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RIGID SETS

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The term "rigid set" appears often in the literature in different contexts and with different meanings. In many cases the notions are unrelated, while in others they refer to related facts. Here we study infinite sets which are rigid according to [2] and analyse the relationship with the notion of rigidity used in [1]. We make several remarks, summarise different results in the area and prove a few new results.

1. INTRODUCTION

The term "rigid set" appears in the literature in many contexts. A few of these are:

(a) in describing situations where endomorphisms, or similar maps, reduce more or less to the identity (for rigid sets in this sense see, for example, [21] for manifolds and [14] for infinite dimensional topological, but not necessarily normed, spaces);

(b) in relation to a famous problem of Mazur concerning isometries for the unit sphere ("transitive" norms, see, for example, [1] and [3] for general results);

(c) by using sequences, as done in [2];

(d) in relation to "intrinsic distances", as done, for example, by K. Borsuk around two decades ago (see [10] for recent results).

We are interested here in rigid sets according to (c), which are connected with rigidity as used in (b).

We now list our definitions. Our settings will be in a complete metric space, that we denote by (E, ρ). According to [1], a subset $A \subset E$ is called *rigid* if it is compact and has the property that

(β) given $a, b \in A$, there exists an isometry T of A onto itself with $Ta = b$.

If a subset A of E satisfies property (β), we say that the group of isometries *acts transitively* on A. This definition of rigidity was given in [1] for a finite dimensional normed space E.

According to [2], a sequence $(x_n) \subset E$ is called *rigid* if its closure is compact and it has the property

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(γ) for all $n \in \mathbb{N}$ and $k \geqslant 1$, $\rho(x_{n+k}, x_n) = \rho(x_{1+k}, x_1)$,

or equivalently,

(γ') for all pairs $i, j \in \mathbb{N}$, $\rho(x_{i+1}, x_{j+1}) = \rho(x_i, x_j)$.

A subset $A = \overline{(x_n)} \subset E$ is called *rigid* if it is the closure of a rigid sequence.

To avoid confusion between these two definitions of rigidity, we adopt the following terminology. We call a set that is compact and satisfies property (β) a *rigid set* and a sequence $(x_n) \subset E$ that satisfies property (γ) a *σ-rigid sequence*. Further, we call the closure of a σ-rigid sequence a *σ-rigid set*, and a σ-rigid set that is compact an *s-rigid set*.

For example, the sequence of all natural numbers, with its natural ordering and metric, is σ-rigid. Of course, being σ-rigid for a sequence is not invariant for reordering. If A is s-rigid and $A = \overline{(x_n)}$, where (x_n) is σ-rigid (and bounded), we say that (x_n) *represents* A.

A *finite* ordered *sequence* $(x_1, \ldots, x_i)$ is called σ-rigid if it satisfies property (γ) whenever $n + k \leqslant i$. A finite set is then called an *s-rigid set* if, for some ordering, it is a σ-rigid sequence. We shall speak of an *infinite sequence* to indicate a sequence (x_n) containing infinitely many different elements.

A set $A \subset E$ is called *equilateral* if $\rho(x, y)$ is independent of $x, y \in A$. An equilateral sequence (x_n) is clearly bounded, closed and σ-rigid. If it is infinite, then it is not s-rigid (since a discrete sequence has no convergent subsequence). Moreover, the isometries act transitively on an equilateral set (in fact, a map only exchanging two points x, y and leaving the others fixed is an isometry).

Given a bounded set A of E, the number $\delta(A) = \sup\{\rho(x, y) : x, y \in A\}$ is called the *diameter* of A.

Let $T : D \to D$ be a map from a subset D of E into itself. The map T is called *nonexpansive* if, for all $x, y \in D$, $\rho(Tx, Ty) \leqslant \rho(x, y)$, and an *isometry* if, for all $x, y \in D$, $\rho(Tx, Ty) = \rho(x, y)$. Let T^k denote the k-fold composition of T with itself. A point $x \in D$ is called a *periodic* point of the map T, of (minimal) period k, if $T^k x = x$ and $T^h x \neq x$ for $h < k$.

Suppose that $x \in D$. The *orbit* of x (under T), denoted by $o(x, T)$, is defined by the formula

$$o(x, T) = (x, Tx, T^2x, T^3x, \ldots),$$

and the *omega limit* set of x, denoted by $\omega(x, T)$, is defined to be the subset of D consisting of all the limit points of the orbit of x, that is,

$$\omega(x, T) = \bigcap_{n \geqslant 1} \overline{(T^n x, T^{n+1}x, T^{n+2}x, \ldots)}.$$

The paper is organised as follows. In Section 2, we recall some simple and known facts. In Section 3, we indicate some other results. In Section 4, we deal with finite rigid sets. In Section 5, we consider the existence of infinite rigid sets in real Banach spaces and we raise some problems. In Section 6, we discuss the rigidity of the unit sphere in Banach spaces. In Section 7, we deal with equilateral sets and orthogonality.

2. Known results

The following facts are well known (see, for example, [**15**] for references).

Lemma 2.1. *Suppose that (E, ρ) is a complete metric space, C is a closed subset of E and $T : C \to C$ is a map. For any x, we have*

(i) *$\omega(x, T)$ is closed, $T\big(\omega(x, T)\big) \subset \omega(x, T)$. Further, $T\big(\omega(x, T)\big) = \omega(x, T)$ if T is continuous and C is compact.*

(ii) *If $\omega(x, T) \neq \emptyset$ and T is nonexpansive, then $\omega(y, T) = \omega(x, T)$ for all $y \in \omega(x, T)$ and T restricted to $\omega(x, T)$ is an isometry of $\omega(x, T)$ onto itself.*

(iii) *If $\omega(x, T)$ is also compact, then there is a commutative group of isometries acting transitively on it.*

From [**2**, Lemma 3.3 and Lemma 4.1], we obtain the following result.

Lemma 2.2. *Suppose that (E, ρ) is a complete metric space, K is a closed subset of E and $T : K \to K$ is a map. We have*

(i) *If K is compact, T is an isometry and, for some $a \in K$, $\omega(a, T)$ is dense in K, then $\rho(T^r x, x) = \rho(T^r y, y)$ for any $x, y \in K$ and all $r \in \mathbb{N}$. Therefore in this case either T is the identity or it has no fixed point.*

(ii) *If K is a compact set and T is nonexpansive and onto, then T is an isometry of K. In particular, under these assumptions on K and T, the set of points of a periodic orbit, as well as the closure of any orbit, is an s-rigid set.*

Remark 2.3. The assumption that "T is surjective" in Lemma 2.2 (ii) is essential. Take, for example,

$$K = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\} \cup \{0\} \subset \mathbb{R}$$

and define T by $T(1/n) = 1/(n+1)$ and $T(0) = 0$.

From [**2**, Lemma 3.2], we have the following result.

Lemma 2.4. *Suppose that $A \subset E$ is compact. The following are equivalent:*

(i) *A is s-rigid.*

(ii) *There exists an isometry $T : A \to A$ and a point $\overline{x} \in A$ such that*

$$\overline{o(\overline{x}, T)} = A.$$

(iii) *There exists an isometry $T : A \to A$ such that for all the points $x \in A$,*

$$\overline{o(x, T)} = A.$$

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This result is also true if $(x_1, x_2, \dots, x_n, \dots)$ consists of a finite number of distinct elements. The isometry T is constructed as follows. If A is an s-rigid set of E represented by $(x_1, x_2, \dots, x_n, \dots)$, then set

$$Tx = \lim_{i\to\infty} x_{n_i+1},$$

where $(x_{n_i})_i$ is a subsequence of $(x_n)_n$ such that $\lim\limits_{i\to\infty} x_{n_i} = x$.

Lemma 2.5. [**13**, Lemma 1]) *Suppose that A is an infinite s-rigid set of E. Then A is a rigid set. Moreover, its corresponding group of isometries is commutative.*

The next result was given for s-rigid sets in [**2**, Lemma 3.4]. For the sake of completeness, we prove here its simple extension to rigid sets in general.

Proposition 2.6. *Suppose that A is a rigid set of E. Then it is diametral, that is, for every $x \in A$ there exists an element $d_x \in A$ such that $\rho(x, d_x) = \delta(A)$.*

Proof: By compactness, there exists a pair y, z in A such that $\rho(y,z) = \delta(A)$. Given $x \in A$, let T be an isometry onto A sending z into x. Let $Ty = w$. We have $\rho(w,x) = \rho(Ty,Tz) = \rho(y,z) = \delta(A)$. ∎

We recall a few properties of "diametral sets" (see [**7**, Proposition 2]).

Proposition 2.7. *Suppose that A is a diametral set of E. We have*

(i) *If $0 < r' < r < \delta(A)$, then*

$$\{x \in A : \rho(y,x) \leqslant r\} \not\subseteq \{x \in A : \rho(z,x) \leqslant r'\}$$

for any $y, z \in A$.

(ii) *A is not symmetric about any of its points, that is, $A - x$ is not symmetric with respect to the origin for any $x \in A$.*

(iii) *A is nowhere dense.*

3. Some other simple results

In general, if A is not an s-rigid set, $\overline{o(x,T)} \neq A$. The next results give information related to Lemma 2.4.

Proposition 3.1. *Suppose that A is a subset of E and $T : A \to A$ an isometry. If, for some $x \in A$, the orbit $o(x,T)$ of x is infinite, then all the elements in $o(x,T)$ are distinct.*

Proof: Suppose there exists a pair n, k such that $T^{n+k}x = T^n x$. We have

$$\rho(T^k x, x) = \rho\big(T^n(T^k x), T^n x\big) = 0$$

and so $T^k x = x$. This implies that

$$T^{k+1}x = Tx, \quad T^{k+2}x = T^2 x, \quad T^{k+3}x = T^3 x, \dots$$

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Therefore the orbit $o(x,T) = (x, Tx, \dots, T^{k-1}x)$ is finite. ∎

We recall that, in general, a compact set must be separable and cannot contain an infinite ε-*discrete* subset for any $\varepsilon > 0$ (in fact, if $\rho(x_i, x_j) \geqslant \varepsilon$ for $i \neq j$, then $\{x_n\}$ cannot contain convergent subsequences). This fact will be used in the proof of the next result.

PROPOSITION 3.2. *Let A be an s-rigid set of E. If A is infinite, then none of its points is isolated.*

PROOF: Suppose that A is represented by the sequence (x_n). Take $x \in A$ and define $T : A \to A$ by $Tx = \lim_{i\to\infty} x_{n_i+1}$, where (n_i) is a sequence with $\lim_{i\to\infty} x_{n_i} = x$. This is a well defined isometry and so, by Lemma 2.4, $A = \overline{o(x,T)}$.

By Proposition 3.1, all the elements of $o(x,T)$ are distinct. Moreover, we know that $A = \overline{\{x, Tx, T^2x, \dots\}}$ satisfies the property (γ).

Let $y = T^{\overline{n}}x \in o(x,T)$ for some $\overline{n} \in \mathbb{N}$. If y is isolated, then there exists $\varepsilon > 0$ such that $\rho(T^{\overline{n}+k}x, T^{\overline{n}}x) \geqslant \varepsilon$ for all k (and $n \in \mathbb{N}$). This shows that all pairs of points in $o(x,T)$ are at a distance $\geqslant \varepsilon$, violating the compactness of A.

Now assume that $y \in A - o(x,T)$. Then there is a sequence (n_i) such that $y = \lim_{i\to\infty} x_{n_i}$, where $x_{n_j} = T^{n_j}x$. If y is isolated, then there exists $\varepsilon > 0$ such that $\rho(\lim_{j\to\infty} x_{n_{j+k}}, y) > \varepsilon$ for all $k > 0$. Thus $\rho(T^{n_{j+k}}x, T^{n_j}x) \geqslant \varepsilon$ for all $k \in \mathbb{N}$, that is, $o(x,T)$ is ε-discrete, and this contradicts the compactness of A. This concludes the proof. ∎

PROPOSITION 3.3. *Suppose that A is an s-rigid set and let $T : A \to A$ be the isometry defined in the proof of Proposition 3.2. Let $x \in A$ and set*

$$q_k = \rho(T^{n+k}x, T^n x), \quad \text{for all } n, k \in \mathbb{N}.$$

Then there is a subsequence q_{k_j} of (q_k) such that $\sum_{j=1}^{\infty} q_j < \infty$. In particular,

$$\liminf_{k\to\infty} q_k = 0.$$

PROOF: If A is finite, then since $A = (x, Tx, T^2x, \dots, T^N x)$ for some $N \in \mathbb{N}$, it follows $q_N = q_{2N} = \dots = 0$ and so the statement holds.

If $A = \overline{(x_n)}$ is infinite, then according to Proposition 3.2, for any point $x \in A$, there is a sequence x_{n_j} such that $\lim_{j\to\infty} x_{n_j} = x$ but $x_{n_j} \neq x$ for all j. Given $\alpha > 0$, let j_1 be the first index such that $\rho(x_{j_1}, x) \leqslant \alpha$. Then, let j_2 be the first index such that

$$\rho(x_{j_2}, x) \leqslant \frac{1}{2}\rho(x_{j_1}, x) \leqslant \frac{\alpha}{2}.$$

Thus

$$\rho(x_{j_2}, x_{j_1}) \leqslant \frac{3}{2}\rho(x_{j_1}, x) \leqslant \frac{3}{2}\alpha.$$

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Proceeding in this way we construct a sequence $\{x_{j_i}\}$ such that

$$\rho(x_{j_{i+1}}, x) \leqslant \frac{1}{2}\,\rho(x_{j_i}, x) \leqslant \frac{\alpha}{2^i},$$

and

$$\rho(x_{j_{1+i}}, x_{j_i}) \leqslant \frac{3}{2}\,\rho(x_{j_i}, x) \leqslant \frac{3}{2}\left(\frac{\alpha}{2^{i-1}}\right) = \frac{3\alpha}{2^i}.$$

This implies that

$$\sum_{i=1}^{\infty} q_{j_{i+1}-j_i} = \sum_{i=1}^{\infty} \rho(x_{j_{i+1}}, x_{j_i}) < \sum_{i=1}^{\infty} \frac{3\alpha}{2^i} = 3\alpha < \infty.$$

Moreover, by setting $k_i = j_{i+1} - j_i$, we see that $\liminf_{k\to\infty} q_k = 0$. ∎

We end this section with some remarks concerning rigid and s-rigid sets.

REMARK 3.4. The notion of "s-rigid set", unlike that of "rigid set", involves some ordering. In fact, reordering of sets does not preserve the property σ.

EXAMPLE 3.5. Let (E, ρ) be the Euclidean plane and consider the points $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (0,1)$, $x_4 = (-1,0)$, $x_5 = (0,-1)$. Then (x_2, x_1, x_3) satisfies property (γ) (we assume that $n + k \leqslant 3$), while (x_1, x_2, x_3) does not. Similarly, (x_2, x_3, x_4, x_5) satisfies (γ) (use indices between 2 and 5), while (x_2, x_4, x_3, x_5) does not.

REMARK 3.6. Rigidity, as well as s-rigidity, is not inherited by subsets. A subset of a rigid set need not be compact or satisfy (γ).

EXAMPLE 3.7. ([**13**, Remark 1]) Let (E, ρ) be the Euclidean plane and

$$A' = \{(1,0), (0,2), (0,0)\} \subset \{(1,0), (1,2), (0,2), (0,0)\} = A.$$

Then A is a rigid set (an isometry sending A to A' exists; we can exchange pairs at the same distance), while A' is not.

The previous remark does not say that it is not possible to find subsets of an s-rigid (or rigid) set that are still s-rigid (or rigid). In fact, for example, if $A = \overline{(x_n)}$ is σ-rigid, then its subsets $A' = \overline{(x_{kn})}_n$ and $A'' = \overline{(x_{k+n})}_n$ ($k \in \mathbb{N}$ fixed) are still σ-rigid.

REMARK 3.8. If A is an s-rigid set, then, by definition, it is σ-rigid. Moreover, by Lemma 2.5, we know that A is also rigid. Now σ-rigid does not imply rigid ((x_2, x_1, x_3) in Example 3.5 is σ-rigid, but not rigid since no isometry sends x_1 to x_2). But also rigid does not imply σ-rigid, (the set A of Example 3.7 is rigid, but not σ-rigid for any ordering).

Another example of a rigid set which is not s-rigid is the following (private communication by R.D. Nussbaum).

EXAMPLE 3.9. Let G be the symmetric group S_n acting on $\mathbb{R}^n$ by permutation of indices and let A be the set of all images of the points $(1, 2, \ldots, n)$ under elements of S_n.

4. Finite rigid sets

We speak of a finite s-rigid set A when we consider property (γ) restricted to a finite subset of $\mathbb{N}$. If a subset A of E consists of only two elements, then (γ) is trivially satisfied. If it has more than two, this is not necessarily true. An example is A' in Example 3.7, or, more generally, a set $A = \{x, y, z\}$ where $\rho(x,y)$, $\rho(x,z)$, and $\rho(y,z)$ are different. In fact, no reordering of A satisfies (γ).

In some cases, it is possible to represent a finite s-rigid set $\{x_1, \dots, x_N\}$ by an s-rigid sequence of the form $(x_1, \dots, x_N, x_1, \dots, x_N, x_1, \dots)$. In this case, $x_{N+j} = x_j$ for all $j \in \mathbb{N}$, and so by property (γ) we should have

$$\rho(x_i, x_1) = \rho(x_N, x_{i-1}) = \rho(x_{N-r}, x_s) = \rho(x_{N-s}, x_r)$$

with $r, s \in \mathbb{N}$ such that

$$i - 1 = r + s.$$

Proposition 4.1. *Let A be an s-rigid set represented by (x_n). The following assertions are equivalent:*

(i) *A is a finite set with N elements.*

(ii) *(x_n) has a periodic point, of period N.*

(iii) *All points of A are periodic of period N.*

Proof: To show that (i) implies (iii), we argue as follows. Since $A = (x_n)$ is finite with N elements, there are two indices i and j $(i > j)$ with $i - j \leqslant N$ such that $x_i = x_j$. By property (γ), for $n = j$ and $k = i - j$, we have

$$0 = \rho(x_i, x_j) = \rho(x_{j+(i-j)}, x_j) = \rho(x_{1+(i-j)}, x_1)$$

that is, $x_{1+(i-j)} = x_1$. Again by property (γ), for $k = i - j$ and n arbitrary, we have $x_{n+(i-j)} = x_n$ for all $n \in \mathbb{N}$. Thus $A = \{x_1, x_2, \dots, x_{i-j}\}$, $N = i - j$.

It is trivial that (iii) implies (ii).

To show that (ii) implies (i), let x_m be an element of period N, that is,

$$x_{m+N} = x_{m+2N} = x_{m+3N} = \dots.$$

Then by property (γ), for $k = N$ and $n = m + N$ it follows that

$$0 = \rho(x_{m+2N}, x_{m+N}) = \rho(x_{1+N}, x_1),$$

that is $x_{1+N} = x_1$. Again by (γ), for $k = N$ and n arbitrary, we have $x_{n+N} = x_n$ for all $n \in \mathbb{N}$.

Repeating the previous argument for $n = m + N$ and $k = 2N, 3N, \dots$, we shall have $x_i = x_{i+N} = x_{i+2N} = x_{i+3N} = \dots$ for all $i \geqslant 2$. Therefore $A = \{x_1, \dots, x_N\}$. Moreover, $x_i \neq x_j$ for $i - j < N$ (otherwise the period of x_m would be $\leqslant i - j$). ∎

Proposition 4.2. *If $A = \{x_1, \dots, x_N\}$ is a finite s-rigid (or σ-rigid) set, which can be represented as*

$$(x_1, \dots, x_N, x_1, \dots, x_N, x_1, \dots),$$

then A is rigid.

Proof: Set $Tx_i = x_{i+1}$ for $i \leqslant N-1$ and $Tx_N = x_1$. Then T maps A onto A and for all $i \in \mathbb{N}$ we have $o(x_i, T) = \omega(x_i, T) = A$. ∎

5. Existence of infinite rigid sets in Banach spaces

Let E be a real vector space and assume that the distance ρ in E is given by a norm $\|\cdot\|$. Let S_E denote the unit sphere of E.

Proposition 5.1. *Let A be an s-rigid set of E, then A is not convex.*

Proof: According to Lemma 2.4, there exists an isometry $T : A \to A$ with the property that $\inf\{\|x - Tx\| : x \in A\}$ is a nonzero constant. This implies that A cannot be convex (see, for example, [**16**, p. 35]). The same conclusion follows also using Schauder's Theorem (see [**16**, p. 15]). ∎

We start now our investigation on the existence of infinite (s-) rigid sets in a Banach space E.

Example 5.2. The unit sphere S of the Euclidean plane E^2 is rigid. Given a and b, a rotation of S sending a into b is an isometry and so it is rigid. Moreover, if the angle between a and b is not a rational multiple of 2π, then the orbit of any point is dense in S. Thus S is also s-rigid.

Remark 5.3. Referring to Remark 3.6, the rigid set S has subsets that are not rigid. For example, any connected proper subset A of S is not rigid (see Proposition 2.6). If A is "less than half of S", this can also be viewed as a consequence of the following result (see [**9**]):

"Let A be a rigid set in a Hilbert space. Given $x_0 \in A$ and an isometry $T : A \to A$ such that $A_1 = \{T^n x_0\}$ is dense in A, then there exists $y \in \overline{\text{co}}(A_1)$ such that for y the best approximation from A is not unique (in practice, if $A \subset S$, then $\overline{\text{co}}(A_1)$ should contain the origin)".

The unit sphere in the Euclidean plane is probably the simplest example of an infinite s-rigid set.

The next result tell us that infinite (s-) rigid sets in finite dimensional normed spaces are rare.

Theorem 5.4. ([**1**]) *If a two dimensional normed space E contains an infinite rigid set A, then A is a circle and the norm is Euclidean.*

Also the following fact is known.

THEOREM 5.5. *If* $\dim E < \infty$ *and the norm is polyhedral, then any rigid set in* E *is finite.*

The result of Theorem 5.5, contained in an unpublished paper [**20**] (see, for example, [**11**]), was reproved in a different way for s-rigid sets (see [**6**]).

Many papers in the area deal with the following problems.

PROBLEM 1. Let $\dim E < \infty$ and assume (s-) rigid sets in E to be finite. Find the optimal upper bounds for the largest cardinality for (s-) rigid sets in E (see, for example, [**15**]).

PROBLEM 2. Find conditions on E implying (s-) rigid sets in E to be finite. In particular, are there infinite-dimensional spaces where no infinite rigid set exists?

A space E contains only finite s-rigid sets if every bounded set satisfying (γ) is finite.

Recall that there exist infinite dimensional Banach spaces containing only finite equilateral sets (see [**17**]).

PROBLEM 3. Are there spaces E containing infinite rigid sets A, but no infinite s-rigid set?

PROPOSITION 5.6.

(i) *Let* X *be a space with a rigid (or s-rigid) set* A. *If* X *embeds isometrically into a space* Y, *then* A *is also rigid (or s-rigid) in* Y.

(ii) *Let* $A \subset X$ *be a rigid (or* σ*-rigid) set and let* $T : A \to Y$ *be an isometry. Then* $T(A)$ *is a rigid (or* σ*-rigid) set of* Y. *In particular, translations preserve such properties.*

This result together with Example 5.2 imply that any space E containing an isometric copy of the Euclidean plane has an infinite rigid set.

EXAMPLE 5.7. $E = L_1[0,1]$ (see [**22**]); $E = L_p$, with p a positive even integer and $\dim E$ sufficiently large (see [**5**]), contain an isometric copy of the Euclidean plane.

The example of the circle S in $\mathbb{R}^2$, embedded in larger spaces, is an example of an infinite (s-) rigid set A such that $\dim(\mathrm{span}[A]) = 2$. It would be interesting to produce "better" examples.

EXAMPLE 5.8. (Private communication by V. Milman) Take $E = \mathbb{R}^2 \oplus \mathbb{R}^2$, with the norm in E determined by the convex hull of the two unit circles in each $\mathbb{R}^2$. A rigid set is given by the two chosen unit circles. We may rotate one of the two circles (leaving the other fixed), or "reflect" one circle on another. All these are transformations that preserve the norm and send any point on these circles to another point.

REMARK 5.9. More generally, consider the space $X \oplus Y$. Let A be a rigid set in X and B a rigid set in Y. If $\|a\|_X = c_1$ for all $a \in A$ and $\|b\|_Y = c_2$ for all $b \in B$, then $A \cup B$ is rigid in the space $X \oplus Y$ endowed with a "monotone" norm (for example, in $(X \oplus Y)_1$).

In fact, if $\{T^n\overline{a}\}$ is dense in A and $\{T^n\overline{b}\}$ is dense in B (for a pair $\overline{a} \in A$ and $\overline{b} \in B$), then $\overline{a}, \overline{b}, T\overline{a}, T\overline{b}, T^2\overline{a}, T^2\overline{b}, \ldots$, gives the correct isometry.

EXAMPLE 5.10. (See [6].) The space c_0 has infinite rigid sets that lie on its unit sphere.

The previous example implies that all Banach spaces in which c_0 can be embedded isometrically have infinite s-rigid sets.

PROBLEM 4. Find an example of an infinite s-rigid set in l_1.

Let assume now that the space where we work is "centreable", in the sense that for any set A, there exists a point $c \in X$ such that

$$\sup_{a\in A} \rho(c, a) = \frac{\delta(A)}{2}.$$

Among Banach spaces, these can be characterised as $\mathcal{P}_1$-spaces (see [4]; see [8] for another characterisation). These are the spaces which are isometric to the space of all continuous functions on an extremally disconnected compact Hausdorff space X (that is, the closure of each open set in X is again open in X), with the sup norm.

Recall that every Banach space can be embedded isometrically in a $\mathcal{P}_1$-space.

PROPOSITION 5.11. *Let A be a rigid set in a $\mathcal{P}_1$-space X. Then A is contained in a circle of X.*

PROOF: Since X is centreable, there exists $c \in X$ such that

$$A \subset B\Big(c, \frac{\delta(A)}{2}\Big).$$

If we had $\|x - c\| < \big(\delta(A)/2\big)$ for some $x \in A$, then we would have

$$\sup_{y\in A} \|x - y\| \leqslant \|x - c\| + \sup_{y\in A} \|c - y\| < \frac{\delta(A)}{2} + \frac{\delta(A)}{2},$$

and this violates Proposition 2.6. Thus $A \subset S\big(c, \big(\delta(A)/2\big)\big)$. ∎

COROLLARY 5.12. *Let A be a rigid set of X. Then there exists some y in a $\mathcal{P}_1$-embedding Y of X such that $A \subset S\big(Y, \big(\delta(A)/2\big)\big) \cap X$.*

REMARK 5.13. Since l^∞ is a $\mathcal{P}_1$-space, the fact that the rigid set of c_0 mentioned in Example 5.10 contains only points of the unit sphere is not surprising. Indeed, in that example, the centre given by the corollary (in l^∞) also belongs to c_0.

6. RIGIDITY OF THE UNIT SPHERE

We recalled in Example 5.2 that the unit sphere in the Euclidean plane is rigid. In fact, we have the following theorem.

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Theorem 6.1. ([**1**, Theorem 6.1]) *Let E be a smooth, strictly convex normed space, $\dim E < \infty$ and odd. Then, if the unit sphere is rigid, E is Euclidean.*

The following result is classical.

Theorem 6.2. *If $\dim E < \infty$ and for every $u, v \in S_E$ there exists a linear isometry T from E onto E such that $Tu = v$, then the norm of E is Euclidean.*

In general, a norm satisfying the hypothesis of the above theorem is called *transitive*. In infinite-dimensional Banach spaces, it is known that the property stated in Theorem 6.2 does not characterise Hilbert spaces in the nonseparable case, while it is still open whether the implication is true in the separable case.

Concerning the relations among the assumptions of Theorems 6.1 and 6.2, the following question was raised in [**18**] (also for $\dim X$ and $\dim Y$ finite).

Let X, Y be two Banach spaces and assume that there exists an isometry T from S_X onto S_Y. Can we extend T to a linear, or affine, map from X to Y? Partial results are known in some special spaces (see [**19**]). It is also known that for several spaces X, an isometry of S_X onto S_X maps antipodal points to antipodal points (see [**12**]). This question is of interest for *finite dimensional spaces*, for otherwise S is not a compact set.

7. Equilateral sets and orthogonality

Suppose that $p > 0$ and let (x_n) be a sequence satisfying

$$(p,k) \qquad \|x_i - x_j\|^p = \|x_i\|^p + \|x_j\|^p, \quad \text{for } |i-j| = k.$$

This condition can be considered as a kind of orthogonality, and, of course, it does not imply anything about property (γ).

Example 7.1. In the space l^2, an orthogonal sequence satisfying $(2,k)$ for all $k \in \mathbb{N}$ is not s-rigid.

Let (x_n) be a sequence satisfying the condition that

$$(kk) \qquad \|x_{n+k} - x_n\| \text{ is constant, for all } n \in \mathbb{N}.$$

This condition still does not imply property (γ).

Example 7.2. Let (x_r) be the sequence in c_0 defined by $x_n = (0,0,\dots)$ if n is even, and $x_{4n-3} = e_{2n-1} - e_{2n}$ and $x_{4n-1} = e_{2n-1} + e_{2n}$ if n is odd, where e_n denotes the sequence $(0,0,\dots,0,1,0,\dots)$, with 1 in the n^{th} place. This sequence is not σ-rigid. In fact, it satisfies $(1,1)$ and (11) (for any $n \in \mathbb{N}$, $\|x_{n+1} - x_n\| = \|x_{n+1}\| + \|x_n\| = 1$), but, depending on n, $\|x_{n+2} - x_n\|$ is 0, 1 or 2.

Lemma 7.3. *Let (x_n) be a sequence that satisfies condition (p,k) for some $p > 0$ and condition (kk) for $k = 1, 2$. Then, for all $n \in \mathbb{N}$, $\|x_n\|$ is constant.*

PROOF: Let start considering the elements x_1, x_2, x_3, x_4 of our sequence. Conditions $(p, 1)$ and (11) imply that $\|x_1\| = \|x_3\|$ and $\|x_2\| = \|x_4\|$. Since $(p, 1)$ and (22) give $\|x_1\| = \|x_2\|$, we have $\|x_1\| = \|x_2\| = \|x_3\| = \|x_4\|$. By starting now with x_4, x_5, x_6, x_7 and repeating the same argument, we get $\|x_4\| = \|x_5\| = \|x_6\| = \|x_7\|$. Proceeding in this way, we therefore obtain $\|x_i\| = \|x_j\|$ for all pairs i, j. ∎

COROLLARY 7.4. *Let (x_n) be a sequence satisfying the hypothesis of the previous Lemma. Then (x_n) is equilateral.*

PROOF: The condition (p, k) and $\|x_i\| = c$ (a constant), imply

$$\|x_i - x_j\| = 2^{1/p} \cdot c,$$

which is a constant, for all $i, j \in \mathbb{N}$. The result follows. ∎

PROPOSITION 7.5. *Let E be an Hilbert space. Any σ-rigid subset A of E consisting of pairwise orthogonal elements is equilateral.*

PROOF: This follows using property (γ) and condition $(2, k)$, for all k. ∎

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