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Degeneracy of 2-Forms and 3-Forms

L. M. Fehér, A. Némethi and R. Rimányi

Abstract. We study some global aspects of differential complex 2-forms and 3-forms on complex manifolds. We compute the cohomology classes represented by the sets of points on a manifold where such a form degenerates in various senses, together with other similar cohomological obstructions. Based on these results and a formula for projective representations, we calculate the degree of the projectivization of certain orbits of the representation $\Lambda^k\mathbb{C}^n$.

1 Introduction

1.1 2-Forms

Let ω be a generic complex differential 2-form on a complex manifold M. Then we can stratify M according to the corank of ω_x at $x \in M$ by the subsets $S_i := \{x \in M \mid \operatorname{corank} \omega_x = i\}$, which are clearly empty when $n-i$ is odd. Our goal is the understanding of the cohomological obstructions to $\bigcup_{i\geq r} S_i = \varnothing$, for any r. That is, we want to describe the cohomological obstructions to the existence of a 2-form which everywhere drops rank by less than r. This set of obstructions consists of those cohomology classes which are "universally supported" [FP98, Ch. 4] on the locus $\overline{S_r}$. These classes form an ideal $O_{\Sigma^{<r}}$ in the Chern classes of the manifold. They were studied originally in [Pra88] in the context of polynomials universally supported on skew-symmetric degeneracy loci, where a certain explicit description of the ideal was given using Schur P-polynomials, see also [FP98, Ch. 4], [PR96]. In Theorem 3.1 we give another explicit description of this ideal, using Schur determinants and fewer generators than in [Pra88].

To put the result of this theorem in context, let us suppose that ω is a non-degenerate 2-form on an even dimensional manifold M, *i.e.,* one for which $S_0 = M$. Then ω yields an isomorphism between TM and T^*M, so $c_i(TM) = c_i(T^*M) = (-1)^i c_i(TM)$, that is $c_i(TM) = 0$ for i odd. We will find that these classes generate $O_{\Sigma^{<1}}$. This is, of course, not surprising. The question is how it generalizes to greater r. In Theorem 3.1 we will show that $O_{\Sigma^{<r}}$ is generated by Schur polynomials in Chern classes indexed by partitions of type odd $>$ even $>$ odd $> \cdots$.

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1.2 3-Forms

Our goal is a similar analysis for 3-forms. If ω is a generic k-form on a complex n-manifold M, then we can stratify M according to the orbits of the representation of $GL_n(\mathbb{C})$ on $\Lambda^k(\mathbb{C}^n)$. It is known that this representation has finitely many orbits only in the cases covered by 1.1 and in the (new) cases $k = 3$, $n = 6, 7, 8$.

For example, in the case of 3-forms on complex 6-manifolds there are 5 orbits of the action of $GL_6(\mathbb{C})$ on $\Lambda^3(\mathbb{C}^6)$: σ_0, σ_1, σ_5, σ_{10} and σ_{20} (indices being the codimensions of the orbits). The corresponding ideals have a large number of generators which cannot be organized as nicely as in case 1.1. Notice also that in the geometric applications those homogeneous elements of the ideal whose degrees are higher than the dimension of the manifold are not relevant. Therefore, for $n = 6$, we list only those homogeneous generators of the obstruction ideals which have degree not greater than 6. They appear only in the ideals σ_0, $\sigma_0 \cup \sigma_1$. The first ideal is thus the collection of characteristic classes which are obstructions to the existence of a 3-form on a 6-manifold which is everywhere generic (these forms are called *stable* in [Hit01]). The other ideal is the collection of obstructions to the existence of a 3-form which is only "mildly" degenerate.

For $n = 7$ and $n = 8$, we compute the ideal of σ_0 only. In the same spirit we describe the obstructions to the existence of a complex Spin_7 structure on complex 8-manifolds.

1.3 In both cases, the elements of the obstruction ideals have geometric meanings. The most straightforward is the meaning of the least degree element, *cf.* Theorem 2.1(3). These polynomials are called *universal classes of degeneracy loci* in algebraic geometry, or *Thom polynomials* in singularity theory.

In the case of 2-forms, the least degree element of $O_{\Sigma^{<r}}$ is the Poincaré dual or the Thom polynomial of $\overline{S_r}$. In this way, for 2-forms, we recover the results of [FR, HT84, JLP81, Pra90].

The other elements of the obstruction ideal are called *derived Thom polynomials* by Kazarian [Kaz97]. They also carry geometric meanings. In the case of 2-forms, these interpretations are slightly artificial, and we do not discuss them here. But we provide a geometric characterization of a higher degree element in the case $k = 3$ and $n = 6$ (Theorem 4.5 and Remark 4.6).

2 Review on Thom Polynomials and Obstruction Ideals of Group Actions

In this section we review the notions of Thom polynomials and obstruction ideals for group actions from [FR] (the theory of Thom polynomials is strongly motivated by [Kaz97]).

Let G act on the vector space V with *finitely many orbits*.

If η is an invariant closed variety of V, one defines the Thom polynomial $\mathrm{Tp}(\eta)$ of η as the Poincaré dual of the fundamental class of η in the equivariant cohomology $H^*_G(V;\mathbb{Z})$.

Similarly, let τ be a union of orbits, usually an open one. Then the obstruction ideal $O_\tau \subset H^*_G(V;\mathbb{Z}) = H^*(BG;\mathbb{Z})$ of τ is defined as

$$O_\tau = \ker\bigl(H^*_G(V;\mathbb{Z}) \to H^*_G(\tau;\mathbb{Z})\bigr),$$

where the morphism is induced by the inclusion $\tau \subset V$.

This "innocent" definition has many advantages. First of all, it is geometric: elements in this ideal restrict to 0 on τ, hence are supported on the complement of τ. In particular, for a bundle whose structure group is G and fiber is V, the G-characteristic classes from O_τ are obstructions to the existence of a section everywhere inside τ. Second, this ideal in many cases is computable (*cf.* the first two parts of the next theorem) provided that one can identify the corresponding stabilizer subgroups. But, in fact, the main point is that it contains all the information about Thom polynomials (modulo a sign, see the last part of the next theorem), which are in general hardly computable. This also explains the role of the next result.

Theorem 2.1

(1) *If τ is an orbit, then $O_\tau = \ker(H^*(BG;\mathbb{Z}) \to H^*(BG_\tau;\mathbb{Z}))$, where G_τ is the stabilizer (isotropy) subgroup of any point in τ;*
(2) *if the orbit stratification of V satisfies the Euler condition (see below), then $O_{\tau_1\cup\tau_2} = O_{\tau_1} \cap O_{\tau_2}$;*
(3) *If τ is the complement of the closure of an orbit η, then $H^{<\operatorname{codim}\eta}(BG;\mathbb{Z}) \cap O_\tau = 0$ and $H^{\operatorname{codim}\eta}(BG,\mathbb{Z}) \cap O_\tau$ is generated by the Thom polynomial of η.*

The stratification satisfies the Euler condition if the equivariant Euler class of any orbit η is not a zero-divisor in $H^*(BG_\eta;\mathbb{Z})$. This condition appeared in [AB83] as a sufficient condition for G-perfectness. The Euler condition will hold for all the representations we consider in this paper.

Theorem 2.1 shows that in order to carry out the calculations we need to determine the stabilizer subgroups only up to homotopy equivalence (*e.g.*, we can work with U_n instead of $GL_n(\mathbb{C})$).

3 Degeneracy of 2-Forms

In this section consider the representation $\Lambda^2(\mathbb{C}^n)$ of $GL_n := GL_n(\mathbb{C})$. It is well known that the orbits are characterized by the corank r, *i.e.*, every 2-form can be identified with one of the following matrices $\frac{n-r}{2}H \oplus 0_{r\times r}$, where $H = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let the orbit with corank r be called Σ^r. Its codimension is $\binom{r}{2}$ if $r \geq 2$ and is 0 if $r = 0$ (n even) or $r = 1$ (n odd).

Theorem 3.1 *Let Δ_λ denote the Schur polynomial associated with the partition λ as in* [FP98, (1.5)], evaluated at the universal Chern classes of BGL_n, and let $n - r$ be even. Then, in $H^*(BGL_n;\mathbb{Q})$ one has

(1) $O_{\Sigma^{<r}} = \langle \Delta_{i_{r-1},i_{r-2},\dots,i_2,i_1} \,|\, i_{r-1} > i_{r-2} > \cdots > i_1,\, i_{\text{odd}} \text{ is odd},\ i_{\text{even}} \text{ is even}\rangle$;
(2) $\operatorname{Tp}(\Sigma^r) = \Delta_{r-1,r-2,\dots,2,1}$, if $r \geq 2$ (otherwise it is 1).

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Proof First we apply Theorem 2.1(1) for the orbit $\tau = \Sigma^s$ for some s with $n - s$ even. The stabilizer subgroup of a representative from Σ^s, *e.g.*, the one given above, is clearly $G_s = Sp_{(n-s)/2} \times GL_s$ (modulo homotopy equivalence), with $H^*BG_s = \mathbb{Z}[p_i, a_j]$, where $i = 1, \dots, (n-s)/2$ is the index set of the characteristic (Pontryagin) classes of $Sp_{(n-s)/2}$ and $j = 1, \dots, s$ of the Chern classes of GL_s. Let us use the convention $c_0 = a_0 = p_0 = 1$. From the inclusion $G_s \subset G := GL_n$ we can also read off the induced ring homomorphism $H^*(BG) = \mathbb{Z}[c_1, \dots, c_n] \to H^*(BG_s) = \mathbb{Z}[p_i, a_j]$:

$$c_k \mapsto \sum_{j=0}^{(n-s)/2} p_j a_{k-2j}. \tag{1}$$

It is convenient to codify this homomorphism in the form

$$\sum_{k=0}^{n} c_k t^k = \Big(\sum_{i=0}^{s} a_i t^i\Big)\Big(\sum_{j=0}^{(n-s)/2} p_j t^{2j}\Big),$$

where t is a free variable. We shorten this into $c(t) = a(t) \cdot p(t)$. Obviously, one can consider the morphism $\varphi\colon \mathbb{C}^s \times \mathbb{C}^{(n-s)/2} \to \mathbb{C}^n$ given by $((a_i)_{i=1}^s, (p_j)_{j=1}^{(n-s)/2}) \mapsto (c_k)_{k=1}^n$, where c_k is given by (1) (and $c_0 = a_0 = p_0 = 1$). Then φ induces a homomorphism $\varphi_*\colon \mathbb{C}[c] \to \mathbb{C}[a, p]$, which is the complexification of the previous homomorphism $H^*(BG) \to H^*(BG_s)$.

Let $\mathcal{K}$ be the kernel of φ_*, and $\mathcal{I} = \mathcal{I}_{n,s}$ be the ideal in $\mathbb{C}[c]$ generated by the Schur polynomials $\Delta_{i_{s+1}, i_s, \dots, i_1}$, where $i_{s+1} > i_s > \cdots > i_1$, and i_{odd} is odd, i_{even} is even. Our first goal is to prove that $\mathcal{K} = \mathcal{I}$ in $\mathbb{C}[c]$.

Step 1: $\sqrt{\mathcal{I}} = \mathcal{K}$.

If $V(\mathcal{J})$ denotes the zero set of an ideal $\mathcal{J}$, then clearly $\overline{\operatorname{Im}\varphi} = V(\mathcal{K})$.

Let us analyze first

$$\operatorname{Im}\varphi = \Big\{ (c_0, \dots, c_n) : c_0 = 1, c(t) = a(t)p(t) \quad \text{for some } a(t) \text{ and } p(t) \text{ with } a_0 = p_0 = 1 \Big\}.$$

Eliminating $p(t)$ from the equations $c(t) = a(t)p(t)$ and $c(-t) = a(-t)p(t)$, one gets

$$c(t)a(-t) = c(-t)a(t), \tag{E}$$

which is equivalent to the system of equations

$$C \cdot \mathbf{a} = 0, \tag{$S_{n,s}$}$$

where $C = C_{n,s} = (c_{2i-j})_{i=1,\dots,\frac{n+s}{2}; j=1,\dots,s+1}$, and $\mathbf{a} = ((-1)^j a_{j-1})_{j=1,\dots,s+1}$.

If the system $(S_{n,s})$ has a non-zero solution $\mathbf{a}$, then clearly all the maximal minors of C vanish. These minors are the Schur polynomial $\Delta_{i_{s+1},\dots,i_1}$ introduced above.

Conversely, if all these minors vanish, then $(S_{n,s})$ has a solution $\mathbf{a} \neq 0$, or equivalently, (E) has a non-zero solution $a(t)$. First we show that this solution $a(t)$ can be replaced by another solution $a'(t)$ of (E) which additionally satisfies $a'(0) \neq 0$ (hence by normalization $a'(0) = 1$). Indeed, write $a(t) = t^m \cdot a'(t)$ with $a'(0) \neq 0$ (since $a \neq 0$, this is possible). Analyzing the coefficient of t^m in (E), one gets that m is even. Then dividing the equation (E) by t^m, we get that a' itself satisfies (E). So, we replace a by a'.

Next, we verify that this new $a(t)$ can be replaced by another solution $a'(t)$ of (E) which has the property that $p(t) := c(t)/a'(t)$ is a polynomial. Set $a^*(t) := a(-t)$. Let $d(t)$ be the greatest common divisor in $\mathbb{C}[t]$ of $a(t)$ and $a^*(t)$. Notice that $1 \pm t\alpha \mid a$ if and only if $1 \mp t\alpha \mid a^*$, hence $d(t) \in \mathbb{C}[t^2]$. Therefore, if $a'(t) := a(t)/d(t)$, then $a'(t)$ satisfies (E) as well, $a'(0) \neq 0$, and $a'(t)$ and $a'(-t)$ are relative prime. Then from (E) one gets that $a'(t) \mid c(t)a'(-t)$, hence $a'(t) \mid c(t)$. Take $p(t) := c(t)/a'(t)$, then again (E) (applied for a') guarantees that p is an even polynomial. Let the degree of p be $2l$. In fact, it can happen that $2l > n - s$. Set $r := (2l - n + s)/2$ and let q be the product of r distinct factors of p of type $1 + \alpha t^2$. Then replace the pair (a', p) by $(a'q, p/q)$. Clearly, their product is still $c(t)$, they have the right degrees, and the second one is even.

In conclusion, for any $c \in V(\mathcal{I})$, we can find (a, p) such that $\varphi(a, p) = c$. In other words, $\operatorname{Im}\varphi = V(\mathcal{I})$. In particular, $\operatorname{Im}(\varphi)$ is closed and $V(\mathcal{I}) = V(\mathcal{K})$. Since $\mathcal{K}$ is reduced, one gets $\sqrt{\mathcal{I}} = \mathcal{K}$. Moreover (since the source of φ is irreducible) we get that $\sqrt{\mathcal{I}}$ is prime.

In fact, one can analyze very precisely the set $\varphi^{-1}(c)$ for any fixed $c \in V(\mathcal{I})$: one has to consider all the possible factorizations of the fixed $c(t)$ in the form $c(t) = a(t)p(t)$ with the additional restrictions about the degrees of a and p, and p should be even. Here there is some freedom to switch some of the roots of a and p, but clearly $\varphi^{-1}(c)$ is finite for any c, *e.g.*, $\varphi^{-1}(0) = (0, 0)$ ($1 = c(t) = a(t)p(t)$ clearly implies $a(t) = p(t) = 1$). Hence, φ is quasi-finite.[1]

Since φ is quasi-finite:

$$\operatorname{codim}(\operatorname{Im}\varphi \subset \mathbb{C}^n) = (n - s)/2. \tag{2}$$

Step 2: $\sqrt{\mathcal{I}} = \mathcal{I}$.

Consider the "general" matrix X with free variables $(x_{ij})_{i=1,\dots,(n+s)/2;\, j+1,\dots,s+1}$. Let $\mathcal{J}$ be the ideal generated by all the minors of X of rank $s + 1$. From [CDP80] one has:

$$\operatorname{codim}(V(\mathcal{J}) \subset \mathbb{C}^{(s+1)(s+n)/2}) = (n - s)/2; \tag{3}$$

$$\mathcal{J} \subset \mathbb{C}[x_{ij}] \text{ is prime and } \mathbb{C}[x_{ij}]/\mathcal{J} \text{ is Cohen–Macaulay.} \tag{4}$$

Consider the space $\mathbb{C}^n$ (with coordinates c_k) introduced in Step 1. Then

$$\mathbb{C}^n \subset \mathbb{C}^{(s+1)(s+n)/2}$$

[1] In fact, φ is (a) weighted homogeneous, (b) finite (which follows from (a) and $\varphi^{-1}(0) = 0$), proper, birational (isomorphism above the set of those $c(t)$'s of degree n, for which $\gcd(a, a^*) = 1$); but we will not need these facts.

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can be realized by $(s+1)(s+n)/2-n$ hyperplane sections $\{H_\alpha\}$. Equations (2) and (3) guarantee that $\{H_\alpha\}$ is an M-sequence in $\mathbb{C}[x_{ij}]/\mathcal{J}$. Hence, by the general theory of Cohen–Macaulay rings, we get that $\mathbb{C}[x]/(\mathcal{J}+\mathbb{C}\langle H_\alpha\rangle) = \mathbb{C}[c]/\mathcal{I}$ is Cohen–Macaulay. Since $\sqrt{\mathcal{I}}$ is prime, we get that $\mathcal{I}$ is a primary ideal associated with $\sqrt{\mathcal{I}}$.

Since $\mathbb{C}[c]/\mathcal{I}$ has no embedded components, the equality $\mathcal{I} = \sqrt{\mathcal{I}}$ can be tested in *any* point $P \in V(\mathcal{I})$: If in a local ring $\mathbb{C}\{c_i - c_i(P)\}_{i=1,\dots,n}$ one has $\mathcal{I}_P = \sqrt{\mathcal{I}_P}$, then $\mathcal{I} = \sqrt{\mathcal{I}}$.

We will consider a special point P. First assume that $n > s+2$. It is not difficult to show that there exists a point P such that the matrix C evaluated at P has the following property: if one deletes its last row and column, then the remaining matrix has rank s. For example, if one takes for $a(t)$ a polynomial (with $a_0 = 1$) of degree s such that $a(t)$ and $a(-t)$ have no common zeros, and $p(t) \equiv 1$, then $c(t) = c(t)p(t) = a(t)$ provides a point $P \in V(\mathcal{I})$ with this property.

Now, we wish to analyze by induction (over n) the ideal $\mathcal{I}_{n,s}$ at the point P, and conclude $(\mathcal{I}_{n,s})_P = \sqrt{(\mathcal{I}_{n,s})_P}$. Recall that $\mathcal{I}_{n,s}$ is the ideal generated by the $(s+1)$-minors of $C_{n,s}$. We distinguish two types of minors. The first group consists of minors which do not involve the last row. The ideal generated by them is denoted by $\mathcal{I}^*_{n,s} \subset \mathbb{C}[c_1,\dots,c_n]$. The others are exactly those which involve the last row of $C_{n,s}$. Recall that this last row has the form $(0,\dots,0,c_n,c_{n-1})$.

We consider the ideal $\mathcal{I}_{n,s} + \langle c_n\rangle$ at P. Any minor of the second type, modulo $c_n = 0$, has the form $c_{n-1}\cdot\delta$, where δ is an s-minor of $C_{n,s}$ not involving the last row and column. At the point P, one of the minors δ is invertible (because of the choice of P). Hence

$$(\mathcal{I}_{n,s} + \langle c_n\rangle)_P = (\mathcal{I}^*_{n,s} + \langle c_n, c_{n-1}\rangle)_P.$$

Notice that $\mathcal{I}^*_{n,s} + \langle c_n, c_{n-1}\rangle \subset \mathbb{C}[c_1,\dots,c_n]$ can be identified with

$$\mathcal{I}_{n-2,s} \subset \mathbb{C}[c_1,\dots,c_{n-2}].$$

Now, we can conclude that $(\mathcal{I}_{n,s})_P = \sqrt{(\mathcal{I}_{n,s})_P}$ by induction. Indeed, by the inductive step, we can assume that $(\mathcal{I}_{n,s} + \langle c_n\rangle)_P$ is reduced. Notice that $\dim \mathbb{C}[c]/\mathcal{I}_{n,s} + \langle c_n\rangle = \dim \mathbb{C}[c]/\mathcal{I}_{n,s} - 1$, hence c_n is not a zero divisor in $(\mathbb{C}[c]/\mathcal{I}_{n,s})_P$. This, and the fact that $(\mathcal{I}_{n,s} + \langle c_n\rangle)_P$ is reduced, imply that $(\mathcal{I}_{n,s})_P$ itself is reduced.

In order to run the induction, we have to verify that if $n = s+2$ then the Cohen–Macaulay variety $\mathbb{C}[c]/\mathcal{I}_{s+2,s}$ is reduced. We proceed as above. We fix a point P such that the determinant of the matrix obtained from C by deleting its last row and column is non-zero at P. Then clearly $(\mathcal{I} + \langle c_n\rangle)_P = \langle c_n, c_{n-1}\rangle$, *i.e.*, it is smooth. Since $\dim \mathbb{C}[c]/\mathcal{I} + \langle c_n\rangle = \dim \mathbb{C}[c]/\mathcal{I} - 1$, one gets that $\mathcal{I}_P$ is reduced.

In conclusion, $\mathcal{I} = \mathcal{K}$ in $\mathbb{C}[c]$. By standard argument, $\mathcal{I} = \mathcal{K}$ over $\mathbb{Q}$ as well. In other words (*cf.* Theorem 2.1(1)), one has:

$$O_{\Sigma^s} = \langle \Delta_{i_{s+1},i_s,\dots,i_2,i_1} \mid i_{s+1} > i_s > \cdots > i_1;\ i_{\mathrm{odd}} \text{ is odd}, i_{\mathrm{even}} \text{ is even}\rangle.$$

Clearly, if $s_1 > s_2$ then $O_{\Sigma^{s_1}} \subset O_{\Sigma^{s_2}}$, so according to Theorem 2.1(2), we have

$$\begin{aligned} O_{\Sigma^{<r}} &= \bigcap_{s<r} O_{\Sigma^s} = O_{\Sigma^{r-2}} \\ &= \langle \Delta_{i_{r-1},i_{r-2},\dots,i_2,i_1} \mid i_{r-1} > i_{r-2} > \cdots > i_1;\ i_{\mathrm{odd}} \text{ is odd}, i_{\mathrm{even}} \text{ is even}\rangle. \end{aligned}$$

The proof of the first statement is complete.

According to Theorem 2.1(3), the Thom polynomial of Σ^r is a least degree generator of the ideal just computed, *i.e.*, it is a constant times $\Delta_{r-1,\dots,2,1}$. The constant can be set by applying the so-called "principal equation" for Thom polynomials, see [FR, Theorem 3.5]; details are left to the reader. ■

Corollary 3.2 *Let ω be a generic 2-form on a complex manifold M with complex cotangent bundle T^*M. Set $c_i := c_i(T^*M)$. Then the cohomology class represented by the set $\overline{S_r}$ of points x where ω_x drops rank by at least r is $\Delta_{r-1,r-2,\dots,2,1}$. If any element in the ideal above is not 0 then the set $\overline{S_r}$ can not be empty.*

Remark 3.3 The second part of Theorem 3.1 has already been known, see [FR, HT84, JLP81]. Another description of the ideal using Schur P-polynomials and more generators was given in [Pra88], see also [PR96], [FP98, Ch.4].

4 Degeneracy of 3-Forms on 6-Manifolds

Now let us turn to the representation $\Lambda^3(\mathbb{C}^6)$ of $GL_6 = GL_6(\mathbb{C})$. The description of the orbits were known by Segre; for a modern account see [Don77].

Theorem 4.1 *Let $e_1,\dots,e_6$ form a basis of $\mathbb{C}^6$. The representation $\Lambda^3(\mathbb{C}^6)$ of GL_6 has 5 orbits σ_0, σ_1, σ_5, σ_{10}, σ_{20} (where the indices are the codimensions), with representatives*

$$\omega_0 = e_{123} + e_{456}, \quad \omega_1 = e_{126} + e_{135} + e_{234}, \quad \omega_5 = e_1 \wedge (e_{23} + e_{45}),$$
$$\omega_{10} = e_{123}, \quad \omega_{20} = 0,$$

where $e_{ij\dots}$ means $e_i \wedge e_j \wedge \cdots$.

In order to apply Theorem 2.1, we need to know (at least up to embedded homotopy equivalence) the stabilizer subgroups G_c (where $c = 0, 1, 5, 10, 20$) of these representatives. The case ω_0 is clarified in [Hit00], the other cases are standard (and their verification is left to the reader). Below, S_3 denotes the permutation group of three elements.

Theorem 4.2 *The following groups are (modulo embedded homotopy equivalence) the stabilizer subgroups of the above representatives*

(1) $G_0 = (SL_3 \times SL_3) \rtimes \mathbb{Z}_2$ [Hit00]. *The two SL_3's act on e_1, e_2, e_3 and e_4, e_5, e_6 respectively, and $\mathbb{Z}_2$ interchanges these two $\mathbb{C}^3$'s.*
(2) $G_1 = U_1^3 \rtimes S_3$. *For $\alpha, \beta, \gamma \in U_1^3$ the action is via the diagonal matrices $(\alpha, \beta, \gamma, \bar\beta\bar\gamma, \bar\gamma\bar\alpha, \bar\alpha\bar\beta)$. The symmetric group S_3 permutes e_1, e_2, e_3 and e_4, e_5, e_6 simultaneously.*
(3) $G_5 = U_1^2 \times Sp_2$. *For $(\alpha, \beta) \in U_1^2$ the action is $(\alpha, \bar\alpha, 1, \bar\alpha, 1, \beta)$. Sp_2 acts on the e_2, e_3, e_4, e_5 the standard way.*
(4) $G_{10} = SL_3 \times GL_3$. *The group SL_3 acts on e_1, e_2, e_3 and GL_3 acts on the remaining coordinates.*
(5) $G_{20} = GL_6$.

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Remark 4.3 One has the following test to check whether we found all the symmetries (*cf.* [AB83]). The orbit stratification of $\Lambda^3(\mathbb{C}^6)$ induces a filtration of this vector space, which yields a spectral sequence converging to $H^*(BGL_6)$. Since the stratification is GL_6-perfect, this spectral sequence degenerates at $E_1^{*,*}$, hence we must have (*cf.* also with [FR, Sect. 10]):

$$\dim H^i(BGL_6) = \sum_{j\in\{0,1,5,10,20\}} \dim H^{i-j}(BG_j).$$

Now we have all the input to compute the obstruction ideals. In fact, as we already explained in the introduction, we will consider only their truncation modulo all the homogeneous generators of degree > 6. Notice that ≤ 6 degree generators appear only in the cases σ_0 and $\sigma_0 \cup \sigma_1$ (because of Theorem 2.1(3), and the fact that the degree of $\mathrm{Tp}(\eta)$ is the codimension of η).

Theorem 4.4 *Using rational coefficients, the obstruction ideals of σ_0 and $\sigma_0 \cup \sigma_1$, modulo terms of degree > 6, are the following:*

(1) $O_{\sigma_0} = \langle c_1 \rangle$,
(2) $O_{\sigma_0\cup\sigma_1} = \langle q_5, q_6\rangle$, *where* $q_5 = c_1^5 + c_1c_2^2 + 2c_1^2c_3 - 4c_1c_4$ *and* $q_6 = c_1^3c_3 + c_1c_2c_3 + 2c_1^2c_4 - c_1^4c_2 - 2c_1c_5 - c_1^2c_2^2$.

Proof The ideal O_{σ_0} is the kernel of the homomorphism $H^*BGL_6 \to H^*BG_0$. This homomorphism, according to the description above, is $\mathbb{Q}[c_1, c_2, c_3, c_4, c_5, c_6] \to \mathbb{Q}[a_2, a_3, b_2, b_3]$, where

$$c_1 \mapsto 0, \quad c_2 \mapsto a_2 + b_2, \quad c_3 \mapsto a_3 + b_3,$$
$$c_4 \mapsto a_2b_2, \quad c_5 \mapsto a_2b_3 + a_3b_2, \quad c_6 \mapsto a_3b_3.$$

The kernel of this homomorphism is the ideal $\langle c_1, c_3^2c_4 - c_2c_3c_5 + c_5^2 + c_2^2c_6 - 4c_4c_6\rangle$, which proves the first statement. (Here and in other concrete algebraic calculations we used the computer algebra package Macaulay2 [GS]). To prove the second statement we need to intersect this ideal with the kernel of the map

$$\mathbb{Q}[c_1, c_2, c_3, c_4, c_5, c_6] \to \mathbb{Q}[a, b, c]^{S_3},$$

given by mapping c_i to the i-th elementary symmetric polynomial of $a, b, c, -b - c, -c - a, -a - b$. Computation shows that this intersection is generated by degree 5, 6, 7, 10 polynomials, the first two being q_5 and q_6. ■

Above we used rational coefficients because of the $\mathbb{Z}_2$ factor in the first stabilizer group. The disadvantage of this fact is that the above theorem identifies the Thom polynomials of $\bar\sigma_1$ and $\bar\sigma_5$ (as the least degree elements of corresponding obstruction ideals, *cf.* Theorem 2.1) only as rational multiples of c_1 and q_5. But these rational multiples can be determined using the so called "principal equation" of [FR, (3.5)]: $\mathrm{Tp}(\bar\sigma_1) = 2c_1$, $\mathrm{Tp}(\bar\sigma_5) = q_5$.

The next theorem provides a geometric interpretation of the "derived Thom polynomial" q_6.

Theorem 4.5 *Let ω be a generic 3-form on a complex 6-manifold M. Then the set $S \subset M$ of points where ω is equivalent to the normal form ω_5 (see above) is a smooth Riemann surface. Then $\{v \in T^*_x M \mid x \in S, v \wedge \omega_x = 0\} \to S$ is a line bundle, whose degree is $q_6(c(T^*M))$.*

Proof The bundle $\Lambda^3(T^*M)$ has structure group GL_6 over M. However, when we restrict this bundle to S the structure group reduces to $G_5 = U_1^2 \times Sp_2$, with different characteristic classes. For instance, this restricted bundle has two degree 2 characteristic classes a and b corresponding to the two copies of U_1. Notice also that the next orbit $\bar{\sigma}_{10}$ comes in codimension 10, hence the genericity of ω implies that only types ω_c, $c = 0, 1, 5$ can appear, and S is smooth. Therefore, we can consider the (Gysin) push-forward of any linear combination of a and b. These will clearly be in the degree 6 part of $O_{\sigma_0 \cup \sigma_1}$, *i.e.*, in $\mathbb{Q} \cdot c_1 q_5 + \mathbb{Q} \cdot q_6$. (In fact, the spectral sequence, Remark 4.3, shows that they will span it.)

So we need to compute $i_! a$ and $i_! b$ for $i : BG_5 \subset BGL_6$. We will use an extension of the method of "restriction equations" of [FR] by restricting the equation $i_! a = \alpha \cdot c_1 q_5 + \beta \cdot q_6$ to BG_5. If we use the notation $H^*(Sp_2) = \mathbb{Z}[k, l]^{D_4}$ (D_4 is generated by $k \leftrightarrow -k$, $l \leftrightarrow -l$, $k \leftrightarrow l$) then the Chern classes restrict to the elementary symmetric polynomials of a, $k-a$, $-k$, $l-a$, $-l$, b. This gives the restriction of the left-hand side. To compute the restriction of the right-hand side we determine the normal direction of the orbit of ω_5 at ω_5: $e_{246}, e_{256}, e_{346}, e_{356}, e_{456}$. So the left-hand side restricts to $i^*\, i_! 1, a = a \cdot i^*\, i_! 1 = a \cdot e$ where e is the equivariant Euler class of the normal space to ω_5, *i.e.*, $e = (l+k-2a+b)(-l+k-a+b)(-k+l-a+b)(-k-l+b)(-a+b)$. If we write this out, it is a system of linear equations in α and β with the only solution $\alpha = 0$, $\beta = 1$. So we obtain that $i_! a = q_6$. (Similarly we would obtain that $i_! b = c_1 q_5 + q_6$.) Since a is the first Chern class of the line bundle over S corresponding to the e_1 direction, and this e_1 direction can be characterized as $\{v \mid v \wedge \omega_5 = 0\}$, the theorem follows. ∎

Remark 4.6 (a) Theorem 4.5 shows geometrically that q_6 is indeed in the ideal $O_{\sigma_0 \cup \sigma_1}$, just like q_5, for which $q_5(c(T^*M)) = \text{Poincaré dual}([S]) \in H^5(M)$. The extra information in the description of the ideal is that these two generators are enough.

(b) At this point it is appropriate to explain/exemplify via q_6 the meaning of "derived Thom polynomial". For this, consider a 6-manifold, for which the class q_5 vanishes but the class q_6 not. Then we obtain that for this manifold every 3-form must have σ_5-points, although the σ_5-points represent a 0-homologous cycle, hence homologically cannot be detected. For more on derived Thom polynomials, see [SS99].

5 3-Forms on 7-Manifolds and 8-Manifolds

In this section we present some obstructions to the existence of certain forms on 7-manifolds and 8-manifolds. They fit naturally with the earlier results, and also have some relevance in the theory of manifolds with special holonomy [Joy00].

Stable 3-Forms on 7-Manifolds

The representation $\Lambda^3(\mathbb{C}^7)$ of GL_7 has finitely many orbits, in particular there is an open orbit σ_0. As a consequence, it makes sense to talk about 3-forms on a closed 7-manifold which are generic everywhere (*stable 3-forms* [Hit01]). In particular, the elements of the obstruction ideal of σ_0 (evaluated at the Chern classes of the complex cotangent bundle) are obstructions to the existence of stable 3-forms on a complex 7-manifold M^7.

Theorem 5.1 *Using rational coefficients, the obstruction ideal of the open orbit of the representation $\Lambda^3(\mathbb{C}^7)$ of GL_7 is $O_{\sigma_0} = \langle c_1, c_3, c_5, c_7, c_2^2 - 4c_4\rangle$.*

Proof It is well known that the stabilizer subgroup of a generic 3-form on $\mathbb{C}^7$ is the exceptional Lie group $G_2 \times \mathbb{Z}_3$ [Her83], where G_2 acts by the representation with highest weight 2·short root+long root, see [FH91, 22.3]. So the sought obstruction ideal is the kernel of the map from $\mathbb{Q}[c_1, c_2, c_3, c_4, c_5, c_6, c_7]$ to $\mathbb{Q}[a, b]$ mapping c_i to the i-th elementary symmetric polynomial of the roots of this representation, *i.e.,* of $0, a, -a, b + a, -b - a, b + 2a, -b - 2a$. The kernel is the ideal above. ■

Corollary 5.2 *The elements of this ideal evaluated at the Chern classes of the cotangent bundle of M^7 are obstructions to the existence of a (complex) G_2-structure on M^7. (On G_2-structure on M^7 we mean a reduction of the structure group of TM to G_2.)*

Stable 3-Forms on 8-Manifolds

There are finitely many orbits of the representation $\Lambda^3(\mathbb{C}^8)$ of GL_8, so there is an open orbit here, too. The computation is as above with the only difference that we need to use the adjoint representation corresponding to the root system A_2. Since the stabilizer subgroup, here PSL_2, is not simply connected, we only get the result with rational coefficients.

Theorem 5.3 *Using rational coefficients the obstruction ideal of the open orbit of the representation $\Lambda^3(\mathbb{C}^8)$ of GL_8 is $O_{\sigma_0} = \langle c_1, c_3, c_5, c_7, c_8, c_2^2 - 4c_4\rangle$.*

Spin_7 Structure on Complex 8-Manifolds

The existence of a complex Spin_7 structure on an 8-manifold M^8 is equivalent to the existence of a certain degenerated 4-form on M^8 with stabilizer subgroup Spin_7, see [Joy, 1.3]. Here the representation of Spin_7 is the one whose highest weight is the long root of the root system B_3. The weights of this representation are $\gamma, \beta - \gamma, \alpha - \beta + \gamma, -\alpha + \gamma$ and their opposites. So the obstruction ideal is the kernel of the map from $\mathbb{Z}[c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8]$ to $\mathbb{Z}[\alpha, \beta, \gamma]$ mapping c_i to the i-th elementary symmetric polynomial of the above 8 weights. So we obtain

Theorem 5.4 *The obstruction ideal of the orbit of Ω_0 of* [Joy, 1.3] *in the representation $\Lambda^4(\mathbb{C}^8)$ of GL_8 is $O_{\Omega_0} = \langle c_1, c_3, c_5, c_7, c_2^4 - 8c_2^2c_4 + 16c_4^2 - 64c_8\rangle$.*

Corollary 5.5 *The elements of this ideal evaluated at the Chern classes of the cotangent bundle of M^8 are obstructions to the existence of a complex* Spin_7*-structure on M^8.*

6 Projective Thom Polynomials and Degree Calculations

If a group G acts linearly on a vector space V then this action ρ induces an action $\mathbb{P}\rho$ of G on the projective space $\mathbb{P}V$. If the image of ρ contains the scalars then there is a bijection between the orbits of $\mathbb{P}\rho$ and the non zero orbits of ρ. Assume that η is an invariant subset of ρ with complex codimension d, and let $\mathbb{P}\eta$ be the corresponding invariant subset of $\mathbb{P}\rho$. Then the projective Thom polynomial $\mathrm{Tp}(\mathbb{P}\eta)$ (*i.e.,* the Poincaré dual of $[\mathbb{P}\eta]$) is an element in $H_G^{2d}(\mathbb{P}V;\mathbb{Z}) \cong H^{2d}(BG;\mathbb{Z})[\xi]/\prod \xi - \beta_i$, where β_i are the weights of the representation ρ. In particular, $\mathrm{Tp}(\mathbb{P}\eta)$ can be written as $\sum p_i\xi^i$ for some $p_i \in H^{2(d-i)}(BG;\mathbb{Z})$. It is easy to see that p_0 is the "affine" Thom polynomial $\mathrm{Tp}(\eta)$ and p_d is the degree of the closure of $\mathbb{P}\eta$ in $\mathbb{P}V$:

$$p_0 = \mathrm{Tp}(\eta), \quad p_d = \deg(\mathbb{P}\eta). \tag{5}$$

In fact, one of the main applications of the projective Thom polynomial is that by their help one can calculate the degree of certain varieties.

The above description suggests that (strangely enough) the projective Thom polynomial formally contains more informations than the affine one. The main general result of this section is that this is not the case: a simple substitution into the affine Thom polynomial $\mathrm{Tp}(\eta)$ provides the projective Thom polynomial.

To state the result we need to give names to the generators of $H^*(BG)$. Let $m\colon U_1^n \to G$ be a (coordinatized) maximal torus of G and let α_i are the corresponding roots. Hence, by the Borel theorem (or splitting principle) $\mathrm{Tp}(\eta)$ is a polynomial in the roots α_i and $\mathrm{Tp}(\mathbb{P}\eta)$ is a polynomial in the roots α_i and ξ. We assume that the image of ρ contains the scalars, *i.e.,* there is a homomorphism $\varphi : GL_1 \to G$ and a non zero integer q such that $\rho \circ \varphi(\lambda) = \lambda^q v$ for all $v \in V$, $\lambda \in GL_1$. We assume that $\mathrm{Im}\, m \supset \mathrm{Im}\, \varphi|_{U_1}$ so we have a homomorphism $\tilde\varphi\colon U_1 \to U_1^n$ such that $\varphi|_{U_1} = m \circ \tilde\varphi$. The homomorphism $\tilde\varphi$ is necessarily of the form $\tilde\varphi(t) = (t^{w_1}, \dots, t^{w_n})$ where $t \in U_1$ and w_i are integers. Notice that the choice of φ is not unique.

Theorem 6.1 *Let $\rho\colon G \to GL(V)$ be a representation of the Lie group G such that the image of ρ contains the scalars. Let α_i, q, w_i be as above and let η be an invariant subset of ρ. Then*

$$\mathrm{Tp}(\mathbb{P}\eta)(\alpha_1, \dots, \alpha_n, \xi) = \mathrm{Tp}(\eta)(\alpha_1 + \frac{w_1}{q}\xi, \dots, \alpha_n + \frac{w_n}{q}\xi).$$

Proof The idea of the proof is that we relate the projective Thom polynomial to an affine Thom polynomial for a different group. Let $\tilde G = GL \times G$ and $\tilde\rho = \hom(\rho_1, \rho)$ acting on $\hom(\mathbb{C}, V)$ (where ρ_1 denotes the standard representation of GL_1 on $\mathbb{C}$). Since $\hom(\mathbb{C}, V)$ is naturally isomorphic to V we can think $\tilde G$ acting on V. Moreover the invariant subsets of $\tilde\rho$ are the same as of ρ. We have a map $Q\colon H^*(B\tilde G;\mathbb{Z}) \to H_G^*(\mathbb{P}V;\mathbb{Z})$ induced by the classifying map of the $\tilde\rho$-bundle $\hom(\tau, B_G V) \to B_G\mathbb{P}V$.

Here B_G denotes the Borel construction, *i.e.*, $B_G\mathbb{P}V = EG\times_G\mathbb{P}V$, τ is the tautological line bundle, and with some abuse of notation we denote by B_GV the pull back of the universal ρ-bundle. It is easy to see that $H^*(B\tilde{G};\mathbb{Z}) \cong H^*(BG;\mathbb{Z})[\xi]$ and Q is simply the factorization with the relation $\prod \xi - \beta_i$.

Below Tp^H (respectively Tp^ρ) denotes the (affine or projective) Thom polynomial associated with the action of the group H (via the representation ρ).

Using the above notations, one has:

Proposition 6.2 *For any invariant subset η of ρ*

$$Q(\mathrm{Tp}^{\tilde{G}}(\eta)) = \mathrm{Tp}^G(\mathbb{P}\eta).$$

Proof The bundle $\hom(\tau, B_GV)$ has a canonical section σ, coming from the inclusion of the fiber of the tautological bundle into the vector space V. The set of points in $B_G\mathbb{P}V$ where σ hits an η-point in the fiber can be identified with $B_G\mathbb{P}\eta$. Therefore, the claim reduces to the definition of $\mathrm{Tp}^G(\mathbb{P}\eta)$. ■

Hence we can concentrate on calculating $\mathrm{Tp}^{\tilde{G}}(\eta)$. To do that we choose the maximal torus of $\tilde{G}$ of the form $\tilde{m}(t_0, t_1, \dots, t_n) = (t_0, m(t_1, \dots, t_n))$. The key observation is that restricted to these maximal tori the representation $\tilde{\rho}$ "almost" factors through ρ. We define a homomorphism $\kappa\colon U_1^{n+1} \to U_1^n$:

$$\kappa(t_0, t_1, \dots, t_n) := (t_0^{w_1}t_1^q, \dots, t_0^{w_n}t_n^q) = \tilde{\varphi}(t_0)(t_1^q, \dots, t_n^q).$$

Then $\rho \circ \kappa(t_0, t_1, \dots, t_n) = \rho(\tilde{\varphi}(t_0)(t_1^q, \dots, t_n^q)) = \rho(\tilde{\varphi}(t_0)\rho(t_1^q, \dots, t_n^q)) = (t_0 \cdot \rho(t_1, \dots, t_n))^q = (\tilde{\rho} \circ \tilde{m}(t_0, t_1, \dots, t_n))^q$.

Now we use the following general fact:

Proposition 6.3 *Let $h\colon K \to G$ be a homomorphism of Lie groups and $\sigma := \rho \circ h$. Then*

(1) *if $\eta \subset V$ is ρ-invariant, then η is σ-invariant, too.*
(2) $\mathrm{Tp}^\sigma(\eta) = (Bh)^* \mathrm{Tp}^\rho(\eta)$ *where $Bh : BK \to BG$ is induced by h.*

Proposition 6.3 is an obvious consequence of the fact that the universal σ-bundle is the pull back of the universal ρ-bundle via Bh (you may consider this as the definition of Bh).

Applying Proposition 6.3 twice finishes the proof of Theorem 6.1. Indeed, first if $h = \kappa$, then $(Bh)^*(\alpha_i) = q\alpha_i + w_i\xi$, then if $h(x) = x^q$, then $(Bh)^*(\alpha_i) = q\alpha_i$. ■

Corollary 6.4 *Let $\deg(\mathbb{P}\eta)$ be the degree of $\mathbb{P}\eta$ in $\mathbb{P}V$. Using the notation of Theorem 6.1, one has:*

$$\deg(\mathbb{P}\eta) = \mathrm{Tp}(\eta)(w_1/q, \dots, w_n/q).$$

In other words, knowing the Thom polynomial of an (affine) orbit, we can calculate the degree of the projectivized orbit by substituting w_i/q into the Chern roots.

Example 6.5 Consider the action of $GL_n \times GL_p$ on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$, given by $(A, B) \cdot X := BXA^{-1}$. Then orbits correspond to coranks. The projectivized orbit of corank r matrices is exactly the so-called corank-r determinantal variety. The Thom polynomial of Σ^r is given by the so-called Giambelli–Thom–Porteous formula, and one can choose $q = 1$, $w_i = 0, 0, \dots, 0, 1, 1, \dots, 1$ (or $q = 1$, $w_i = (-1, -1, \dots, -1, 0, 0, \dots, 0)$). Thus our theorem recovers the formula for the degree of determinantal varieties given in [Ful98, 14.4.14].

Example 6.6 All the Thom polynomial computations of this paper can be translated into degree calculations. In particular we can recover the calculations of [HT84] (about an inaccuracy of that paper see [FP98, p.78]) for the degree of the degeneracy loci $\bar{S}_r$. For simplicity, we provide the details only in the following two cases.

Proposition 6.7

(i) *If n is even then the dual of the Grassmannian* $Gr_2(\mathbb{C}^n)$ *in* $\mathbb{P}(\Lambda^2\mathbb{C}^n)$ *has codimension* 1 *and degree* $n/2$ [Las81, Hol79].

(ii) *If n is odd then the dual of the Grassmannian* $Gr_2(\mathbb{C}^n)$ *in* $\mathbb{P}(\Lambda^2\mathbb{C}^n)$ *has codimension* 3 *and degree* $n(n+1)(n-1)/24$ *(for small values of n see* [Hol79]*).*

Proof The cone over the Grassmannian (via the Plücker embedding) is the smallest stratum, *i.e.,* Σ^{n-2}, so the cone over its dual is the largest stratum, *i.e.,* Σ^2 if n even and Σ^3 if n is odd. The Thom polynomials of these are $\Delta_1 = c_1$ and $\Delta_{2,1} = c_1c_2 - c_3$ respectively. Their degrees give the complex codimensions of the dual of the Grassmannians: 1 and 3 respectively. According to Corollary 6.4 we get the degrees if we substitute $1/2$ into the Chern roots. Hence if n is even then the degree is $\frac{1}{2} + \cdots + \frac{1}{2}$ (n terms) $= n/2$, while if n is odd then degree is

$$\det \begin{pmatrix} \binom{n}{2}\frac{1}{4} & \binom{n}{3}\frac{1}{8} \\ 1 & n\frac{1}{2} \end{pmatrix} = n(n+1)(n-1)/24. \qquad \blacksquare$$

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Department of Analysis
Eotvos University
Budapest
Hungary
e-mail: lfeher@math-inst.hu

Department of Mathematics
The Ohio State University
e-mail: nemethi@math.ohio-state.edu

Department of Mathematics
University of North Carolina at Chapel Hill
e-mail: rimanyi@email.unc.edu