho}(1;\beta_n,P\acute{e})} \int_0^\infty e^{\beta_n t} f(\xi - t) dt - \sum_0^\infty \frac{f(\rho;\alpha_n,P\acute{e})}{\frac{\partial^2 f}{\partial p \partial \rho}(1;\alpha_n,P\acute{e})} \int_{-\infty}^0 e^{\alpha_n t} f(\xi - t) dt \,,$$

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provided the integrals converge. This is assured provided $f(\xi) \to 0$ as $\xi \to -\infty$ and $e^{-P\acute{e}\xi} f(\xi) \to 0$ as $\xi \to +\infty$.

In conclusion, it should be noted that the mean mixed temperature is approximately $\frac{2}{P\acute{e}} e^{-P\acute{e}\xi}$ as $\xi \to -\infty$, so that for small values of the Péclet number the preheating is significant.

Table 1. Coefficients in the expansion $\alpha_n = \alpha_{n_0} + \frac{2}{3} P\acute{e} + \alpha_{n_2} P\acute{e}^2 + \ldots$

n	α_{n_2}	α_{n_3}
1	.086185	.007492
2	.040714	.001101
3	.027098	.000425
4	.020409	.000241
5	.016399	.000166
6	.013718	.000128
7	.011796	.000106
8	.010348	.000092

Table 2. Eigenvalues and associated values for $P\acute{e} = 1$

n	p	$\frac{\partial^2 f}{\partial p \partial \rho}$	$f(1, p)$	$\int_0^1 \rho(1-\rho^2)f(\rho, p)d\rho$
0	.976444	-.522499	1.130204	.266027
1	4.599192	1.76722	-.367443	.071025
2	7.725007	-2.22246	.285927	-.009920
3	10.868041	2.63160	-.241590	.004183
4	14.011195	-2.98704	.212951	-.002194
5	17.153958	3.30490	-.142540	.001317
6	20.296420	-3.59544	.177056	-.000863
7	23.438673	3.86344	-.164974	.000602
8	26.580782	-4.11404	.154773	-.000439
0	0.0	.5	1.0	.250000
1	-3.248102	-1.46838	-.433917	.041103
2	-6.389522	2.02160	.314249	-.013073
3	-9.533755	-2.46479	-.257924	.005374
4	-12.677305	2.84126	.223871	-.002722
5	-15.820260	-3.17464	-.200492	.001583
6	-18.962828	3.47528	.183178	-.001012
7	-22.105147	-3.75190	-.169693	.000692
8	-25.247299	4.00950	.158809	-.000497

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Table 3. Eigenvalues and associated values for $P\acute{e} = .5$

n	p	$\frac{\partial^2 f}{\partial p \partial \rho}$	$f(1, p)$	$\int_0^1 \rho(1-\rho^2)f(\rho, p)d\rho$
0	.496964	-.252658	1.031577	.253931
1	4.189290	1.63066	-.384859	.062720
2	7.359486	-2.15969	.293000	-.011268
3	10.513719	2.58405	-.245641	.004511
4	13.662200	-2.94696	.215659	-.002327
5	16.808107	3.27047	-.194513	.001382
6	19.952650	-3.56350	.178576	-.000899
7	23.096386	3.83407	-.166012	.000623
8	26.239607	-4.08669	.155777	-.000453
0	0.0	.25	1.0	.250000
1	-3.520312	-1.49256	-.419445	.047622
2	-6.692551	2.05956	.307236	-.012776
3	-9.846934	-2.50077	-.253820	.005115
4	-12.995464	2.87414	.221122	-.002593
5	-16.141394	-3.20495	-.198491	.001516
6	-19.285951	3.50346	.181638	-.000974
7	-22.429695	-3.77832	-.168461	.000668
8	-25.572922	4.03444	.157795	-.000482

Table 4. Temperature distribution for $P\acute{e} = 1$

ξ	$\theta(\xi, 0)$	$\theta(\xi, .5)$	$\theta(\xi, 1)$	θ_m
-2.0	.271	.287	.307	.289
-1.0	.751	.759	.817	.765
-0.5	1.126	1.217	1.352	1.233
-0.25	1.367	1.502	1.789	1.541
-0.125	1.489	1.641	2.131	1.703
0.0	1.6	1.8	∞	1.845
0.125	1.695	1.848	2.333	1.910
0.25	1.771	1.907	2.178	1.944
0.5	1.883	1.966	2.066	1.977
1.0	1.974	1.994	2.012	1.996
2.0	1.999	2.000	2.000	2.000

Table 5. Temperature distribution for $P\acute{e} = .5$

ξ	$\theta(\xi, 0)$	$\theta(\xi, .5)$	$\theta(\xi, 1)$	θ_m
-2.0	1.465	1.485	1.511	1.488
-1.0	2.399	2.438	2.488	2.433
-0.5	3.022	3.102	3.218	3.116
-0.25	3.334	3.456	3.721	3.492
-0.125	3.479	3.616	4.085	3.676
0.0	3.6	3.8	∞	3.832
0.125	3.709	3.845	4.308	3.904
0.25	3.789	3.907	4.158	3.941
0.5	3.899	3.967	4.054	3.977
1.0	3.981	3.995	4.009	3.996
2.0	3.999	4.000	4.000	4.000

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