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# RUMOR PROCESSES ON ℕ

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**Abstract**

We study four discrete-time stochastic systems on ℕ, modeling processes of rumor spreading. The involved individuals can either have an active or a passive role, speaking up or asking for the rumor. The appetite for spreading or hearing the rumor is represented by a set of random variables whose distributions may depend on the individuals. Our goal is to understand—based on the distribution of the random variables—whether the probability of having an infinite set of individuals knowing the rumor is positive or not.

*Keywords:* Coverage of space; epidemic model; disk percolation; rumor model

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## 1. Introduction

Until a few decades ago, epidemic and rumor models were treated under the same class of models. While there is a clear similitude among the status of the individuals in the models (susceptibles are ignorants, immunes are stiflers, and infected are spreaders), the rates at which individuals change their status might be qualitatively different (see [16]). Generally speaking, the production of stiflers is definitely more complex than the production of immune individuals.

Recently, the mathematics of rumors has generated a good deal of interest. The focus used to be on deterministic or stochastic models, modeling homogeneously mixed populations living on spaces with no structure, as in the Maki–Thompson (see [15] and [18]) and Daley–Kendall (see [5] and [17]) models. Possible variations that can be found in the recent literature include competing rumors (see [11]), more than two people meeting at a time (see [10]), moving agents (see [12]) and rumors through tree-like graphs (see [13] and [14]), complex networks (see [9]), grids (see [1]), and multigraphs (see [2]).

Still, the most important question for both models, epidemic and rumor, is, in terms of a rumor model: if a spreader (an individual who wants to see the rumor spread) is introduced into a reservoir of ignorants, under what conditions will the rumor spread to a large proportion of the population, instead of dying out quickly? Another important question is: if the rumor does not die out quickly, what is the final proportion of individuals hit by the rumor?

We study discrete-time stochastic systems on $\mathbb{N} = \{0, 1, 2, \dots\}$ whose dynamics are as follows. First, assume that at time 0 all vertices of ℕ are declared inactive, except for the origin, which is active. The origin immediately exerts an influence on its neighboring vertices,

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activating a contiguous random set of vertices placed on its right. In general, this is the behavior of every vertex when it is activated.

We consider both *homogeneous* and *heterogeneous* versions of what we call the *radius of influence* of a vertex. In the *homogeneous* version, as a rule, in the moment after activation, each vertex behaves in the same (random) manner as the origin, independent of it and of everything else. We also deal with a *heterogeneous* version where each vertex, if activated, has a distinct distribution for its radius of influence.

We say that the process *survives* if the number of vertices activated is infinite. Otherwise we say that the process *dies out*. We call this the *firework process*, associating the activation dynamic of a vertex to a rumor process. Vertices become spreaders as soon as they are activated. The instant after activation, they propagate the rumor and immediately become stiflers.

A possible variation is what we call the *reverse firework process*. In this process a vertex, instead of being hit by a rumor, defines a set of neighbors on its left to which it asks once whether anyone in this set has heard the rumor. We also deal with *homogeneous* and *heterogeneous* versions of this variation. The models are shown to be qualitatively different in some pertinent cases.

Our main interest is to establish whether each process has positive probability of survival, which is equivalent to a rumor propagation. To this end, we use the distribution of the random variable which defines the radius of influence of each active vertex.

The paper is organized as follows. In Section 2 we present the main results. Section 3 brings the proofs of the main results together with auxiliary lemmas and useful inequalities. In Section 4 we present examples where some conditions can be verified.

## 2. Main results

### 2.1. Firework process

Let $\{u_i\}_{i\in\mathbb{N}}$ be a set of vertices of $\mathbb{N}$ such that $0 = u_0 < u_1 < u_2 < \cdots$, and let $\{R_i\}_{i\in\mathbb{N}}$ be a set of independent random variables assuming values in $\mathbb{R}_+$ whose joint distribution is P. The firework process can be formally defined in the following way. At time 0, an explosion of size $R_0$ occurs at the origin, activating all vertices $u_i \le R_0$. As a rule, at every discrete time $t$ all vertices $u_j$ activated at time $t-1$ generate an explosion (whose radius of influence is $R_j$), and they do this just once, activating the vertices $u_i$ (only those vertices which have not been activated before) such that $u_j < u_i \le u_j + R_j$. Observe that, except for the set of vertices $\{u_i\}$, all other vertices are nonactionable, meaning that the random variable associated with them is 0 almost surely.

If, for all $u_j$ activated at time $t-1$, there are no vertices $u_i$ such that $u_j < u_i \le u_j + R_j$, the process *dies out*. This means that the rumor reaches only a finite number of individuals. If, on the other hand, the process never stops, we say that it *survives*, meaning that the rumor reaches an infinite number of individuals. We call the process *homogeneous* if all the $R_i$ have the same distribution and $u_i = i$ for all $i$. Otherwise we call it *heterogeneous*. We focus on the cases $\mathrm{P}(R_i < 1) \in (0, 1)$ for all $i$.

Let us consider the following monotone decreasing event and its limit:

$$V_n = \{\text{the vertex } u_n \text{ is hit by an explosion}\},$$

$$V = \lim_{n\to\infty} V_n.$$

2.1.1. *The homogeneous case.*

**Theorem 2.1.** *For the homogeneous firework process, consider*

$$a_n = \prod_{i=0}^{n} \mathrm{P}(R < i + 1).$$

*Then*

$$\sum_{n=1}^{\infty} a_n = \infty \quad \textit{if and only if} \quad \mathrm{P}(V) = 0.$$

*Moreover,*

$$\mathrm{P}(V) \geq \prod_{j=0}^{\infty}\left[1 - \prod_{i=0}^{j} \mathrm{P}(R < i + 1)\right], \tag{2.1}$$

$$\mathrm{P}(V) \leq 1 - \mathrm{P}(R = 0) - \sum_{k=1}^{\infty}\left[\mathrm{P}(R = k) \prod_{j=0}^{k-1} \mathrm{P}(R \leq j)\right]. \tag{2.2}$$

**Corollary 2.1.** *For the homogeneous firework process, suppose that*

$$L = \lim_{n\to\infty} n\mathrm{P}(R \geq n).$$

*We have*

(i) *if* $L > 1$ *then* $\mathrm{P}(V) > 0$,

(ii) *if* $L < 1$ *then* $\mathrm{P}(V) = 0$,

(iii) *if* $L = 1$ *and there exists* $N$ *such that, for all* $n \geq N$,

$$\mathrm{P}(R \geq n) \leq \frac{1}{n - 2},$$

*then* $\mathrm{P}(V) = 0$.

**Remark 2.1.** Consider a homogeneous firework process with $R$ assuming values in $\mathbb{N}$. Observe that, in this case, if $\mathrm{E}[R] < \infty$ then $L = 0$. Consequently,

$$\mathrm{E}[R] < \infty \quad \Longrightarrow \quad \mathrm{P}(V) = 0.$$

The following result gives a criterion for the distribution of the random variable $R$ to be a power law.

**Corollary 2.2.** *Let* $\alpha > 1$, *and let* $Z_\alpha$ *be an appropriate constant. Consider the homogeneous firework process such that*

$$\mathrm{P}(R = k) = \frac{Z_\alpha}{(k + 1)^\alpha} \quad \textit{for } k \in \mathbb{N}. \tag{2.3}$$

(i) *If* $\alpha < 2$ *then* $\mathrm{P}(V) > 0$.

(ii) *If* $\alpha \geq 2$ *then* $\mathrm{P}(V) = 0$.

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**Remark 2.2.** Observe that, for the homogeneous firework process, if $R$ has a power-law distribution as in (2.3), with $\alpha = 2$, we have

$$\mathrm{E}[R] = \infty \quad \text{and} \quad \mathrm{P}(V) = 0.$$

2.1.2. *The heterogeneous case.*

**Remark 2.3.** Consider the heterogeneous firework process. We can obtain a sufficient condition for $\mathrm{P}(V) = 0$ (and for $\mathrm{P}(V) > 0$) by a coupling argument. Consider

$$\mathrm{P}(R_i \geq k) \leq \Pr(R \geq k) \quad \text{and} \quad \mathrm{P}(R_i \geq k) \geq \Pr(R \geq k)$$

for some random variable $R$ whose distribution Pr satisfies

$$\lim_{n\to\infty} n \Pr(R \geq n) < 1 \quad \text{and} \quad \lim_{n\to\infty} n \Pr(R \geq n) > 1,$$

respectively. Then respectively use Corollary 2.1(ii) and (i).

**Theorem 2.2.** *Consider a heterogeneous firework process whose actionable vertices are at the integer positions* $0 = u_0 < u_1 < u_2 < \cdots$ *such that* $u_{n+1} - u_n \leq m$ *for* $m \geq 1$. *Furthermore, assume that* $\mathrm{P}(R_n < m) \in (0, 1)$ *for all* $n$.

(i) *If* $\sum_{n=0}^{\infty}[\mathrm{P}(R_n < tm)]^t < \infty$ *for some* $t \geq 1$ *then* $\mathrm{P}(V) > 0$.

(ii) *If, for some random variable* $R$*, whose distribution is* Pr*, the limits*

- $\Pr(R \geq k) - \mathrm{P}(R_n \geq k) \leq b_k$ *for all* $k \geq 0$ *and all* $n \geq 0$*,*
- $\lim_{n\to\infty} n[\Pr(R \geq n) - b_n] > m$*,*
- $\lim_{n\to\infty} b_n = 0$*,*

*exist, then* $\mathrm{P}(V) > 0$.

(iii) $\mathrm{P}(V) \geq \prod_{j=0}^{\infty}[1 - \prod_{i=0}^{j} \mathrm{P}(R_{j-i} < (i+1)m)]$.

## 2.2. Reverse firework process

Let $\{u_i\}_{i\in\mathbb{N}}$ be a set of vertices of $\mathbb{N}$ such that $0 = u_0 < u_1 < u_2 < \cdots$, and let $\{R_i\}_{i\in\mathbb{N}}$ be a set of independent random variables assuming values in $\mathbb{N}$ whose joint distribution is P. The reverse firework process can be defined as follows. At time 0, only the origin is activated. At time 1, explosions of size $R_i$ towards the origin occur at all vertices of $\{u_i\}_{i\in\mathbb{N}}$. All vertices $u_i \leq R_i$ are activated. As a rule, at discrete times $t$ the set of vertices $u_j$ which can find an activated vertex at time $t - 1$ within a distance $R_j$ to its left are activated. Let us call this set $A_t$. If, for some $t$, $A_t$ is empty, the process stops. If the process never stops, we say that it survives. We call the process *homogeneous* if all $R_i$ have the same distribution and $u_i = i$ for all $i$, otherwise we call it *heterogeneous*. We focus on the cases $\mathrm{P}(R_i < 1) \in (0, 1)$ for all $i$. Unless stated otherwise, we assume that $u_i = i$ for all $i$.

Let $S$ be the event 'the reverse process survives'.

2.2.1. *The homogeneous case.*

**Theorem 2.3.** *Consider the reverse homogeneous firework process.*

(i) *If* $\mathrm{E}[R] = \infty$ *then* $\mathrm{P}(S) = 1$.

(ii) *If* $\mathrm{E}[R] < \infty$ *then* $\mathrm{P}(S) = 0$.

**Remark 2.4.** For a random variable $R$ having a power-law distribution as in (2.3), the following assertions hold.

- If $1 < \alpha \leq 2$ then $\mathrm{E}[R] = \infty$.
- If $\alpha > 2$ then $\mathrm{E}[R] < \infty$.

In conclusion, if $R$ has a power-law distribution as in (2.3), with $\alpha = 2$, then $\mathrm{P}(V) = 0$ for the homogeneous firework process by Remark 2.2 and $\mathrm{P}(S) = 1$ for the reverse homogeneous firework process.

2.2.2. *The heterogeneous case.*

**Theorem 2.4.** *Consider the reverse heterogeneous firework process. It holds that*

(i) $\sum_{k=1}^{\infty} \mathrm{P}(R_{n+k} \geq k) = \infty$ *for all* $n$ *if and only if* $\mathrm{P}(S) = 1$,

(ii) *if* $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathrm{P}(R_{n+k} < k) < \infty$ *then* $\mathrm{P}(S) > 0$.

**Remark 2.5.** Let $\rho = \sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathrm{P}(R_{n+k} < k)$. Observe that Theorem 2.3 gives more information for the reverse homogeneous firework process, as in that case $\rho$ equals either 0 ($\mathrm{E}[R] = \infty$) or $\infty$ ($\mathrm{E}[R] < \infty$).

**Remark 2.6.** By a coupling argument and Theorem 2.3, we can see that, if there is a random variable $R$, whose distribution is Pr, with $\mathrm{E}[R] < \infty$ or $\mathrm{E}[R] = \infty$ such that $\mathrm{P}(R_n \geq k) \leq \mathrm{Pr}(R \geq k)$ or, respectively, $\mathrm{P}(R_n \geq k) \geq \mathrm{Pr}(R \geq k)$ for all $k$, then we respectively have $\mathrm{P}(S) = 0$ and $\mathrm{P}(S) = 1$.

## 3. Proofs

Next we present some basic facts, starting with Raabe's test (see [3, p. 48] or [7, p. 32]).

**Fact 3.1.** (Raabe's test.) *For* $a_n > 0$, *let us define*

$$L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right).$$

*Then Raabe's test states that*

- *if* $L > 1$, $\sum_{n=1}^{\infty} a_n < \infty$,
- *if* $L < 1$, $\sum_{n=1}^{\infty} a_n = \infty$,
- *if* $L = 1$ *and* $n(a_n/a_{n+1} - 1) \leq 1$ *for large enough* $n$, $\sum_{n=1}^{\infty} a_n = \infty$.

The following result (see [4, p. 422]) is useful for what follows.

**Lemma 3.1.** *Let* $\{a_n\}_{n\geq 1}$ *be a sequence of real numbers in* $(0, 1)$. *Then,*

$$\prod_{n=0}^{\infty}(1 - a_n) = 0 \iff \sum_{n=0}^{\infty} a_n = \infty. \tag{3.1}$$

**Remark 3.1.** Assume that the actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \cdots$ such that $u_{n+1} - u_n \leq m$ for $m \geq 1$. From the definition of $V_n$ we can see that

- $V_{k+1} \supset V_k \cap \{\bigcup_{i=0}^{k} (R_{k-i} \geq (i+1)m)\}$,

- $V_k$ and $\bigcup_{i=0}^{k}(R_{k-i} \geq (i+1)m)$ are increasing events with respect to the realization of the $R_i$s,
- $P(V_n) > 0$ for all $n$.

From the Fortuin–Kasteleyn–Ginibre (FKG) inequality (see [8, p. 34]), it follows that

$$\begin{aligned} P(V_{k+1}) &\geq P\bigg(V_k \cap \bigg\{\bigcup_{i=0}^{k}(R_{k-i} \geq (i+1)m)\bigg\}\bigg) \\ &\geq P(V_k)P\bigg(\bigg\{\bigcup_{i=0}^{k}(R_{k-i} \geq (i+1)m)\bigg\}\bigg) \\ &= P(V_k)\bigg[1 - \prod_{i=0}^{k} P(R_{k-i} < (i+1)m)\bigg]. \end{aligned} \tag{3.2}$$

Then

$$P(V_n) \geq \prod_{j=0}^{n-1}\bigg[1 - \prod_{i=0}^{j} P(R_{j-i} < (i+1)m)\bigg].$$

Therefore,

$$P(V) \geq \prod_{j=0}^{\infty}\bigg[1 - \prod_{i=0}^{j} P(R_{j-i} < (i+1)m)\bigg]. \tag{3.3}$$

Inequality (3.2) becomes an equality if $u_i = mi$ for all $i \in \mathbb{N}$ and some $m \in \mathbb{N}$. From the latter set of displays and (3.1), the next proposition follows.

**Proposition 3.1.** *Consider a heterogeneous firework process whose actionable vertices are at the integer positions* $0 = u_0 < u_1 < u_2 < \cdots$ *such that* $u_{n+1} - u_n \leq m$. *Let* $a_n = \prod_{i=0}^{n} P(R_{n-i} < (i+1)m)$, *and assume that* $P(R_i < m) \in (0, 1)$. *If*

$$\sum_{n=0}^{\infty} a_n < \infty \quad \textit{then} \quad P(V) > 0. \tag{3.4}$$

### 3.1. Firework process

*Proof of Theorem 2.1.* Assume that $\sum_{n=0}^{\infty} a_n < \infty$. From Proposition 3.1, with $m = 1$, we have $P(V) > 0$.

Assume now that $\sum_{n=0}^{\infty} a_n = \infty$. First consider the event

$$C = \{\text{there exists } n \text{ such that, for all } u_i > n, \text{ there exists } x \text{ such that } x < u_i \leq x + R_x\}.$$

In words this means that, from some vertex $x \in \mathbb{N}$, all vertices belong to the radius of influence of some other vertices. Note that such vertices have not necessarily been activated.

Next, consider the following event:

$$B(u_n) = \{u_n > x + R_x \text{ for all } x < u_n\}.$$

In words, the vertex $u_n$ does not belong to the radius of influence of any vertex to its left.

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Assuming that all random variables $R_i$ have the same distribution as $R$ and that $u_i = i$ ($B_n = B(u_n)$) (they are independent by definition),

$$\mathrm{P}(B_n) = \mathrm{P}\biggl(\bigcap_{i=1}^{n}[R_{n-i} < i]\biggr) = \prod_{i=1}^{n}\mathrm{P}(R < i) = a_{n-1}.$$

Conditional independence of the $B_i$s, i.e. for $i > j$,

$$\begin{aligned}\mathrm{P}(B_i \cap B_j) &= \mathrm{P}\biggl(\bigcap_{k=1}^{i-j}[R_{i-k} < k] \cap \bigcap_{k=1}^{j}[R_{j-k} < k]\biggr)\\ &= \prod_{k=1}^{i-j}\mathrm{P}(R < k)\prod_{k=1}^{i}\mathrm{P}(R < k)\\ &= \mathrm{P}(B_{i-j})\mathrm{P}(B_j),\end{aligned}$$

ensures that the $B_i$s satisfy the definition of a renewal event given in [6, p. 308]. So, from the fact that $\sum_{n=1}^{\infty}\mathrm{P}(B_n) = \infty$, we can rely on Theorem 2 of [6, p. 312] to see that

$$\mathrm{P}(B_n \text{ infinitely often}) = 1.$$

From this we conclude that $\mathrm{P}(V) = 0$, as

$$V^{\mathrm{c}} \supset C^{\mathrm{c}} \supset \{B_n \text{ infinitely often}\}.$$

Inequality (2.1) follows from (3.3) and inequality (2.2) follows from the fact that

$$V^{\mathrm{c}} \supseteq \bigcup_{k=0}^{\infty}\biggl[R_0 = k,\ \bigcap_{j=1}^{k}[R_j \le k-j]\biggr].$$

*Proof of Corollary 2.1.* Observe that, as $a_n = \prod_{i=0}^{n}\mathrm{P}(R < i+1)$,

$$\frac{a_n}{a_{n+1}} - 1 = \frac{\mathrm{P}(R \ge n+2)}{\mathrm{P}(R < n+2)}.$$

Therefore, from the fact that $R$ is almost surely finite,

$$\lim_{n\to\infty} n\biggl(\frac{a_n}{a_{n+1}} - 1\biggr) = \lim_{n\to\infty} n\mathrm{P}(R \ge n). \tag{3.5}$$

Equation (3.5), Raabe's test, and Theorem 2.1 yield (i), (ii), and (iii).

*Proof of Corollary 2.2.* Observe that

$$\frac{1}{(\alpha-1)(n+1)^{\alpha-1}} = \int_{n+1}^{\infty}\frac{1}{x^{\alpha}}\,\mathrm{d}x \le \sum_{j=n+1}^{\infty}\frac{1}{j^{\alpha}} \le \int_{n+1}^{\infty}\frac{1}{(x-1)^{\alpha}}\,\mathrm{d}x = \frac{1}{(\alpha-1)n^{\alpha-1}}.$$

Then

$$\frac{Z_\alpha}{(\alpha-1)}\frac{1}{(n+1)^{\alpha-1}} \le \mathrm{P}(R \ge n) \le \frac{Z_\alpha}{(\alpha-1)}\frac{1}{n^{\alpha-1}}.$$

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Consequently,

$$\lim_{n\to\infty} n\mathrm{P}(R \geq n) = \begin{cases} +\infty & \text{if } \alpha < 2, \\ \dfrac{6}{\pi^2} & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha > 2. \end{cases}$$

The conclusion follows from Corollary 2.1.

*Proof of Theorem 2.2.* Let

$$a_n = \prod_{j=0}^{n} \mathrm{P}(R_{n-j} < (j+1)m).$$

For the proof of part (i), note that, as

$$\sum_{n=t}^{\infty} [\mathrm{P}(R_n < tm)]^t < \infty$$

implies that

$$\sum_{n=t}^{\infty} \Big[\max_{j\in\{0,\dots,t-1\}} \{\mathrm{P}(R_{n-j} < tm)\}\Big]^t < \infty,$$

and as, for $n \geq t$,

$$a_n \leq \prod_{j=0}^{t-1} \mathrm{P}(R_{n-j} < tm) \leq \Big[\max_{j\in\{0,\dots,t-1\}} \{\mathrm{P}(R_{n-j} < tm)\}\Big]^t,$$

the series whose coefficients are $a_n$ converges. So we can use (3.4) in order to obtain the result.

For the proof of part (ii), let

$$r_n = \prod_{j=0}^{n} [\Pr(R < (j+1)m) + b_{(j+1)m}].$$

As

$$n\left(\frac{r_n}{r_{n+1}} - 1\right) = \frac{n[\Pr(R \geq (n+2)m) - b_{(n+2)m}]}{\Pr(R < (n+2)m) + b_{(n+2)m}},$$

from the hypothesis,

$$\lim_{n\to\infty} n\left(\frac{r_n}{r_{n+1}} - 1\right) > 1.$$

However, by the first assumption we have $a_n \leq r_n$; therefore, by Raabe's test, the series whose coefficients are $a_n$ is convergent and so we can use Proposition 3.1 to obtain the desired result.

The proof of part (iii) follows from (3.3).

### 3.2. Reverse firework process

First consider the following variation of the homogeneous firework process. Instead of having just the origin activated at time 0, we consider that all vertices to its left are also activated at time 0. The set of independent random variables which defines the radius of influence of all vertices is $\{F_i\}_{i\in\mathbb{Z}}$, all of which have the same distribution $R$, the random variable which defines the reverse homogeneous firework process.

For this variation of the homogeneous firework process, let us define the events

$$\mathcal{V}_n = \{\text{the vertex } n \text{ is hit by an explosion}\}, \qquad \mathcal{V} = \{\text{the process survives}\}.$$

By analogy, '*to survive*' in this variation means to hit infinitely many vertices of $\mathbb{N}$. It follows that

$$\mathcal{V} = \bigcap_{n=0}^{\infty}\bigcup_{j=0}^{\infty}[F_{n-j} \geq j+1]. \tag{3.6}$$

**Proposition 3.2.** *If* $\mathrm{E}[R] < \infty$ *then* $\mathrm{P}(\mathcal{V}) = 0$.

*Proof.* Let us define the events

$$\mathcal{A}_n = \bigcup_{i=-\infty}^{n-1}\{F_i \geq 2n-i\} \quad \text{and} \quad \mathcal{B}_n = \bigcup_{i=n}^{2n-1}\{F_i \geq 2n-i\}.$$

Observe that

$$\mathcal{V}_{2n} \subseteq \mathcal{V}_n \cap [\mathcal{A}_n \cup \mathcal{B}_n].$$

Therefore,

$$\mathrm{P}(\mathcal{V}_{2n}) \leq \mathrm{P}(\mathcal{A}_n) + \mathrm{P}(\mathcal{B}_n)\mathrm{P}(\mathcal{V}_n).$$

Now

$$\mathrm{P}(\mathcal{A}_n) \leq \sum_{i=-\infty}^{n-1}\mathrm{P}(F_i \geq 2n-i) = \sum_{i=n+1}^{\infty}\mathrm{P}(F_{2n-i} \geq i) = \sum_{i=n+1}^{\infty}\mathrm{P}(R \geq i) \to 0$$

and

$$\mathrm{P}(\mathcal{B}_n) = \mathrm{P}\bigg(\bigcup_{i=n}^{2n-1}\{F_i \geq 2n-i\}\bigg) = 1 - \prod_{i=n}^{2n-1}\mathrm{P}(F_i < 2n-i) \leq 1 - \prod_{i=1}^{\infty}\mathrm{P}(R < i).$$

Then, (3.1) and $\mathrm{E}[R] < \infty$ guarantee the existence of $\lambda \in (0,1)$ such that

$$\mathrm{P}(\mathcal{B}_n) \leq \lambda \quad \text{for all } n.$$

So, as for the homogeneous case $\mathrm{P}(\mathcal{A}_n) \geq \mathrm{P}(\mathcal{A}_{n+1})$,

$$\lim_{n\to\infty}\mathrm{P}(\mathcal{V}_n) = 0,$$

and this implies that $\mathrm{P}(\mathcal{V}) = 0$ as $\mathcal{V}_{n+1} \subset \mathcal{V}_n$.

*Proof of Theorem 2.3.* Let $\{R_i\}_{i\in\mathbb{N}}$ be a set of independent random variables distributed as $R$. Observe that

$$S = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]. \tag{3.7}$$

Using the FKG inequality (see [8, p. 34]) and the fact that intersections of increasing events is an increasing event, we have

$$\mathrm{P}\bigg(\bigcap_{n=0}^{n_0} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\bigg) \geq \prod_{n=0}^{n_0} \mathrm{P}\bigg(\bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\bigg)$$

for all $n_0$. By taking the limit $n_0 \to \infty$ and using the continuity of probability, we obtain

$$\mathrm{P}\bigg(\bigcap_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\bigg) \geq \prod_{n=0}^{\infty} \mathrm{P}\bigg(\bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\bigg).$$

Therefore,

$$\mathrm{P}(S) \geq \prod_{n=0}^{\infty} \bigg[1 - \prod_{j=1}^{\infty} [1 - \mathrm{P}(R_{n+j} \geq j)]\bigg]. \tag{3.8}$$

To prove part (i), note that, from the hypothesis,

$$\sum_{j=1}^{\infty} \mathrm{P}(R \geq j) = \infty. \tag{3.9}$$

Now, (3.1) and (3.9) imply that

$$\prod_{j=1}^{\infty} [1 - \mathrm{P}(R \geq j)] = 0.$$

It then follows from (3.8) that $\mathrm{P}(S) = 1$.

We now turn to the proof of part (ii). By Proposition 3.2, (3.6), and the fact that $R_i$ and $F_i$ have the same distribution,

$$\mathrm{P}\bigg(\bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n-j} \geq j+1]\bigg) = 0. \tag{3.10}$$

On the other hand, as the $R_i$ are all distributed as $R$,

$$\mathrm{P}\bigg(\bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n-j} \geq j+1]\bigg) = \mathrm{P}\bigg(\bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n+j+1} \geq j+1]\bigg),$$

and, therefore, by (3.7) and (3.10), $\mathrm{P}(S) = 0$.

*Proof of Theorem 2.4.* To prove part (i), assume that $\sum_{k=1}^{\infty} \mathrm{P}(R_{n+k} \geq k) = \infty$ for all $n$. Then, by (3.1), we have

$$\prod_{k=1}^{\infty} [1 - \mathrm{P}(R_{n+k} \geq k)] = 0 \quad \text{for all } n.$$

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Therefore, by (3.8), $\mathrm{P}(S) = 1$. On the other hand, as $\mathrm{P}(S) \le 1 - \prod_{k=1}^{\infty} \mathrm{P}(R_{n+k} < k)$ for all $n$, if $\mathrm{P}(S) = 1$, we have

$$\prod_{k=1}^{\infty} [1 - \mathrm{P}(R_{n+k} \ge k)] = 0 \quad \text{for all } n.$$

Now, from (3.1),

$$\sum_{k=1}^{\infty} \mathrm{P}(R_{n+k} \ge k) = \infty \quad \text{for all } n.$$

We now turn to the proof of part (ii). Since $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathrm{P}(R_{n+k} < k) < \infty$, it follows from (3.1) that

$$\prod_{n=0}^{\infty} \left[ 1 - \prod_{k=1}^{\infty} [1 - \mathrm{P}(R_{n+k} \ge k)] \right] > 0.$$

Then, by (3.8) we have $\mathrm{P}(S) > 0$.

## 4. Final remarks and examples

We have considered two discrete propagation phenomena, and modeled both the homogeneous and heterogeneous versions. While the firework process models a phenomenon where there is at all times a finite number of individuals trying to spread information to an infinite group of individuals, the reverse firework process models a phenomenon where there is always an infinite number of individuals willing and working towards hearing information from a finite quantity of informed individuals. As a consequence of their definitions, while the set of individuals who has heard the rumor grows with time in both models, it is a connected set in the reverse firework process, but may have 'holes' that will never be filled in the firework process. Our results show that the two versions are qualitatively different.

Considering the homogeneous firework process, Remark 2.1 shows that the information will not be spread to an infinite number of individuals if $\mathrm{E}[R]$ is finite. To have a *radius of influence* with infinite expectation is also no guarantee that the information will reach an infinite number of individuals, as Remark 2.2 shows. Besides, the probability of not reaching an infinite amount of individuals is at least $\mathrm{P}(R = 0)$. Conversely, in the reverse homogeneous firework process, to have an infinite expectation guarantees almost surely that the information will spread among an infinite amount of individuals, as Theorem 2.3 points out. Furthermore, in the case where the radius of influence has a power-law distribution as in (2.3), the process works in an opposite direction, as Remark 2.4 shows for $\alpha = 2$. The processes agree for $R$ with finite expectation.

Next we present some final examples pointing to some extreme cases. In what follows, we assume that $\{b_n\}_{n \in \mathbb{N}}$ is a nonincreasing sequence convergent to 0, such that $0 < b_n < 1$ for all $n$.

**Example 4.1.** In the heterogeneous firework process it is possible for the expectation of the radius of influence to be infinite for all vertices and for the process to die out almost surely.

(i) $\mathrm{P}(R_n = 0) = 1 - b_n$ and $\mathrm{P}(R_n = k) = b_{n+k-1} - b_{n+k}$ for $k \ge 1$.

(ii) $\sum_{n=0}^{\infty} b_n = \infty$.

(iii) $\lim_{n \to \infty} n b_n = 0$.

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Observe that $E[R_n] = \infty$ for all $n$ from (ii). Furthermore, $P(V) = 0$ from (iii) because

$$P(V_n) \leq \sum_{k=0}^{n-1} P(R_k \geq n-k) = \sum_{k=0}^{n-1} b_{n-1} = (n-1)b_n.$$

**Example 4.2.** In the heterogeneous firework process it is possible for the expectation of the radius of influence to be finite for all vertices and for the process to survive with positive probability. Assume that $\sum_{n=0}^{\infty} b_n < \infty$, while

- $P(R_n = 0) = b_n$,
- $P(R_n = 1) = 1 - b_n$.

Then $E[R_n] < 1$ for all $n$ and $P(V) > 0$ by Theorem 2.2(i) with $m = t = 1$.

**Example 4.3.** In this example we present a family of radii $\{R_n\}$ where $P(S) = 1$ for the reverse heterogeneous firework process, while $P(V) = 0$ for the heterogeneous firework process. For this, consider the sequences $\{b_n\}$ and $\{R_n\}$ such that

(i) $P(R_n = 0) = 1 - b_n$ and $P(R_n = n) = b_n$,

(ii) $\sum_{n=0}^{\infty} b_n = \infty$,

(iii) $\lim_{n\to\infty} nb_n = 0$.

Observe that, even though $\lim_{n\to\infty} E[R_n] = 0$ and $\lim_{n\to\infty} P(R_n = 0) = 1$, from Theorem 2.4 and (ii), it holds for the reverse heterogeneous firework process that $P(S) = 1$. In the opposite direction,

$$P(V_n) \leq \sum_{k=0}^{n-1} P(R_k \geq n-k) = \sum_{k=\lceil n/2\rceil}^{n-1} P(R_k = k) \leq \left\lceil \frac{n}{2} \right\rceil b_{\lceil n/2\rceil},$$

and by (iii) we have $P(V) = 0$ for the heterogeneous firework process.

It is worth noting that this is also an example where the reverse heterogeneous firework process has finite expectation of the radius of influence for all vertices and nevertheless survives almost surely.

The sequence $b_n = ((n+2)\log(n+2))^{-1}$ for instance would satisfy the statements of Examples 4.1 and 4.3.

**Example 4.4.** In contrast to Example 4.3 in this example we present a family of radii $\{R_n\}$ where $P(S) = 0$ for the reverse heterogeneous firework process, while $P(V) > 0$ for the heterogeneous firework process. For this, consider the sequences $\{b_n\}_{n\geq 2}$ and $\{R_n\}$ such that

(i) $P(R_0 = 2) = P(R_1 = 0) = 1$,

(ii) $P(R_n = 0) = 1 - P(R_n = 1) = b_n$ for $n \geq 2$,

(iii) $\sum_{n=2}^{\infty} b_n < \infty$.

While it is easy to see that $P(S) = 0$, $P(V) > 0$ follows from Theorem 2.2(i).

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