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SOME GENERALIZATIONS OF RAMANUJAN'S SUM

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0. Introduction. Ramanujan's well known trigonometrical sum $C(m, n)$ defined by

$$C(m, n) = \sum_{x} e^{(2\pi imx/n)}$$

where x runs through a reduced residue system (mod n), had been shown to occur in analytic problems concerning modular functions of one variable, by Poincaré [**4**]. Ramanujan, independently later, used these trigonometrical sums in his remarkable work on representation of integers as sums of squares [**6**]. There are various generalizations of $C(m, n)$ in the literature (some also to algebraic number fields); see, for example, [**9**] which gives references to some of these. Perhaps the earliest generalization to algebraic number fields is due to H. Rademacher [**5**]. We here consider a novel generalization involving matrices.

In his important work on the analytic theory of quadratic forms and modular functions, Siegel [**7**] had introduced the notion of coprimality for matrices. Christian [**2**], in an interesting paper, generalized the notion of Euler's totient function and proved elementary properties of it. Since $C(m, n)$ for $m = n$ has the value $\phi(n)$, it seems interesting to study the Ramanujan sum for matrices. It turns out that this sum has multiplicative properties. However evaluating it seems a difficult problem except in the case of 2 rowed matrices. We however show how one can evaluate these sums for matrices of higher order provided the modulus is of a special type. These sums occur, in a rather complicated way, in the Fourier expansions of Eisenstein series [**7**].

Siegel's modular forms have been generalized in several ways. An important class is the so-called Hermitian modular forms due to H. Braun [**1**]. We take the simple example of the Hermitian pairs over the ring of Gaussian integers and evaluate the sums also in the case of two rowed matrices.

It is possible to generalize these results to the case of algebraic number fields. About these and other aspects we shall report on another occasion.

1. Coprime residue classes. Let $n \geqq 1$ be a natural number. We shall consider integral n-rowed matrices. For any such matrix A let A' denote its transpose. A matrix U is said to be *unimodular* if $|U| = \pm 1$ so that U^{-1} is also an integral matrix.

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We call two integral matrices C and D a *symmetric pair* if

(1) $CD' = DC'$.

If U and V are unimodular matrices, it is clear that UCV and UDV'^{-1} are also a symmetric pair. We call two matrices C and D *coprime* if for every rational vector $\mathbf{x}$, $\mathbf{x}'C$, $\mathbf{x}'D$ integral implies $\mathbf{x}$ is integral. In this case it follows that the matrix

$$P = (CD)$$

of n rows and $2n$ columns has rank n and that there exist integer matrices X and Y such that

$$(2) \qquad CX + DY = E = \begin{pmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & 1 \end{pmatrix},$$

E being the n-rowed unit matrix.

Let C be an integral non-singular n-rowed matrix. We say that two integral n-rowed matrices D_1 and D_2 are in the same *residue class* mod C if

(3) $C^{-1}D_1 \equiv C^{-1}D_2 \pmod 1$

i.e., $C^{-1}(D_1 - D_2)$ is an integral matrix. It is possible to determine the exact number of residue classes mod C. This had been done by Eisenstein.

A residue class mod C is said to be a *symmetric residue class* if any representative D of that class satisfies

(4) $C^{-1}D \equiv (C^{-1}D)' \pmod 1$.

If D_1 is any other representative of that residue class then $C^{-1}(D - D_1) \equiv 0 \pmod 1$ and so

$$C^{-1}(D - D_1) \equiv (C^{-1}(D - D_1))' \pmod 1.$$

Therefore from (4) it follows that

$$C^{-1}D_1 \equiv (C^{-1}D_1)' \pmod 1.$$

A residue class represented by D is called a *coprime residue class* if C and D form a coprime pair. Then it is clear from (3) that C and D_1 also form a coprime pair since $D_1 = D + CT$ for some integral matrix T. We shall consider only coprime, symmetric residue classes mod C.

Let D run through a complete system of coprime symmetric residue classes mod C. We assert that for unimodular U and V, UDV'^{-1} run through a complete residue system mod UCV which are coprime and symmetric. That UCV and UDV'^{-1} are coprime, symmetric residue classes mod UCV are easily verified. If D_1 and D_2 are distinct coprime

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symmetric residue classes mod C, then $UD_1V'^{-1}$, $UD_2V'^{-1}$ are distinct mod UCV; for otherwise

$$(UCV)^{-1}(UD_1V'^{-1} - UD_2V'^{-1})$$

which equals $V^{-1}C^{-1}(D_1 - D_2)V'^{-1}$ cannot be $\equiv 0 \pmod 1$ since it would mean

$$C^{-1}(D_1 - D_2) \equiv 0 \pmod 1.$$

Since by the elementary divisor theorem U and V can be chosen so that

$$(5) \qquad UCV = \begin{pmatrix} f_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & f_n \end{pmatrix} = F$$

where $f_i | f_{i+1}$, $i = 1, 2, \cdots, n-1$, we shall consider only coprime, symmetric residue classes mod F.

We shall denote by $F_1, F_2, \cdots$ matrices of the type (5), which are called *elementary divisor matrices*. Christian has proved [**2**]

LEMMA 1. *Let F_1 and F_2 be two elementary divisor matrices which are coprime. Let D_1 run through a complete system of coprime symmetric residue classes* mod F_1 *and D_2 similarly for F_2. Then $D = F_1D_2 + F_2D_1$ runs through a complete system of coprime symmetric residue classes* mod F_1F_2.

This shows that if $\phi_n(F)$ denotes the number of coprime, symmetric residue classes mod F, then

LEMMA 2. *If F_1 and F_2 are two coprime elementary divisor matrices, then*

$$\phi_n(F_1 \cdot F_2) = \phi_n(F_1) \cdot \phi_n(F_2).$$

This lemma reduces the computation of $\phi_n(F)$ to the case where

$$F = [p^{a_1}, \cdots, p^{a_n}], \; 0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n,$$

p being a prime number.

It is to be remarked that everything that has been mentioned above goes through in any commutative integrity domain, with unit, which has unique factorization into prime elements. For Lemma 2 it would be necessary to assume that the residue class ring modulo any element is a finite ring.

2. The trigonometric sum $W(F, T)$ and its evaluation for $n = 2$. We now introduce the trigonometrical sums.

Let C be an integral non-singular n-rowed matrix and T any integral

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symmetric matrix. Consider the sum

(6) $$W(C, T) = \sum_{D} e^{2\pi i\sigma(C^{-1}DT)}$$

where σ denotes matrix trace and D runs through a complete system of coprime symmetric residue classes mod C. It is clear that the sum is independent of the choice of representatives D in the residue class mod C. For if

$$C^{-1}D \equiv C^{-1}D_1 \pmod 1,$$

then

(7) $$\sigma(C^{-1}(D - D_1)T) = \sigma(LT) \equiv 0 \pmod 1)$$

L being an integral matrix.

Let U and V be two unimodular matrices. Then

$$W(UCV, T) = \sum_{D_1 (\text{mod} UCV)} e^{2\pi i\sigma((UCV)^{-1}D_1T)}$$

where the summation is over a complete system of coprime, symmetric residues mod UCV. But if D runs through a complete system of coprime, symmetric residues mod C, then UDV'^{-1} runs through a complete system mod UCV. Thus

$$W(UCV, T) = \sum_{D} e^{2\pi i\sigma((UCV)^{-1}(UDV'^{-1})T)}$$

which gives

$$W(UCV, T) = \sum_{D} e^{2\pi i\sigma(C^{-1}DV'^{-1}TV^{-1})}.$$

Therefore we have the relation

(8) $$W(UCV, T) = W(C, V'^{-1}TV^{-1}).$$

Since for any C, U and V can be so chosen that

$$UCV = F = \begin{pmatrix} f_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & f_n \end{pmatrix}$$

we get

(9) $$W(F, T) = \sum_{D} e^{2\pi i\sigma(FD^{-1}T)}$$

where D runs through a complete system of coprime, symmetric residues mod F.

We call $W(F, T)$ the *Ramanujan sum* associated to F and T.

We shall now evaluate this sum for $n > 1$. Because of Lemmas 1 and 2, we have

LEMMA 3. *If F_1 and F_2 are elementary divisor matrices which are coprime, then*

$$W(F_1, T) \cdot W(F_2, T) = W(F_1F_2, T).$$

We can therefore consider the case, p a prime and

$$F = \begin{pmatrix} p^{a_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & p^{a_n} \end{pmatrix}, \quad 0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n.$$

We shall only consider the case $n = 2$ as for $n \geqq 3$ the computations seem difficult.

Put

$$F = \begin{pmatrix} p^{a_1} & 0 \\ 0 & p^{a_2} \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix}$$

integral symmetric. Let

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}.$$

Then

$$F^{-1}D \equiv (F^{-1}D)' \pmod 1$$

so that

$$\begin{pmatrix} p^{-a_1}d_1 & p^{-a_1}d_2 \\ p^{-a_2}d_3 & p^{-a_2}d_4 \end{pmatrix} \equiv \begin{pmatrix} p^{-a_1}d_1 & p^{-a_2}d_3 \\ p^{-a_1}d_2 & p^{-a_2}d_4 \end{pmatrix} \pmod 1$$

which gives

(10) $\quad d_3 \equiv p^{-a_1+a_2}d_3 \pmod 1.$

Now, we have

(11) $\quad W(F, T) = \sum_{d_1, d_3, d_4} e^{2\pi i (p^{-a_1}d_1 t_1 + 2p^{-a_1}d_2 t_2 + p^{-a_2}d_4 t_4)}$

where d_1, d_3, d_4 satisfy the following conditions:

(12) $\quad d_1 \pmod{p^{a_1}}$, $d_2 \pmod{p^{a_1}}$, $d_4 \pmod{p^{a_2}}$

subject to the condition that F and D are coprime.

We shall consider three cases:

Case (i). $0 = a_1 < a_2$. Then

(13) $\quad W(F, T) = \sum_{d_3, d_4} \exp(2\pi i p^{-a_2} d_4 t_4)$

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d_3, d_4 satisfying (12) and F and D coprime. This means that the matrix

$$(14)\quad \begin{pmatrix} 1 & 0 & d_1 & d_2 \\ 0 & p^{a_2} & p^{a_2-a_1}d_2 & d_4 \end{pmatrix}$$

can be completed to a matrix with determinant prime to p. In this case we have

$$(d_4, p) = 1.$$

This gives at once

$$(15)\quad W(F, T) = C(p^{a_2}, t_4)$$

where $C(m, n)$ is the usual Ramanujan sum.

Case (ii). $0 < a_1 < a_2$. Then $W(F, T)$ is given by (11) and (12) and we have to look at the condition (14). It leads to

$$(16)\quad d_1 d_4 \not\equiv 0 \pmod{p}.$$

Since d_1, d_2, d_4 are independent, we have

$$W(F, T) = \Bigg(\sum_{\substack{d_1 \pmod{p^{a_1}} \\ (d_1, p)=1}} \exp(2\pi i d_1 t_1/p^a)\Bigg)$$
$$\times \Bigg(\sum_{d_2 \pmod{p^{a_1}}} \exp(4\pi i d_2 t_2/p^{a_1})\Bigg) \times \Bigg(\sum_{\substack{d_4 \pmod{p^{a_2}} \\ (d_4, p)=1}} \exp(2\pi i d_4 t_4/p^{a_2})\Bigg).$$

The second factor is zero unless p^{a_1} divides $2t_2$. In this case the value of $W(F, T)$ is

$$(17)\quad W(F, T) = C(p^{a_1}, t_1) \cdot C(p^{a_2}, t_4) \cdot p^{a_1}.$$

Case (iii). $0 < a_1 = a_2$. The condition (14) now is given by the matrix

$$\begin{pmatrix} p^a & 0 & d_1 & d_2 \\ 0 & p^a & d_2 & d_4 \end{pmatrix}, \quad a = a_1 = a_2 > 0$$

having

$$(18)\quad d_1 d_4 - d_2^2 \not\equiv 0 \pmod{p}.$$

We write the sum (11) as

$$(19)\quad W(F, T) = W_1 + W_2$$

where

$$W_1 = \sum_{\substack{d_1, d_2, d_4 \\ p \mid d_1}}, \quad W_2 = \sum_{\substack{d_1, d_2, d_4 \\ p \nmid d_1}}.$$

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Let us evaluate W_1. By (18), since $p|d_1$, and d_4 is arbitrary, $p \nmid d_2$ and so we have

$$W_1 = \left(\sum_{\substack{d_1 \pmod{p^a} \\ p|d_1}} \exp(2\pi i d_1 t_1/p^a)\right) \times \left(\sum_{\substack{d_2 \pmod{p^a} \\ p\nmid d_2}} \exp(2\pi i \cdot 2d_2 t_2/p^a)\right) \times \left(\sum_{d_4 \pmod{p^a}} \exp(2\pi i d_4 t_4/p^a)\right).$$

The last factor vanishes unless $p^a|t_4$ when its value is p^a and similarly the first factor vanishes unless $p^{a-s}|t_1$ in which case its value is p^{a-s}, where p^s, $s \leqq a$, is the higher power of p dividing d_1. Thus

$$(20) \quad W_1 = \begin{cases} 0 & , \text{if } p^a \nmid t_4 \quad \text{or} \quad p^{a-s} \nmid t_1, \quad 1 \leqq s < a \\ p^{2a-s} C(p^a, 2t_2), & \text{otherwise.} \end{cases}$$

Let us now consider the sum (20). We can write

$$W_2 = W_3 + W_4$$

where

$$W_3 = \sum_{\substack{d_1, d_2, d_4 \\ p\nmid d_1, p|d_4}}, \quad W_4 = \sum_{\substack{d_1, d_2, d_4 \\ p\nmid d_1 d_4}}.$$

Let us consider W_3. Using (11) we have

$$W_3 = \left(\sum_{\substack{d_1 \pmod{p^a} \\ p\nmid d_1}} \exp(2\pi i d_1 t_1/p^a)\right) \times \left(\sum_{\substack{d_2 \pmod{p^a} \\ p\nmid d_2}} \exp(2\pi i \cdot 2d_2 t_2/p^a)\right) \times \left(\sum_{\substack{d_4 \pmod{p^a} \\ p|d_4}} \exp(2\pi i d_4 t_4/p^a)\right).$$

As before we have

$$(21) \quad W_3 = \begin{cases} 0 & , \text{if } p^{a-s} \nmid t_4, \quad 1 \leqq s < a \\ p^{a-s} C(p^a, t_1) \cdot C(p^a, 2t_2), & \text{otherwise.} \end{cases}$$

We shall finally consider W_4. Write

$$W_4 = W_5 + W_6$$

where

$$W_5 = \sum_{\substack{d_1, d_2, d_4 \\ p\nmid d_1 d_4, p|d_2}}, \quad W_6 = \sum_{\substack{d_1, d_2, d_4 \\ p\nmid d_1 d_2 d_4}}.$$

Again W_5 is easy to evaluate. A simple computation gives

$$(22) \quad W_5 = \begin{cases} 0 & , \text{if } p^{a-s} \nmid 2t_2, \quad 1 \leqq s < a \\ p^{a-s} C(p^a, t_1) \cdot C(p^a, t_4), & \text{otherwise.} \end{cases}$$

As for W_6 we see that since $d_1 d_4 - d_2^2 \not\equiv 0 \pmod{p}$ and $p \nmid d_1 d_2 d_4$, p has necessarily to be odd. At this stage, for the sake of simplicity, we

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shall assume that $t_2 = 0$. We shall again split the sum W_6 as

$$W_6 = W_7 + W_8 + W_9, \text{ where}$$
$$W_7 = \sum_{\substack{d_1=r\\ d_4=n}}, \quad W_8 = \sum_{\substack{d_1=n\\ d_4=r}}, \quad W_9 = \sum_{\substack{d_1,d_4=n \text{ or}\\ d_1,d_4=r}}$$

where r denotes quadratic residue and n denotes quadratic non residues mod p^a. It should be noted that in W_7 and W_8, d_2 is arbitrary prime to p with the conditions made; namely $d_1d_4 - d_2^2 \not\equiv 0 \pmod p$. Thus

$$(23)\quad \left.\begin{aligned} W_7 &= \frac{\phi(p^a) - 1}{2}\left(\sum_n \exp(2\pi i n t_4/p^a)\right)\left(\sum_r \exp(2\pi i r t_1/p^a)\right)\\ W_8 &= \frac{\phi(p^a) - 1}{2}\left(\sum_r \exp(2\pi i r t_4/p^a)\right)\left(\sum_n \exp(2\pi i n t_1/p^a)\right)\end{aligned}\right\}$$

Under the conditions in the summation in W_9, d_1d_4 is always a quadratic residue. Therefore d_2 should run over those d_2 such that $d_1d_4 - d_2^2 \not\equiv 0 \pmod p$. For given d_1, d_4 there are clearly 2 values of d_2. Since $t_2 = 0$, the summation over d_2 for a given d_1, d_4 will give

$$(\phi(p^a) - 2) \exp(2\pi i d_1 t_1/p^a) \exp(2\pi i d_4 t_4/p^a)$$

and therefore

$$(24)\quad \begin{aligned} W_9 = (\phi(p^a) - 2)\Big\{&\left(\sum_n \exp(2\pi i n t_1/p^a)\right)\left(\sum_n \exp(2\pi i n t_4/p^a)\right)\\ &+ \left(\sum_r \exp(2\pi i r t_1/p^a)\right)\left(\sum_r \exp(2\pi i r t_4/p^a)\right)\Big\}\end{aligned}$$

The sums inside the bracket can be summed by the use of classical results.

We have thus evaluated $W(F, T)$ in Case 3.

3. Evaluation of special cases of $W(F, T)$ **for** $n > 3$. Although it seems to be difficult to evaluate $W(F, T)$ in the general case $n \geqq 3$, we shall nevertheless obtain some results when F and T have special forms. For simplicity we shall assume that

$$(25)\quad T = \begin{pmatrix} t_1 & & & 0\\ & \cdot & & \\ & & \cdot & \\ 0 & & & t_n \end{pmatrix}.$$

Since F is diagonal, F has the form

$$(26)\quad F = \begin{pmatrix} p^{a_1}E_1 & & & 0\\ & \cdot & & \\ & & \cdot & \\ 0 & & & p^{a_k}E_k \end{pmatrix}, \quad 0 \leqq a_1 < \ldots < a_k$$

where E_i is the unit matrix of order m_i, and

$$m_1 + m_2 + \cdots + m_k = n.$$

We shall now assume that

(27) $m_i = 1$ or 2, $1 \leqq i \leqq k$.

Split D up in the same form $D = (D_{ki})$ where D_{ij} is a matrix of m_i rows and columns. The symmetry condition then gives

(28) $p^{-a_i}D_{ij} \equiv p^{-a_j}D'_{ji} \pmod 1$

for all i, j. Thus D_{ii} is symmetric mod 1 and since for $i < j$, $a_i < a_j$ we get

(29) $D'_{ji} \equiv p^{a_j - a_i}D_{ij} \pmod{p^{a_j}}$

which means that

(30) $D_{ji} \equiv 0 \pmod{p^{a_j - a_i}}$.

Put

$$T = \begin{pmatrix} T_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & T_k \end{pmatrix}$$

where T_i is a diagonal matrix of order m_i. Then

$$(F^{-1}DT) = \sum_{j=1}^{k} \sigma(p^{-a_j}D_{jj}T_j).$$

Since we require the exponent mod 1, we take D_{jj} symmetric integral and by (20) D_{ij} is arbitrary integral but D_{ji} satisfies (30). Then

(31) $$W(F, T) = \sum \exp\left(2\pi i \sum_{j=1}^{k} \sigma(p^{-a_j}D_{jj}T_j)\right)$$

where the summation is over

$$D_{ij} \pmod{p^{a_i}},\ i \leqq j$$

which satisfy the condition of coprimality of F and D. We distinguish two cases:

Case (i). $a_1 = 0$. In this case because of (30) the only condition is

(32) $$\left(\prod_{i=2}^{k} |D_{ii}|, p\right) = 1.$$

Case (ii). $a_1 > 0$. Then we have by (30) again

(33) $$\left(\prod_{i=1}^{k} |D_{ii}|, p\right) = 1.$$

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From (31),

$$(34) \quad W(F, T) = \lambda \prod_{i=1}^{k} \sum_{D_{ii}} \exp\,(2\pi i \sigma(p^{-a_i} D_{ii} T_i))$$

where λ is a number that comes from summation over D_{ij}, $i < j$. This number λ can be easily seen to be

$$(35) \quad \lambda = p^{\mu}, \quad \mu = \sum_{1 \leqq i < j \leqq k} m_i m_j.$$

From (34) we see how to evaluate the product. Since $m_i = 1$ or 2, each one of the terms in the product is an ordinary Ramanujan sum ($m_i = 1$) or a generalized Ramanujan sum of type 3 discussed in the previous section with $m_i = 2$. Note that the corresponding T_i is a diagonal matrix and so the considerations of the previous section apply.

4. Ramanujan sums involving matrices of Gaussian integers. We shall now give another generalization which is related to the theory of Hermitian modular functions. We shall only illustrate the case of the field of Gaussian numbers, $\mathbf{Q}(\sqrt{-1})$.

We consider the ring of n rowed matrices with Gaussian integers as entries. The Gaussian integers form a principal ideal ring. We denote this ring by $\mathfrak{o}$. Two matrices C and D form a Hermitian pair if

$$(36) \quad C\bar{D}' = D\bar{C}'$$

where bar denotes complex conjugation and $'$, as before, denotes the transpose. C and D are coprime if for any vector $\mathbf{x}' = (x_1, \cdots, x_n)$ with elements in $\mathbf{Q}(\sqrt{-1})$, $\mathbf{x}'C$, $\mathbf{x}'D$ are in $\mathfrak{o}$ imply $\mathbf{x}$ has entries in $\mathfrak{o}$.

Let C be a non-singular matrix over $\mathfrak{o}$, i.e., with entries in $\mathfrak{o}$. Two matrices D_1 and D_2 are in the same residue class mod C if

$$C^{-1}(D - D_2) \equiv 0 \pmod{\mathfrak{o}};$$

that is that this matrix has entries in $\mathfrak{o}$. We can define a Hermitian residue class as that defined by D such that

$$(37) \quad C^{-1}D \equiv (\overline{C^{-1}D})' \pmod{\mathfrak{o}}.$$

We denote by $\tilde{\phi}_n(C)$ the number of coprime Hermitian residue classes mod C. Since for unimodular matrices U and V, (that is matrices with elements in $\mathfrak{o}$ and whose inverses also have elements in $\mathfrak{o}$) suitably chosen

$$UCV = F = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix}$$

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where $f_1, \cdots, f_n$ are in V. We see that $\tilde{\phi}_n(C) = \tilde{\phi}_n(F)$ and has the multiplicative property with regard to F. $\tilde{\phi}_n(F)$ can be evaluated in much the same way as Christian has done with regard to $\phi_n(F)$.

We can now define the Ramanujan sum

$$(38) \quad \tilde{W}(F, T) = \sum_D e^{\pi i \sigma(F^{-1}DT)}$$

where D runs through a complete system of coprime Hermitian matrices satisfying (37) mod F and T is a Hermitian matrix with elements in V. It again has the multiplicativity property and one needs to evaluate $\tilde{W}(F, T)$ only when

$$F = \begin{pmatrix} p^{a_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & p^{a_n} \end{pmatrix}$$

where $0 \leqq a_1 \leqq a_2 \cdots \leqq a_n$ and (p) is a prime ideal in $\mathfrak{o}$. It should be noted that σ here denotes the algebraic trace of the matrix trace, the algebraic trace of any element $\alpha \in \mathbf{Q}(\sqrt{-1})$ being $\alpha + \bar{\alpha}$.

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