A LINEAR RELATION BETWEEN E-FUNCTIONS

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§ 1. *Introductory.* The formula to be proved is

$$\sum_{r=0}^{2n} {}^{2n}C_r \frac{(b\,;r)(2x)^{-r}}{(\tfrac{1}{2}b+\tfrac{1}{2}-n\,;r)} E\left(\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}r,\ 1+\tfrac{1}{2}r,\ \tfrac{1}{2}b+\tfrac{1}{2}-n+r,\ \alpha_1+r,\ \ldots,\ \alpha_p+r : x\\ \tfrac{1}{2}+\tfrac{1}{2}b+r,\ \tfrac{1}{2}-n+r,\ 1+r,\ \rho_1+r,\ \ldots,\ \rho_q+r\end{matrix}\right)$$

$$=\frac{(2n)!\,2^{-2n}}{n!\,(\tfrac{1}{2}-\tfrac{1}{2}b\,;n)} E(p\,;\alpha_r:q\,;\rho_s:x). \qquad \ldots\ldots\ldots(1)$$

The formulae required in the proof are the Barnes Integral

$$E(p\,;\alpha_r:q\,;\rho_s:x)=\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)} x^{\zeta}\,d\zeta \ldots\ldots\ldots(2)$$

and Whipple's Formula (1)

$$F\left(\begin{matrix}\alpha,\beta\quad,\ \gamma\,;1\\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2},\ 2\gamma\end{matrix}\right)=\frac{\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2})\Gamma(\tfrac{1}{2})\Gamma(\gamma+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2})\Gamma(\tfrac{1}{2}\beta+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\alpha+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\beta+\tfrac{1}{2})}. \ \ldots\ldots\ldots(3)$$

§ 2. *Proof of the Formula.* From (2) the E-function on the left of (1) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}r-\zeta)\Gamma(1+\tfrac{1}{2}r-\zeta)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}-n+r-\zeta)\Pi\Gamma(\alpha_t+r-\zeta)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}b+r-\zeta)\Gamma(\tfrac{1}{2}-n+r-\zeta)\Gamma(1+r-\zeta)\Pi\Gamma(\rho_s+r-\zeta)} x^{\zeta}\,d\zeta.$$

Here replace ζ by $\zeta+r$, note that

$$\Gamma(\tfrac{1}{2}-\tfrac{1}{2}r-\zeta)\Gamma(1-\tfrac{1}{2}r-\zeta)=\Gamma(\tfrac{1}{2})\Gamma(1-r-2\zeta)2^{r+2\zeta}$$

$$=\frac{\Gamma(\tfrac{1}{2})\Gamma(1-2\zeta)2^{r+2\zeta}}{(-1)^r(2\zeta\,;r)}=\frac{2^r\Gamma(\tfrac{1}{2}-\zeta)\Gamma(1-\zeta)}{(-1)^r(2\zeta\,;r)},$$

and it can be seen that the L.H.S. of (1) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\tfrac{1}{2}-\zeta)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}-n-\zeta)\Pi\Gamma(\alpha_t-\zeta)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}b-\zeta)\Gamma(\tfrac{1}{2}-n-\zeta)\Pi\Gamma(\rho_s-\zeta)} F\left(\begin{matrix}-2n,\ b\ ,\ \zeta\,;1\\ \tfrac{1}{2}b+\tfrac{1}{2}-n,\ 2\zeta\end{matrix}\right) x^{\zeta}\,d\zeta.$$

Now, by (3), the generalised hypergeometric function is equal to

$$\frac{\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2})\Gamma(\zeta+\tfrac{1}{2})\Gamma(\zeta+\tfrac{1}{2}-\tfrac{1}{2}b+n)}{\Gamma(\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})\Gamma(\zeta+n+\tfrac{1}{2})\Gamma(\zeta-\tfrac{1}{2}b+\tfrac{1}{2})},$$

and, noting that $\Gamma(\tfrac{1}{2}+\zeta)\Gamma(\tfrac{1}{2}-\zeta)=\pi/\cos\zeta\pi$,

$$\Gamma(\tfrac{1}{2}-n-\zeta)\Gamma(\tfrac{1}{2}+n+\zeta)=\pi/\cos(n+\zeta)\pi,$$

$$\Gamma(\tfrac{1}{2}-n-\zeta+\tfrac{1}{2}b)\Gamma(\tfrac{1}{2}+n+\zeta-\tfrac{1}{2}b)=\pi/\cos(n+\zeta-\tfrac{1}{2}b)\pi,$$

$$\Gamma(\tfrac{1}{2}-\zeta+\tfrac{1}{2}b)\Gamma(\tfrac{1}{2}+\zeta-\tfrac{1}{2}b)=\pi/\cos(\zeta-\tfrac{1}{2}b)\pi,$$

it is found that the expression reduces to

$$\frac{\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})}\times \text{R.H.S. of (2)};$$

and from this the result follows.

Alternative proof. When $p \leqq q$ the E-functions in (1) can be expressed as generalised hypergeometric functions. On picking out the terms in x^{-m} on the left and summing by means of formula (3) the term in x^{-m} on the right is obtained. The restriction on p can then be removed by applying the formula (2)

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1} E(p\,;\alpha_r : q\,;\rho_s : x/\lambda)\, d\lambda = E(p+1\,;\alpha_r : q\,;\rho_s : x), \qquad (4)$$

repeatedly, if necessary.

REFERENCES

(1) Whipple, F. J. W., *Proc. Lond. Math. Soc.* (2), 23 (1923), p. 113.
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