ON SUPPLEMENTS IN FINITE GROUPS

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Let G be a finite group. If N denotes a normal subgroup of G, a subgroup S of G is called a supplement of N if we have $G = SN$. For every normal subgroup of G there is always the trivial supplement $S = G$. The existence of a non-trivial supplement is important for the extension theory, i.e., for the description of G by means of N and the factor group G/N. Generally, a supplement S is the more useful the smaller the intersection $S \cap N$. If we have even $S \cap N = 1$, then S is called a complement for N in G. In this case G is a splitting extension of N by S.

A number of theorems state that a given subgroup S of G is a complement of a suitable normal subgroup N. A well known example is the following theorem of Burnside: If S is a Sylow subgroup of G which is contained in the centre of its normalizer then G contains a normal subgroup N of which S is a complement, i.e. $G = SN$ and $S \cap N = 1$. A paper by D. G. Higman [1], for instance, contains a generalization of this theorem. Another generalization of the theorem of Burnside has been obtained by the author [2] and under much weaker conditions by G. Zappa [3].

The theorems in [2] and [3] are based upon a special property which a system of coset representatives of a subgroup may have. Let H be a subgroup of G and let

$$G = \sum_{r \in R} Hr$$

denote the decomposition of G into cosets with respect to H. If the system R of coset representatives has the property:

$$h^{-1}Rh = R \quad \text{for any} \quad h \in H,$$

then R is called a *distinguished* system of coset representatives. The main theorem in [3] deals with Hall subgroups H, i.e., subgroups H whose order is prime to their index $[G : H]$. It states:

Let H be a nilpotent Hall subgroup of G possessing a distinguished system of coset representatives. Then G contains a normal subgroup N such that $G = HN$, $H \cap N = 1$.

In the present note, we shall generalize the theorem of Zappa by giving a condition under which a subgroup H is a supplement of a suitable normal subgroup N and an upper bound for the intersection $H \cap N$.

Let $r_1, \cdots, r_n$ denote a system of coset representatives of G with respect to H. So we have $n = [G:H]$ and

$$G = \sum_{\nu=1}^{n} Hr_\nu.$$

Transforming $r_1, \cdots, r_n$ by the elements of H we obtain

(1) $$h^{-1}r_\nu h = c_{\nu,h}r_{\nu h} \qquad (\nu = 1, \cdots, n;\, h \in H),$$

where the $c_{\nu,h}$ are in H and $r_{1h}, \cdots, r_{nh}$ form a permutation of $r_1, \cdots, r_n$, depending on h. The mappings

$$r_\nu \to c_{\nu,h}r_{\nu h} \qquad (\nu = 1, \cdots, n)$$

yield an intransitive monomial representation of H, the coefficients of which belong also to H. The subgroup C of H which is generated by all $c_{\nu,h}(\nu = 1, \cdots, n;\, h \in H)$ shall be called the *coefficient group* belonging to the system $r_1, \cdots, r_n$ of coset representatives. The distinguished systems of coset representatives are exactly those for which the corresponding coefficient group consists of the unit element alone.

It is easy to see that C is always a normal subgroup of H. For if $k \in H$ we have

$$k^{-1}h^{-1}r_\nu hk = k^{-1}c_{\nu,h}kk^{-1}r_{\nu h}k = k^{-1}c_{\nu,h}kc_{\nu h,k}r_{\nu hk}$$

and on the other hand

$$(hk)^{-1}r_\nu(hk) = c_{\nu,hk}r_{\nu hk}.$$

Hence

$$k^{-1}c_{\nu,h}kc_{\nu h,k} = c_{\nu,hk},$$
$$k^{-1}c_{\nu,h}k = c_{\nu,hk}c_{\nu h,k}^{-1} \in C.$$

This proves that C is a normal subgroup of H.

THEOREM. *Let H be a subgroup of G and let C denote the coefficient group belonging to a system R of coset representatives of G with respect to H. If H/C is nilpotent and if $[G:H]$ is prime to $[H:C]$ then G contains a normal subgroup N such that $G = HN$ and $H \cap N \subseteqq C$.*

Using the terminology of [1], the proof of this theorem may be sketched as follows: From our condition it follows that C is chained to H in G. So Theorem 3.1 of [1] is valid, and Corollary 3.5 yields the theorem. We shall give a detailed proof, however.

If V, W are subgroups of G and $W \subseteqq V$, then $(W, V)^*$ shall denote the subgroup of W generated by all those commutators

$$(w, v) = wvw^{-1}v^{-1} (w \in W, v \in V)$$

which are contained in W. Obviously, $(W, V)^*$ contains the commutator subgroup W' of W, hence $(W, V)^*$ is a normal subgroup of W.

For a set π of prime numbers we shall denote by $P(\pi)$ the subgroup of G which is generated by all those elements of G whose orders are not divisible by any prime in π.

Let U be a subgroup of G and T a normal subgroup of U. We assume that π contains all prime divisors of $[U : T]$ and write

$$P(\pi) = P, \quad P \cap U = A, \quad P \cap T = B.$$

LEMMA 1.

$$x^{[P:A]} \in (A, P)^* B \quad \text{for each} \quad x \in A.$$

PROOF. The transfer of P into A is a homomorphism τ of P into the factor group A/A'. In order to compute the image x^τ of an element x in P we may use the formula

$$x^\tau = A' \prod_{\lambda=1}^{l} t_\lambda x^{f_\lambda} t_\lambda^{-1}.$$

Here the t_λ are suitable elements in P, the f_λ are integers, and

$$t_\lambda x^{f_\lambda} t_\lambda^{-1} \in A, \quad f_1 + \cdots + f_l = [P : A].$$

In particular, if x is in A we have

$$t_\lambda x^{f_\lambda} t_\lambda^{-1} x^{-f_\lambda} = (t_\lambda, x^{f_\lambda}) \in A,$$

hence

$$t_\lambda x^{f_\lambda} t_\lambda^{-1} = (t_\lambda, x^{f_\lambda}) x^{f_\lambda} \equiv x^{f_\lambda} \text{ mod. } (A, P)^*.$$

So we find

$$x^\tau \equiv x^{f_1 + \cdots + f_l} = x^{[P:A]} \text{ mod. } (A, P)^*.$$

Now A' is contained in $(A, P)^* B$, for A' is even a subgroup of $(A, P)^*$. There exists therefore a natural homomorphism ν of A/A' onto $A/(A, P)^* B$. Then $\sigma = \tau\nu$ is a homomorphism of A into $A/(A, P)^* B$ such that

$$(2) \qquad x^\sigma \equiv x^{[P:A]} \text{ mod. } (A, P)^* B \quad (x \in A).$$

The order of the factor group $A/(A, P)^* B$ divides $[A : B]$, and $[A : B] = [P \cap U : P \cap T]$ divides $[U : T]$. Since π contains all prime divisors of $[U : T]$, all prime divisors of the order of $A/(A, P)^* B$ are in π. Hence, since $x \in A \subseteqq P$, it follows from the definition of P that $\sigma = 0$. So (2) yields

$$1 \equiv x^{[P:A]} \quad \text{mod.} \quad (A, P)^* B,$$

which proves the lemma.

The main step towards the proof of our theorem is the following lemma, asserting that the divisibility theorem 3.1 of [1] holds, if the conditions of our theorem are satisfied.

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LEMMA 2. Let the conditions of the theorem be satisfied and let π denote the set of all prime divisors of $[H : C]$. Then every prime divisor of $[P(\pi) \cap H : P(\pi) \cap C]$ divides $[P(\pi) : P(\pi) \cap H]$.

PROOF. Let $H^{(\mu)}$ denote the μ-th term of the lower central series of H, i.e.

$$H^{(0)} = H,$$

$H^{(\mu+1)} =$ the subgroup of H which is generated by all commutators

$$(h^{(\mu)}, h) \quad \text{with} \quad h^{(\mu)} \in H^{(\mu)}, h \in H \qquad (\mu = 0, 1, \cdots).$$

Since H/C is nilpotent there exists an integer m such that $H^{(m)} \subseteqq C$. Writing

$$H_\mu = H^{(\mu)} C \qquad (\mu = 0, 1, \cdots, m)$$

we obtain the series

$$H = H_0 \supset H_1 \supset \cdots \supset H_m = C.$$

Here every H_μ is a normal subgroup of H. The subgroup $(H_\mu, G)^*$ is generated by all those commutators

$$h_\mu g h_\mu^{-1} g^{-1} \quad (h_\mu \in H_\mu, g \in G)$$

which are contained in H_μ. Writing $g = hr (h \in H, r \in R)$ we have in view of (1) and since C is a normal subgroup of H

$$\begin{aligned} h_\mu g h_\mu^{-1} g^{-1} &= h_\mu h r h_\mu^{-1} r^{-1} h^{-1} \\ &= h_\mu h h_\mu^{-1} c_1 r_1 r^{-1} h^{-1} \\ &= h_\mu h h_\mu^{-1} h^{-1} c_2 r_2 r_3^{-1}, \end{aligned}$$

where c_1, c_2 are in C and r_1, r_2, r_3 in R. If the last product is contained in H_μ, it follows that $r_2 = r_3$ and furthermore

$$h_\mu g h_\mu^{-1} g^{-1} = h_\mu h h_\mu^{-1} h^{-1} c_2 \in H^{(\mu+1)} C = H_{\mu+1}.$$

Hence we have

$$(H_\mu, G)^* \subseteqq H_{\mu+1}. \tag{3}$$

We write $P(\pi) = P$,

$$P \cap H_\mu = T_\mu \qquad (\mu = 0, 1, \cdots, m),$$

in particular $P \cap H = T_0$, $P \cap C = T_m$. Then $T_{\mu+1}$ is a normal subgroup of T_μ and, by (3),

$$(T_\mu, P)^* \subseteqq T_{\mu+1} \qquad (\mu = 0, 1, \cdots, m-1). \tag{4}$$

Since π contains all prime divisors of $[T_\mu : T_{\mu+1}]$, Lemma 1 can be applied and yields in view of (4)

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(5) $$x^{[P:T_\mu]} \in (T_\mu, P)^* T_{\mu+1} = T_{\mu+1} \quad \text{for every} \quad x \in T_\mu.$$

Now we prove that every prime divisor of $[P : T_\mu]$ also divides $[P : T_0] = [P : P \cap H]$. This proposition being true for $\mu = 0$ we may proceed by induction. We have $[P : T_{\mu+1}] = [P : T_\mu][T_\mu : T_{\mu+1}]$. By (5), the index $[T_\mu : T_{\mu+1}]$ cannot be divisible by any prime different from those dividing $[P : T_\mu]$. So $[P : T_{\mu+1}]$ contains only such prime divisors which divide $[P : T_\mu]$. Hence, if we assume that every prime divisor of $[P : T_\mu]$ divides $[P : P \cap H]$, the same is true for $[P : T_{\mu+1}]$. For $\mu = m$ we obtain that every prime divisor of $[P : P \cap C]$ divides $[P : P \cap H]$. This proves Lemma 2.

Using Lemma 2 it is easy to prove our theorem.

Since π is the set of all primes dividing $[H : C]$ and since, by hypothesis, $[G : H]$ is prime to $[H : C]$, no prime divisor of $[G : H]$ is contained in π. It follows that $G = HP(\pi)$. For let q be a prime which is not in π, then $P(\pi)$ contains the Sylow q-subgroups of G. On the other hand for a prime $p \in \pi$ the index $[G : H]$ is not divisible by p, so the order of H must be divisible by the same power of p as the order of G. Hence $HP(\pi)$ has the same order as G.

By Lemma 2, every prime divisor of $[P(\pi) \cap H : P(\pi) \cap C]$ divides $[P(\pi) : P(\pi) \cap H]$. On the other hand $[P(\pi) \cap H : P(\pi) \cap C]$ divides $[H : C]$ and hence is prime to

$$[G : H] = [HP(\pi) : H] = [P(\pi) : P(\pi) \cap H].$$

We have therefore $[P(\pi) \cap H : P(\pi) \cap C] = 1$, hence $P(\pi) \cap H \subseteqq C$, which proves the theorem.

The assumption that H/C is nilpotent can probably by replaced by a weaker one (cf. [3]). The following example shows, however, that it would not be sufficient to assume only that H/C is solvable. Let G be the symmetric group of degree 5 on $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$; H the subgroup which leaves α_5 unchanged, and let R consist of 1, (α_1, α_5), (α_2, α_5), (α_3, α_5), (α_4, α_5). Then we have $C = 1$, hence R is a distinguished system of coset representatives. However, G contains no normal subgroup of order 5.

References

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