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BOUNDED ORBITS OF POSITIVELY BOUNDED SYSTEMS

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Abstract. In this paper, we discuss the structure of the global attractor of a positively bounded system. In particular, we are concerned with the existence of connecting orbits and the relation between maximal elliptic sectors and connecting orbits. For the systems with two singular points a necessary and sufficient condition for the existence of connecting orbits is given.

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1. Introduction. This article deals with the structure of the set of all bounded orbits for a positively bounded system in the plane R^2. We consider the planar system:

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y), \tag{E}$$

where X and Y are continuous, and assume that solutions of arbitrary initial value problems are unique. Suppose that the vector field $V = (X, Y)$ defines a flow $f(p, t)$. For two isolated singular points p_1 and p_2 of System (E), if there exists a $p \in R^2$ such that $\lim_{t\to-\infty} f(p, t) = p_1$ and $\lim_{t\to+\infty} f(p, t) = p_2$, then the set $f(p, R) = \{f(p, t)|t \in R\}$ is called a *connecting orbit* from p_1 to p_2 (or a heteroclinic orbit). The existence of connecting orbits plays an important role in the studies of shock-wave solutions [**3**, **4**]. On the other hand, as special invariant sets, such orbits are crucial objects of the global structure of dynamical systems. Now the study for the existence of such orbits is still going on (see [**6**, **8**, **10**, **11**, **12**] and references therein). Among those papers (e.g., [**8**, **10**, **11**, **12**]), a crucial condition for the existence of connecting orbits is the absence of singular closed orbits.

The system (E) is said to be *positively bounded*, if for each $p \in R^2$ there exists an $r = r(p) > 0$ such that the positive semi-orbit $\gamma^+(p) = \{f(p, t)|t \in [0, +\infty)\}$ lies in the closed disc $B_r = \{z \in R^2|d(z, O) \leq r\}$, where O is the origin and d is the ordinary metric on R^2. In [**8**, **11**] it is proved that for a positively bounded system (E) with a finite number of singular points, if there are no closed orbits and singular closed orbits, then the system has a connecting orbit. In this paper, we strengthen this result, and using limit sets of subsets we prove that the set of all bounded orbits is simply connected and compact. Furthermore, we discuss the relation between connecting orbits and homoclinic orbits. It is shown that the number of connecting orbits is closely related to the maximal elliptic sectors. For the system (E) with two singular points we give a necessary and sufficient condition for the existence of connecting orbits.

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2. Preliminaries. Let $f : R^2 \times R \to R^2$ be the flow defined by the vector field $V = (X, Y)$ of System (E). For $A \subset R^2$ and $I \subset R$ we denote $A \cdot I = \{f(p, t)|p \in A, t \in I\}$, in particular $p \cdot t = f(p, t)$. A set S is invariant under f provided $S \cdot R = S$. Throughout the paper for $A \subset R^2$, ClA, ∂A and Int$A$ denote respectively the closure, boundary and interior of A. Also, B_r always denotes the closed disc $\{p \in R^2|d(p, O) \le r\}$ with radius $r > 0$ and center O (the origin).

DEFINITION 2.1. A simple closed curve is called a *singular closed orbit* if it is the union of alternating non-closed whole orbits and singular points, and it is contained in the ω(or α)-limit set of an orbit.

DEFINITION 2.2. For a singular point p in R^2, an orbit $\gamma(q) = q \cdot R$ $(q \ne p)$ is called a *homoclinic orbit* with respect to p provided $\lim_{t\to-\infty} q \cdot t = \lim_{t\to+\infty} q \cdot t = p$.

NOTATION. If L is a homoclinic orbit with respect to p, in this paper we always let D_L denote the interior of the bounded region surrounded by $L \cup \{p\}$.

The following famous theorem is fundamental in the qualitative theory of planar systems (see [**7**, p. 154]).

POINCARÉ-BENDIXSON THEOREM. *Assume that $\gamma^+(p)$ is bounded and $\omega(p)$ contains a finite number of singular points. Then $\omega(z)$ is a singular point, or a closed orbit, or a connected set composed of some singular points and some orbits whose positive semi-orbit and negative semi-orbit tend respectively to a singular point in $\omega(z)$.*

PROPOSITION 2.3. *Suppose that the system* (E) *has a homoclinic orbit L with respect to a singular point p. If there exist no singular points in D_L, then any orbit passing through a point in D_L is homoclinic with respect to p.*

Proof. Take a point $z \in D_L$. Since D_L is an invariant and bounded set, we get $\omega(z) \ne \emptyset$. If there exists a regular point $q \in \omega(z)$, let J denote a transversal at q of the flow. Then the positive semi-orbit $\gamma^+(z)$ crosses J in the same direction successively at t_i with $0 < t_1 < t_2 < \cdots$ $(t_i \to +\infty)$ and $z \cdot t_i$ tends monotonously to q along J (see [**7**, chapter 7]). Thus a simple closed curve consisting of the solution arc $z \cdot [t_i, t_{i+1}]$ and a segment of J between $z \cdot t_i$ and $z \cdot t_{i+1}$ surrounds a bounded region $Z \subset D_L$. Hence a (negative or positive) semi-orbit of $z \cdot t_i$ or $z \cdot t_{i+1}$ lies in Z; from the Poincaré-Bendixson Theorem it follows that there exists at least a singular point in Z, giving a contradiction. Thus we conclude $\omega(z) = \{p\}$. Similarly $\alpha(z) = \{p\}$ holds, and now it follows that the orbit $\gamma(z)$ is homoclinic with respect to p. This completes the proof.

DEFINITION 2.4. For two homoclinic orbits L_1 and L_2 with respect to a singular point p, if $L_1 \subset D_{L_2}$ holds or there exists another homoclinic orbit L with respect to p satisfying $L_1 \cup L_2 \subset D_L$, we say that L_1 and L_2 are in the *same class*. By a maximal elliptic sector we mean the union of $\{p\}$ and the set consisting of all homoclinic orbits in the same class with respect to p.

DEFINITION 2.5. If A is a subset of R^2, the *ω-limit set* of A is defined to be the set $\omega(A) = \bigcap_{t\ge0} \mathrm{Cl}\{A \cdot [t, \infty)\}$ (see [**2**] for the definition of $\omega(A)$ in a more general setting and its basic properties).

LEMMA 2.6 [**2**]. *If $A \cdot [0, +\infty)$ is compact and A is connected, then $\omega(A)$ is a compact and connected set. Furthermore it is the maximal invariant set in $A \cdot [0, +\infty)$.*

LEMMA 2.7. *Assume that all singular points lie in* Int B_r *for an* $r > 0$. *If the system* (E) *is positively bounded, then* $\omega(B_r)$ *is compact and connected.*

Proof. By Lemma 2.6 we only need to prove that there is a $\lambda > 0$ such that for every $p \in B_r$ the semi-orbit $\gamma^+(p) = p \cdot R^+$ is contained in B_λ, i.e., $B_r \cdot [0, +\infty) \subset B_\lambda$. Actually, we may suppose that there are no closed orbits outside IntB_r. Now for each $p \in \partial B_r$, since $\gamma^+(p)$ is bounded, it follows that $\omega(p)$ is a closed orbit or contains a singular point. Thus, if $\gamma^+(p)$ leaves B_r at p, we assert that $\gamma^+(p)$ meets with ∂B_r again at time $t_p > 0$, since all singular points lie in IntB_r and each closed orbit enters IntB_r. If $\gamma^+(p)$ does not leave B_r at p, we define $t_p = 0$. By the compactness of the circle ∂B_r, there is a $T > 0$ such that for each $p \in \partial B_r$, $0 \leq t_p \leq T$ holds. So we conclude that $B_r \cdot [0, +\infty) \subset B_r \cdot [0, T]$. Because $B_r \cdot [0, T]$ is compact, it is easy to find a $\lambda > 0$ such that $B_r \cdot [0, T] \subset B_\lambda$. The proof is complete.

THEOREM 2.8. *For a positively bounded system* (E), *if the set of singular points is bounded and there exist no closed orbits outside* B_δ *for some* $\delta > 0$, *then the set of all bounded orbits is simply connected and compact.*

Proof. We take an $r > 0$ such that all the singular points and closed orbits lie in IntB_r. Since the ω-limit set of each p in R^2 is a closed orbit or contains at least a singular point, we have $\gamma^+(p) \cap B_r \neq \emptyset$ and $\omega(p) \cap \text{Int}B_r \neq \emptyset$. Now if the orbit $\gamma(p)$ is bounded, we assert that $\gamma(p) \subset B_r \cdot [0, +\infty)$. In fact, as the above argument we also have $\alpha(p) \cap \text{Int}B_r \neq \emptyset$ for the bounded orbit $\gamma(p)$. Thus for every $p \cdot t \in \gamma(p)$ there exists a $\tau > 0$ such that $(p \cdot t) \cdot (-\tau) \in B_r$, and it follows that $p \cdot t \in B_r \cdot \tau \subset B_r \cdot [0, +\infty)$, so $\gamma(p) \subset B_r \cdot [0, +\infty)$ holds. Since $\omega(B_r)$ is the maximal invariant set in $B_r \cdot [0, +\infty)$, we get $\gamma(p) \subset \omega(B_r)$. On the other hand, by Lemma 2.7 all the orbits in $\omega(B_r)$ are bounded, so $\omega(B_r)$ is just the set of all bounded orbits, and it is compact and connected. The simple connectedness is directly derived from the fact that $\omega(B_r)$ is the maximal invariant set in the simply connected set $B_r \cdot [0, +\infty)$. In fact, any loop C in $\omega(B_r)$ surrounds a bounded region which is also contained in $\omega(B_r)$, so C is contractible.

REMARK. Actually the set of all bounded orbits is a global attractor, since by Lemma 2.7 it is easy to see that $\omega(B_r)$ attracts each bounded subset (see [**9**]).

Finally, we recall the concept of first prolongational limit set (see [**1**]). The set $J^+(x) = \{y \in R^2 |\text{there exists a sequence } \{x_i\} \text{ in } R^2 \text{ and a sequence } \{t_i\} \text{ in } R^+ \text{ such that } x_i \to x, t_i \to +\infty \text{ and } x_i \cdot t_i \to y\}$ is called the *first positive prolongational limit set of* x.

3. Connecting orbits.

In the sequel, we suppose that the system (E) has only a finite number of singular points $\{p_1, p_2, \ldots, p_n\}$ $(n \geq 2)$.

LEMMA 3.1. *For* $x \in R^2$, *if* $\omega(x)$ *is a compact set with two or more singular points, then there exists at least one connecting orbit in* $\omega(x)$.

Proof. From the Poincaré-Bendixson Theorem, it follows that for each $p \in \omega(x)$ both $\omega(p)$ and $\alpha(p)$ are singular points. Thus if there exist no connecting orbits in $\omega(x)$, all the orbits in $\omega(x)$ are homoclinic orbits and singular points. If $\omega(x)$ contains only a finite number of homoclinic orbits, obviously it is contradictory to the connectedness of $\omega(x)$. Let $\rho = \min\{d(p_i, p_j) | 1 \leq i < j \leq n\} > 0$ and define $C_i = \{z \in R^2 | d(z, p_i) = \rho/2\}$. Now we see that only a finite number of homoclinic orbits in $\omega(x)$ intersect each C_i (see [**7**, Lemma 8.2]), it is also contradictory to the connectedness of $\omega(x)$. So we are done.

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Corollary 3.2. *For $x \in R^2$, if $J^+(x)$ is a compact set with two or more singular points and without closed orbits in it, and also each homoclinic orbit in $J^+(x)$ is not contained in the limit set of a point in $J^+(x)$, then there exists at least one connecting orbit in $J^+(x)$.*

Proof. For $p \in J^+(x)$, $\omega(p)$ is not a closed orbit. If $\omega(p)$ is not a singular point, $\omega(p)$ has two or more singular points, otherwise $\omega(p)$ contains a homoclinic orbit. Thus, by Lemma 3.1, a connecting orbit lies in $\omega(p) \subset J^+(x)$. A similar argument works for $\alpha(p)$. Now we only need consider the case that for each $p \in J^+(x)$ both $\omega(p)$ and $\alpha(p)$ are singular points, which is just the situation we met in the proof of Lemma 3.1. Observe that now $J^+(x)$ is a compact and connected set, i.e., $J^+(x)$ is a *continuum* (see [**1**]). The proof follows from the connectedness of $J^+(x)$ as in the proof of Lemma 3.1.

Lemma 3.3. *Let L be a homoclinic orbit with respect to a singular point p and suppose D_L contains at least one singular point. If there exist no closed orbits in D_L and the closure of each homoclinic orbit is not a singular closed orbit, then there exists a connecting orbit in D_L.*

Proof. Assume that D_L contains singular points $\{q_1, q_2, \ldots, q_m\}$. Let S_i be the union of $\{q_i\}$ and all the homoclinic orbits with respect to q_i $(i = 1, 2, \ldots, m)$. At first we consider the case $S_i = \{q_i\}$ for some i, i.e., there are no homoclinic orbits with respect to q_i. Since no closed orbit lies in D_L, let $q' \in D_L$ satisfy $q' \cdot t \to q_i$ as $t \to +\infty$ or $t \to -\infty$ (see [**7**, Lemma 8.1]). Without loss of generality we may suppose $q' \cdot t \to q_i$ as $t \to +\infty$; then consider $\alpha(q')$. Certainly $\alpha(q')$ is not a closed orbit, and also $\alpha(q')$ contains no homoclinic orbits. Thus by the Poincaré-Bendixson Theorem, $\alpha(q')$ is a singular point or $\alpha(q')$ is a connected set with two or more singular points, and so $\gamma(q')$ is a connecting orbit or $\alpha(q')$ contains a connecting orbit (Lemma 3.1). Next, we consider the case $S_i \neq \{q_i\}$ for all i. If there exists a point $p' \in \mathrm{Cl}D_L \backslash (\cup_{i=1}^m S_i)$ such that $\gamma(p')$ is not homoclinic with respect to p, by a similar argument as above it follows that $\gamma(p')$ is a connecting orbit or a connecting orbit lies in $\omega(p') \cup \alpha(p')$. If for each $p' \in \mathrm{Cl}D_L \backslash (\cup_{i=1}^m S_i)$, $\gamma(p')$ is homoclinic with respect to p, let $z \in \partial(\cup_{i=1}^m S_i)$ be a regular point and $J^+(z)$ be the first positive prolongational limit set of z. Now $J^+(z)$ is a continuum and contains two or more singular points (one is p). From Corollary 3.2 it follows that there is a connecting orbit in $J^+(x)$. The proof is complete.

Theorem 3.4. *If the positively bounded system (E) with two or more singular points has no closed orbits, and the closure of each homoclinic orbit is not a singular closed orbit, then the system has at least a connecting orbit.*

Proof. Let M be the set of all bounded orbits. By Theorem 2.8 M is compact and connected. It is also an invariant set. We take a regular point $z \in \partial M$ and consider the limit set $\omega(z)$. Of course $\omega(z)$ is not a closed orbit. If there exist regular points and a unique singular point in $\omega(z)$, it follows from the Poincaré-Bendixson Theorem that $\omega(z)$ contains a homoclinic orbit. It is a contradiction. Otherwise, if there exist two or more singular points in $\omega(z)$, it follows from Lemma 3.1 that there exists at least one connecting orbit in $\omega(z)$. If $\omega(z)$ is a singular point, we consider $\alpha(z)$. Similarly we conclude that there is a connecting orbit in $\alpha(z)$ or $\alpha(z)$ is a singular point. In the latter case, $\gamma(z)$ is a connecting orbit if $\omega(z) \neq \alpha(z)$. We consider the case $\omega(z) = \alpha(z) = \{p\}$; now $\gamma(z) = L$ is a homoclinic orbit with respect to p. Since ∂M is an invariant set, it follows $\mathrm{Cl}L = L \cup \{p\} \subset \partial M$. If there exist singular points in D_L, the existence of connecting orbits follows from Lemma 3.3. If there exist no singular points in D_L, collapse $\mathrm{Cl}D_L$ to the point p and continue the above process for the quotient flow. Thus, either we find a connecting orbit or we finally get a quotient space M' of M and a quotient flow $f' = f|_{M' \times R}$ with the same singular points of the flow f such that for any $q \in \partial M'$, $\gamma(q)$ is not homoclinic for f'. Since the connected set M' contains at least two singular points, there exist regular points on $\partial M'$. So now any orbit of a regular point on $\partial M'$ is a connecting orbit, which corresponds to a connecting orbit of the flow f on M. This completes the proof.

LEMMA 3.5. *If the system* (E) *has no closed orbits or singular closed orbits, then each bounded semi-orbit tends to a singular point.*

Proof. This is straightforward from the Poincaré-Bendixson Theorem, since now the limit set of an orbit contains no homoclinic orbits and singular closed orbits.

THEOREM 3.6. *For a positively bounded system* (E) *with n isolated singular points, if there exist no closed orbits, singular closed orbits and homoclinic orbits, then the number of connecting orbits is either* $n-1$ *or uncountable.*

Proof. Choose an $r > 0$ such that $\omega(B_r)$ is the set of all bounded orbits; we at first prove that in $\omega(B_r)$ there exist at least $n-1$ connecting orbits. By Lemma 3.5 we know all the orbits in $\omega(B_r)$ are singular points and connecting orbits. If there are at most $n-2$ connecting orbits in $\omega(B_r)$, it is impossible for them to connect all the n singular points and become a connected set $\omega(B_r)$ (Theorem 2.8). Thus the system (E) has at least $n-1$ connecting orbits. Next, if the system has n connecting orbits in $\omega(B_r)$, then there exist k $(2 \le k \le n)$ singular points and k connecting orbits which constitute a Jordan curve C. Let D be the bounded region surrounded by C; then D is an invariant subset with nonempty interior, and it follows that each orbit in D is a connecting orbit or a singular point. Thus, there are uncountably many connecting orbits in D. This completes the proof.

THEOREM 3.7. *Suppose connecting orbits* H_1, H_2 *and two singular points* $\{p_1, p_2\}$ *constitute a Jordan curve C surrounding a bounded region D, assume that the system* (E) *has no singular points in D and let* σ *be the number of all maximal elliptic sectors in D with respect to* p_1 *or* p_2*. Then the number of connecting orbits in* $\mathrm{Cl}D$ *is infinite or* $\sigma + 1$.

Proof. For each $p \in D$, from an argument similar to the proof of Proposition 2.3 it follows that both $\omega(p)$ and $\alpha(p)$ are singular points, i.e., $\gamma(p)$ is a homoclinic orbit or a connecting orbit. Now assume that the number of connecting orbits in D is finite, and suppose that connecting orbits G_1, G_2 in $\mathrm{Int}D$ and $\{p_1, p_2\}$ circle a simply connected region W containing no other connecting orbits. Thus for any $p \in W$, $\gamma(p)$ is a homoclinic orbit. If there exist two homoclinic orbits L_1, L_2 in W with respect to p_1 and p_2 respectively, we denote by S_i the union of all the maximal elliptic sectors in W with respect to p_i for $i = 1, 2$. Clearly, we have $\partial S_1 \cap \partial S_2 \neq \emptyset$. Take $z \in \partial S_1 \cap \partial S_2$; it is easy to see that $J^+(z)$ contains p_1 and p_2. Now by Corollary 3.2, a connecting orbit lies in $J^+(z) \subset W$, giving a contradiction. So one of S_1 and S_2 is empty. Without loss of generality we assume that all the orbits in W are homoclinic with respect to p_1. Now it is easy to see that all the homoclinic orbits in W are in the same class, i.e., $\mathrm{Int}W \cup \{p_1\}$ is a maximal elliptic sector. Hence, we conclude that if the number of connecting orbits in $\mathrm{Cl}D$ is $\sigma + 1$, there exist just σ maximal elliptic sectors in D.

COROLLARY 3.8. *For two connecting orbits H_1 and H_2 running from p_1 to p_2 in the same direction, if H_1, H_2 and $\{p_1, p_2\}$ surround a region D with no singular points in it, then there exists another connecting orbit H between p_1 and p_2.*

Proof. Suppose, on the contrary, there exist no connecting orbits in Int$D$. It follows from Theorem 3.7 that there is a unique maximal elliptic sector in D. However, according to the continuity of dependence on initial conditions, H_1 and H_2 tend to p_2 in the reverse direction. This contradicts the condition of Corollary 3.8.

In the following, we suppose that all singular points are elementary. The next lemma holds for any planar systems.

LEMMA 3.9. *If p is an elementary singular point, i.e., the Jacobian at p is nonzero, and the system has no closed orbits, and no homoclinic orbit is contained in the limit set of an orbit, then there exist no homoclinic orbits with respect to p.*

Proof. Suppose that there exists a homoclinic orbit L with respect to p. Since p is elementary, it is a saddle point and there are just two positive and two negative base solutions (see [**7**, p. 162]) for a sufficiently small circle C with center p. Now let u and v be the two intersecting points of L and C, which divide C into two parts C_1 and C_2. Further, according to the behavior of solutions near a saddle point we have that the remaining two base solutions lie in D_L or $R^2\backslash D_L$, and without loss of generality we may suppose that no base solutions intersect C_1. By the continuity of dependence on initial conditions, an orbit passing through a point $z_1 \in C_1$ (sufficiently near u) will intersect C_1 at z_2 and z_3 successively. Since the system has no closed orbits, we have $z_1 \neq z_3$. Thus the solution arc from z_1 to z_3 and a segment of C_1 constitute a Jordan curve H. Let W be the torus region surrounded by H and $L \cup \{p\}$. Since in W there are no closed orbits and singular points, a semi-orbit of z_1 or z_3 lies in W and tends to ∂D_L, so its limit set contains L. This is a contradiction.

THEOREM 3.10. *For a positively bounded system (E) with n elementary singular points, if there exist no closed orbits and singular closed orbits, then the number of connecting orbits is either $n - 1$ or uncountable.*

Proof. It follows from Lemma 3.9 that the system (E) has no homoclinic orbits. So by Theorem 3.6 we know that the theorem is true.

4. Systems with two singular points. In this section we discuss systems with two singular points. Let S always be the set of all bounded orbits.

THEOREM 4.1. *For a positively bounded system (E) with two singular points, if there exist no closed orbits and homoclinic orbits, then S is composed of singular points and connecting orbits.*

Proof. By Theorem 2.8, S is compact and connected. For any $x \in S$, suppose that x is not a singular point, and consider the limit set $\omega(x)$. Firstly we prove that $\omega(x)$ has at most one singular point. If not, since two different singular points separate, it follows from the connectedness of $\omega(x)$ that there exists a regular point p in $\omega(x)$. Let J be a transversal at p and suppose the positive semi-orbit $\gamma^+(x)$ crosses J successively at t_i with $0 < t_1 < t_2 < \cdots$ $(t_i \to +\infty)$ and $x \cdot t_i$ tends to p. Thus the simple closed curve consisting of the solution arc $x \cdot [t_i, t_{i+1}]$ and a segment of J separates $\omega(x)$ from the negative semi-orbit $\gamma^-(x)$. Now it follows from the Poincaré-Bendixson Theorem that there exists another singular point in $\alpha(x)$, a contradiction since we find three singular points. Further, we assert that there exist no regular points in $\omega(x)$, otherwise $\omega(x)$ has a homoclinic orbit. Hence $\omega(x)$ is just a singular point, and also is $\alpha(x)$. Since the system has no homoclinic orbits, $\gamma(x)$ is a connecting orbit and the theorem follows.

THEOREM 4.2. *Assume that the positively bounded system* (E) *has two singular points, but no closed orbits. Then the system has a connecting orbit if and only if for any homoclinic orbit* L *neither* $L \subset \omega(x)$ *nor* $L \subset \alpha(x)$ $(x \in S)$ *holds, i.e., L is not contained in the limit set of a point in S.*

Proof. Suppose that the system has a connecting orbit H between singular points p_1 and p_2. Without loss of generality, let L be a homoclinic orbit about p_1. If $L \subset \omega(x)$ for some regular point $x \in S$ (of course $x \notin H$), let J be a transversal at a point p on L. Then the positive semi-orbit $\gamma^+(x)$ crosses J successively at t_i and $x \cdot t_i$ tends monotonically to q. Since $L \cup H \cup \{p_1, p_2\}$ is an invariant and connected set, thus a simple closed curve consisting of the solution arc $x \cdot [t_i, t_{i+1}]$ and a segment of L separates $\alpha(x)$ and $L \cup H \cup \{p_1, p_2\}$. Now $\alpha(x)$ contains at least a singular point, hence we find three singular points, a contradiction. A similar argument works for the case $L \subset \alpha(x)$.

We shall now prove sufficiency. Each homoclinic orbit L is not contained in the limit set of a point in S. Thus the result follows immediately from Theorem 3.4 and Lemma 3.3.

COROLLARY 4.3[**8**]. *For a positively bounded system* (E)*, if there exist no closed orbits and singular closed orbits then the system has a connecting orbit.*

Proof. Since a homoclinic orbit in the limit set of a point in S is a singular closed orbit, the result follows directly from Theorem 4.2.

THEOREM 4.4. *For a planar system* (E) *with two singular points, if there exist no homoclinic orbits, then the possible numbers of connecting orbits are zero, one and uncountable.*

Proof. Suppose that there exist two connecting orbits $\gamma(p)$ and $\gamma(q)$ between p_1 and p_2. Then $\gamma(p) \cup \gamma(q) \cup \{p_1, p_2\}$ constitutes a simple closed curve surrounding a bounded region W. Since p_1 and p_2 lie in the boundary of W, there exist no singular points and closed orbits in W. Choose a point $z \in W$ arbitrarily. From the proof of Theorem 4.1 it follows that $\gamma(z)$ is a connecting orbit. So all the orbits in W are connecting orbits.

REMARK 4.5. Theorem 4.4 is true for any planar systems. Moreover by Theorem 4.4 if the system has a finite number (more than two) of connecting orbits, then there exists at least a homoclinic orbit. On the other hand, if the system admits homoclinic orbits, we think that the number of connecting orbits can be any positive integer. In the following we give a system with two connecting orbits and another system with uncountable connecting orbits. Systems with a unique connecting orbit are trivial.

EXAMPLE 1. To give an example with two connecting orbits, we consider the Liénard system [**5**, p. 33]:

$$\dot{x} = y - \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right), \qquad \dot{y} = -x^3. \tag{4.1}$$

In the phase-portrait of (4.1) (see [**5**, p. 34]) there is a maximal elliptic sector S consisting of all homoclinic orbits with respect to $O = (0, 0)$. Define a smooth function $\phi : R^2 \to R$ satisfying $\phi(x, y) \geq 0$ and $\phi(x, y) = 0$ only at a point $p \in \text{Int}S$; then the following system has two singular points p and O:

$$\dot{x} = \left[y - \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\right] \cdot \phi(x, y), \qquad \dot{y} = -x^3 \cdot \phi(x, y). \tag{4.2}$$

Now the homoclinic orbit passing through p of (4.1) becomes two connecting orbits and a singular point p of (4.2), and the other orbits of (4.1) remain unchanged.

Example 2. Consider the following planar system in polar coordinates:

$$\dot{r} = r(1 - r), \qquad \dot{\theta} = \sin^2\frac{\theta}{2} + (r - 1 + |r - 1|). \tag{4.3}$$

This system has exactly two singular points, $O = (0, 0)$ and $p = (1, 0)$. The circle $C = \{(r, \theta)|r = 1\}$ is an invariant set, which is composed of a homoclinic orbit L and a singular point p. Obviously the system (4.3) is positively bounded, and for any point x outside the disc $B_1 = \{(r, \theta)|r \leq 1\}$ we have $\omega(x) = C$. Thus C is a singular closed orbit, and the results of [**8**, **12**] do not work. However, the segment $\overline{Op}$ is also invariant, and it is not difficult to see that for any $x \in \text{Int}B_1$ the relation $L \subset \omega(x)$ or $L \subset \alpha(x)$ doesn't hold. By Theorem 4.2 we conclude that the system has a connecting orbit. In fact, any orbit passing through a point in $\text{Int}B_1 \backslash \{O\}$ is a connecting orbit.

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