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Geometry and Arithmetic of Certain Double Octic Calabi–Yau Manifolds

Sławomir Cynk and Christian Meyer

Abstract. We study Calabi–Yau manifolds constructed as double coverings of $\mathbb{P}^3$ branched along an octic surface. We give a list of 87 examples corresponding to arrangements of eight planes defined over $\mathbb{Q}$. The Hodge numbers are computed for all examples. There are 10 rigid Calabi–Yau manifolds and 14 families with $h^{1,2} = 1$. The modularity conjecture is verified for all the rigid examples.

1 Introduction

One of the methods of constructing Calabi–Yau manifolds is to study a double covering of $\mathbb{P}^3$ branched along an octic surface. If the octic is smooth then the double covering is a smooth Calabi–Yau manifold. If the branch locus is singular, one has to resolve the singularities of the double covering. In [4, 1] a sufficient condition for the resolution to produce a Calabi–Yau manifold was given. This condition led to the description of a large class of surfaces (called octic arrangements) for which the double covering has a smooth model that is Calabi–Yau. The resolution of singularities of those double solids was given and the Euler characteristic of the resulting Calabi–Yau manifolds was computed. Much more interesting than the Euler characteristic in that context are other invariants, the Hodge numbers. There are only two nontrivial ones, $h^{1,1}$ and $h^{1,2}$; their difference equals half of the Euler characteristic.

In general, it is very difficult to compute Hodge numbers of threefolds, but for a Calabi–Yau manifold, $h^{1,2}$ equals the number of infinitesimal deformations, so we can apply the results of [3]. In the situation being considered, $h^{1,2}$ equals the sum of the number of equisingular deformations of the branch locus in $\mathbb{P}^3$ and the genera of all curves blown-up during the resolution.

The main goal of this paper is to provide a list of nice examples of double solid Calabi–Yau manifolds. It is possible to write down more than 370 examples. We shall include only a list of 87 examples that correspond to arrangements of eight planes defined over $\mathbb{Q}$. These examples satisfy certain additional properties; for instance, their Picard groups are generated by divisors defined over $\mathbb{Q}$.

We shall pay special attention to the examples with a small number of deformations; we shall give a detailed description of nine rigid Calabi–Yau manifolds and equations of 14 one-dimensional families (with $h^{1,2} = 1$). For all the rigid examples we verify the modularity conjecture and compute the corresponding cusp form. Due

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to the very simple form of our Calabi–Yau manifolds, it is easy to compute the trace of Frobenius on H^2, which made the verification possible. Note that the modularity of all the rigid examples also follows from recent work of Dieulefait and Manoharmayum [5].

2 Double Solid Calabi–Yau Threefolds

Let $X \xrightarrow{\pi} \mathbb{P}^3$ be a double covering of $\mathbb{P}^3$ branched along an octic surface D. If D is smooth then X is a (smooth) Calabi–Yau manifold; if D is singular then X is also singular, and the singularities of X are in one–to–one correspondence with the singularities of D. The singularities of X can be resolved by a sequence of blow-ups of $\mathbb{P}^3$, more precisely there is a sequence of blow-ups with smooth centers $\sigma\colon Y \longrightarrow \mathbb{P}^3$, and a smooth, reduced divisor D^* such that $\sigma(D^*) = D$ and D^* is an even element of the Picard group $\operatorname{Pic}(Y)$ of Y. Then the double covering $\tilde{X}$ of Y branched along D^* is a smooth model of X (for details see, *e.g.*, [6]).

If the resolution can be realized by a sequence of blow-ups of

- double or 3-fold curves,
- 4-fold or 5-fold points,

then $\tilde{X}$ is a smooth Calabi–Yau manifold. It is quite easy to compute the Euler characteristic of $\tilde{X}$, namely every blow-up of a 4-fold or 5-fold point increases the Euler number by 36, whereas every blow-up of a double or triple curve C increases the Euler number by

$$7\deg(\mathcal{O}_X(D)|C) - 6\deg(\bigwedge^2 \mathcal{N}_C).$$

In the case of triple curves and 5-fold points, we should remember that after the blow-up we add the exceptional divisor to the branch locus. This may produce "new" singularities in the branch locus, which also require to be resolved (*cf.* [2])

Now assume that D is an *octic arrangement* as in [2], *i.e.*, a surface $D \subset \mathbb{P}^3$ of degree 8 which is a sum of irreducible surfaces $D_1, \dots, D_r$ with only isolated singular points satisfying the following conditions:

(1) For any $i \neq j$ the surfaces D_i and D_j intersect transversally along a smooth irreducible curve $C_{i,j}$ or they are disjoint;
(2) The curves $C_{i,j}$ and $C_{k,l}$ either coincide, are disjoint or intersect transversally.

We shall call a singular point of D_i an *isolated singular point of the arrangement*. A point $P \in D$ which belongs to p of the surfaces $D_1, \dots, D_r$ we shall call an *arrangement p-fold point*. We say that an irreducible curve $C \subset D$ is a *q-fold curve* if exactly q of the surfaces $D_1, \dots, D_r$ pass through it.

We shall use the following numerical data for an arrangement:

d_i The degree of D_i,
p_q^i Number of arrangement q-fold points lying on exactly i triple curves,
l_3 Number of triple lines,
m_q Number of isolated q-fold points.

Theorem 2.1 ([2]) *If an octic arrangement D contains only*

- *double and triple curves,*
- *arrangement q-fold points, $q = 2, 3, 4, 5$,*
- *isolated q-fold points, $q = 2, 4, 5$,*

then the double covering of $\mathbb{P}^3$ branched along D has a non-singular model $\tilde{X}$ which is a Calabi–Yau threefold.

Moreover if D contains no triple elliptic curves then

$$e(\tilde{X}) = 8 - \sum_i (d_i^3 - 4d_i^2 + 6d_i) + 2\sum_{i<j}(4 - d_i - d_j)d_i d_j - \sum_{i<j<k} d_i d_j d_k \\ + 4p_4^0 + 3p_4^1 + 16p_5^0 + 18p_5^1 + 20p_5^2 + l_3 + 2m_2 + 36m_4 + 56m_5.$$

The ordinary double points (nodes) play a special role in the above theorem. They are resolved by a small resolution (on the double covering). As a consequence, in general $\tilde{X}$ cannot be realized as a double covering, and it is even non-projective (or equivalently non-Kähler). In this case it is easier to study a large resolution of X which is a blow-up of the small resolution at the exceptional lines.

The resolution of singularities is done in the following way:

(1) *Blow-up of isolated singular points:* For points of even multiplicity we take the strict transform of the branch divisor as the new branch divisor; for points of odd multiplicity we take the strict transform of the branch divisor plus the exceptional divisor as the new branch divisor. In the latter case, we get a new double curve (projectivisation of the normal cone).

(2) *Blow-up of arrangement 5-fold points:* We take the strict transform of the branch divisor plus the exceptional divisor as the new branch divisor. This introduces five double lines (lying on the exceptional divisor).

(3) *Blow-up of triple curves:* We take the strict transform of the branch divisor plus the exceptional divisor as the new branch divisor. We get three copies of the blown-up curve as double curves. Moreover every 4-fold point lying on that curve gives rise to a double line.

(4) *Blow-up of arrangement 4-fold points:* We take the strict transform of the branch divisor as the new branch divisor (no new singularities).

(5) *Blow-up of double curves:* We take the strict transform of the branch divisor as the new branch divisor (no other singularities). Observe that arrangement triple points disappear.

Since Y is a blow-up of $\mathbb{P}^3$ we have:

Lemma 2.2 *The Picard group $\operatorname{Pic}(Y)$ is a free abelian group generated by the exceptional divisors and the pullback of a plane in $\mathbb{P}^3$. If D contains no triple elliptic curves, then*

$$\rho(Y) = \operatorname{rank}\operatorname{Pic}(Y) = 1 + \binom{r}{2} + p_4^0 + p_4^1 + 6p_5^0 + 7p_5^1 + 8p_5^2 + l_3 + m_4 + 2m_5.$$

By the Lefschetz theorem on $(1,1)$-forms we have $h^2(Y) = \rho(Y)$. So the above theorem (together with the computation of the Euler characteristic) allows us to compute $h^3(Y)$ and so also $h^{1,2}(Y)$. However under the assumption that the resolution is a sequence of blow-ups of double and 3-fold curves and 4-fold and 5-fold points, using the Leray spectral sequence (see [3, Section 6]), one can prove that $h^{1,2}(Y)$ is the sum of the genera of the blown-up curves. Now, simple computations show:

Lemma 2.3 *If D contains no triple elliptic curves then*

$$h^2(\Omega^1_Y) = 6m_5 + \frac{1}{2}\sum_{i<j} d_i d_j (d_i + d_j - 4) + \binom{r}{2}.$$

Since $\tilde{X}$ is a Calabi–Yau manifold, we have $\Omega^2_{\tilde{X}} \cong \mathcal{T}_{\tilde{X}}$, and so we can use the results of [3] to compute the Hodge number $h^{1,2}(\tilde{X})$.

Theorem 2.4 ([3, Prop. 2.1, Thm. 4.1])

(1) $h^{1,2}(\tilde{X}) = h^{1,2}(Y) + h^1(\mathcal{T}_Y(\log D^*))$,
(2) $h^1(\mathcal{T}_Y(\log D^*))$ *equals the number of equisingular deformations of D in* $\mathbb{P}^3$.

Roughly speaking an equisingular deformation of an octic arrangement is a deformation that preserves the numerical data of the arrangement (number and type of singularities). Clearly equisingular deformations allow a simultaneous resolution and hence give a deformation of the double covering. Much more complicated is the geometric meaning of deformations coming from blow-ups of curves ($H^1(\mathcal{T}_Y)$). Those deformations correspond to ruled surfaces in the Calabi–Yau manifold, their geometry is explained in [15] (see also [12, 13]).

The number $h^1(\mathcal{T}_Y(\log D^*))$ can also be computed from:

Lemma 2.5 ([3, Thm. 4.5, Lemma 4.6])

$$h^1(\mathcal{T}_Y(\log D^*)) = \dim_{\mathbb{C}}(I_{\mathrm{eq}}/Jf)^{(8)},$$

where I_{eq} *is an equisingular ideal of D defined by*

$$I_{\mathrm{eq}} = \bigcap_C \big(I_C^{\mathrm{mult}_C D} + Jf\big),$$

the intersection being taken over all multiple curves and points of the arrangement D, and

$$Jf := \Big(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_3}\Big)$$

is the Jacobian ideal of D.

Using Lemmas 2.3 and 2.5 we can compute the Hodge numbers of $\tilde{X}$ with a computer algebra system. The restriction is that the arrangement should be defined over the rational numbers. (We should be able to factorize the equation and find the triple

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curves, 4-fold and 5-fold points; for isolated 4-fold and 5-fold points this requires use of primary decomposition.) This way we were able to study over 370 arrangements (compute the numerical invariants of the arrangements and the Hodge numbers of the resulting Calabi–Yau manifolds). In the following table we collect the numerical data of 87 configurations of eight planes. The table was verified with a Singular [7] program.

Remark 2.6 Observe that Arrangements 85 and 86 have the same number and type of singularities but different Hodge numbers (which proves that the Hodge numbers are *not* functions of the other numerical data).

3 Arrangements with $h^1\mathcal{T}_{\tilde{X}} \leq 1$

Table 1 contains nine examples of rigid Calabi–Yau manifolds and 14 examples with $h^1\mathcal{T}_{\tilde{X}} = 1$. We shall describe the arrangements for which the resulting Calabi–Yau is infinitesimally rigid (*i.e.*, $h^1\mathcal{T}_{\tilde{X}} = 0$) and give equations of those for which the resulting Calabi–Yau deforms in a one-dimensional family.

3.1 Rigid Arrangements

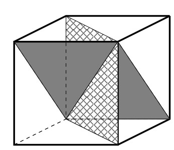
For all rigid Calabi–Yau examples we give equations for the arrangements and list the 4-fold and 5-fold points and triple lines. All but two arrangements can be realized as a cube with two additional planes, in which case we add a geometric description and a picture.

Arrangement no. 2 may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y-t)(y+z-t).$$

It consists of the faces of a cube and additional two planes, each going through four vertices of the cube and intersecting along a diagonal of the cube.

p_5^2 points: (0:1:0:1), (1:0:0:0), (0:1:0:1), (1:0:1:1),
p_4^1 points: (0:0:1:1), (1:0:0:1), (1:1:0:1), (0:1:1:1),
p_4^0 point: (0:1:0:0),
triple lines: $y = x - t = 0, x = y - t = 0,$
$z = y - t = 0, y = z - t = 0.$



Equivalently this arrangement may be described as a tetrahedron and additional four planes going through four edges of the tetrahedron and intersecting in one point. The corresponding equation is

$$xyzt(x+y)(y+z)(z+t)(t+x).$$

Arrangement no. 6 may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y-t)(x+y+z-t).$$

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Table 1: Double coverings of arrangements of eight planes

No	p_3	p_4^0	p_4^1	p_5^0	p_5^1	p_5^2	l_3	$h^{1,2}$	$h^{1,1}$	$e(\tilde{X})$
1	8	0	4	0	0	4	4	1	69	136
2	4	1	4	0	0	4	4	0	70	140
3	20	0	3	0	0	3	3	3	59	112
4	16	1	3	0	0	3	3	2	60	116
5	12	2	3	0	0	3	3	1	61	120
6	8	3	3	0	0	3	3	0	62	124
7	16	0	7	0	0	2	3	3	55	104
8	12	1	7	0	0	2	3	2	56	108
9	13	0	5	0	1	2	3	2	60	116
10	8	2	7	0	0	2	3	1	57	112
11	9	1	5	0	1	2	3	1	61	120
12	12	0	11	0	0	1	3	3	51	96
13	9	0	9	0	1	1	3	2	56	108
14	6	0	7	0	2	1	3	1	61	120
15	18	0	6	1	0	1	2	3	51	96
16	22	0	2	0	2	1	2	3	55	104
17	18	1	2	0	2	1	2	2	56	108
18	14	2	2	0	2	1	2	1	57	112
19	25	0	4	0	1	1	2	4	50	92
20	21	1	4	0	1	1	2	3	51	96
21	17	2	4	0	1	1	2	2	52	100
22	13	3	4	0	1	1	2	1	53	104
23	9	4	4	0	1	1	2	0	54	108
24	28	0	6	0	0	1	2	5	45	80
25	24	1	6	0	0	1	2	4	46	84
26	20	2	6	0	0	1	2	3	47	88
27	16	3	6	0	0	1	2	2	48	92
28	12	4	6	0	0	1	2	1	49	96
29	8	5	6	0	0	1	2	0	50	100

Table 1: Double coverings of arrangements of eight planes

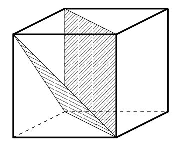
No	p_3	p_4^0	p_4^1	p_5^0	p_5^1	p_5^2	l_3	$h^{1,2}$	$h^{1,1}$	$e(\tilde{X})$
30	18	0	6	0	2	0	2	3	51	96
31	14	1	6	0	2	0	2	2	52	100
32	10	2	6	0	2	0	2	1	53	104
33	21	0	8	0	1	0	2	4	46	84
34	17	1	8	0	1	0	2	3	47	88
35	13	2	8	0	1	0	2	2	48	92
36	24	0	10	0	0	0	2	5	41	72
37	20	1	10	0	0	0	2	4	42	76
38	16	2	10	0	0	0	2	3	43	80
39	34	0	1	0	2	0	1	5	45	80
40	30	1	1	0	2	0	1	4	46	84
41	26	2	1	0	2	0	1	3	47	88
42	22	3	1	0	2	0	1	2	48	92
43	18	4	1	0	2	0	1	1	49	96
44	14	5	1	0	2	0	1	0	50	100
45	32	0	1	1	2	0	1	3	51	96
46	27	0	3	1	1	0	1	4	46	84
47	23	1	3	1	1	0	1	3	47	88
48	19	2	3	1	1	0	1	2	48	92
49	40	0	5	0	0	0	1	7	35	56
50	36	1	5	0	0	0	1	6	36	60
51	32	2	5	0	0	0	1	5	37	64
52	28	3	5	0	0	0	1	4	38	68
53	24	4	5	0	0	0	1	3	39	72
54	20	5	5	0	0	0	1	2	40	76
55	16	6	5	0	0	0	1	1	41	80
56	37	0	3	0	1	0	1	6	40	68
57	33	1	3	0	1	0	1	5	41	72
58	29	2	3	0	1	0	1	4	42	76

Table 1: Double coverings of arrangements of eight planes

No	p_3	p_4^0	p_4^1	p_5^0	p_5^1	p_5^2	l_3	$h^{1,2}$	$h^{1,1}$	$e(\tilde{X})$
59	25	3	3	0	1	0	1	3	43	80
60	21	4	3	0	1	0	1	2	44	84
61	17	5	3	0	1	0	1	1	45	88
62	13	6	3	0	1	0	1	0	46	92
63	36	0	0	2	0	0	0	5	41	72
64	32	1	0	2	0	0	0	4	42	76
65	28	2	0	2	0	0	0	3	43	80
66	24	3	0	2	0	0	0	2	44	84
67	46	0	0	1	0	0	0	7	35	56
68	42	1	0	1	0	0	0	6	36	60
69	38	2	0	1	0	0	0	5	37	64
70	34	3	0	1	0	0	0	4	38	68
71	30	4	0	1	0	0	0	3	39	72
72	26	5	0	1	0	0	0	2	40	76
73	56	0	0	0	0	0	0	9	29	40
74	52	1	0	0	0	0	0	8	30	44
75	48	2	0	0	0	0	0	7	31	48
76	44	3	0	0	0	0	0	6	32	52
77	40	4	0	0	0	0	0	5	33	56
78	36	5	0	0	0	0	0	4	34	60
79	32	6	0	0	0	0	0	3	35	64
80	32	6	0	0	0	0	0	4	36	64
81	28	7	0	0	0	0	0	3	37	68
82	24	8	0	0	0	0	0	2	38	72
83	20	9	0	0	0	0	0	1	39	76
84	20	9	0	0	0	0	0	0	38	76
85	16	10	0	0	0	0	0	1	41	80
86	16	10	0	0	0	0	0	0	40	80
87	8	12	0	0	0	0	0	0	44	88

It consists of the faces of a cube and additional two planes, one through three vertices and the other through four vertices of the cube and intersecting along the diagonal of a face.

p_5^2 points: $(0{:}0{:}1{:}0), (1{:}0{:}0{:}1), (0{:}1{:}0{:}1)$,
p_4^1 points: $(1{:}0{:}1{:}1), (0{:}1{:}1{:}1), (1{:}-1{:}0{:}0)$,
p_4^0 points: $(0{:}0{:}1{:}1), (0{:}1{:}0{:}0), (1{:}0{:}0{:}0)$,
triple lines: $y = x - t = 0, x = y - t = 0, z = x + y - t = 0$.

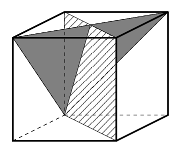


Arrangement no. 23 may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y-t)(x-y+z-t).$$

It consists of the faces of a cube and additional two planes, one through three vertices and the other through four vertices of the cube, having only one of the vertices of the cube in common.

p_5^2 point: $(0{:}0{:}1{:}0)$,
p_5^1 point: $(1{:}0{:}0{:}1)$,
p_4^1 points: $(0{:}1{:}0{:}1), (1{:}0{:}1{:}1), (0{:}1{:}1{:}1), (0{:}1{:}2{:}1)$,
p_4^0 points $(0{:}0{:}1{:}1), (0{:}1{:}0{:}0), (1{:}0{:}0{:}0), (1{:}1{:}1{:}1)$,
triple lines: $y = x - t = 0, x = y - t = 0$.



Arrangement no. 29 may be defined by the equation

$$xyzt(x+y+z+t)(y+z+t)(x-z+t)(x+y+2t).$$

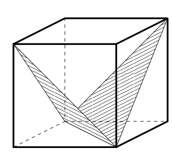
p_5^2 point: $(0{:}-2{:}1{:}1)$,
p_4^1 points: $(0{:}0{:}1{:}-1), (2{:}0{:}1{:}-1), (0{:}1{:}0{:}-1), (1{:}1{:}0{:}-1), (0{:}1{:}-1{:}0)$,
$(1{:}-1{:}1{:}0)$,
p_4^0 points: $(0{:}0{:}1{:}0), (1{:}0{:}0{:}0), (0{:}1{:}0{:}0), (1{:}0{:}0{:}-1), (1{:}-1{:}0{:}0)$
triple lines: $y + z + t = x + y + 2t = 0, x = y + z + t = 0$.

Arrangement no. 44 may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y+z-t)(x-y+z-t).$$

It consists of the faces of a cube and additional two planes through three vertices of the cube intersecting along the diagonal of a face.

p_5^1 points: $(1{:}0{:}0{:}1), (0{:}0{:}1{:}1)$,
p_4^1 point: $(1{:}0{:}-1{:}0)$,
p_4^0 points: $(0{:}1{:}0{:}1), (0{:}0{:}1{:}0), (0{:}1{:}0{:}0), (1{:}0{:}0{:}0)$,
$(1{:}1{:}1{:}1)$,
triple line: $x + z - t = y = 0$.



Arrangement no. 62 may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y+z-2t)(x+y).$$

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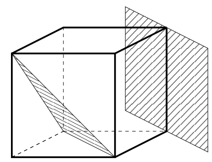
It consists of the faces of a cube and additional two planes: one plane through an edge of the cube and parallel to a diagonal of the cube, and one plane through three vertices of the cube not belonging to the first plane.

p_5^1 point: $(0{:}0{:}1{:}0)$,
p_4^1 points: $(0{:}0{:}0{:}1), (0{:}0{:}2{:}1), (0{:}0{:}1{:}1)$,
p_4^0 points: $(1{:}1{:}0{:}1), (1{:}0{:}1{:}1), (0{:}1{:}0{:}0)$,
$(1{:}0{:}0{:}0), (0{:}1{:}1{:}1), (1{:}-1{:}0{:}0)$,
triple line: $x = y = 0$.



Arrangement no. 84 may be defined by the equation

$$xyzt(x + y + z + t)(2x + 2z + t)(2y + 2z + t)(x + y + 2z + 2t).$$

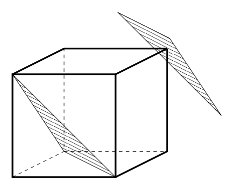
p_4^0 points: $(1{:}0{:}0{:}0), (0{:}1{:}0{:}0), (0{:}1{:}-1{:}0), (1{:}0{:}-1{:}0), (1{:}1{:}0{:}-2)$,
$(1{:}1{:}-1{:}0), (0{:}0{:}1{:}-1), (1{:}-1{:}0{:}0), (0{:}0{:}1{:}-2)$.

Arrangement no. 86 may be defined by the equation

$$(x - t)(x + t)(y - t)(y + t)(z - t)(z + t)(x + y + z + t)(x + y + z - 3t).$$

It consists of the faces of a cube and an additional two parallel planes: one through three vertices of the cube and the second through one. The 4-fold points are: four vertices, three points at infinity which are the intersection of parellel edges of the cube, and three points of intersection at infinity of a pair of parallel faces of the cube and the additional two planes.

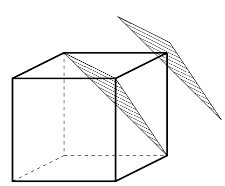
p_4^0 points: $(1{:}1{:}-1{:}-1), (1{:}-1{:}1{:}-1), (1{:}-1{:}-1{:}1)$,
$(1{:}0{:}0{:}0), (0{:}1{:}0{:}0), (0{:}0{:}1{:}0)$,
$(0{:}1{:}-1{:}0), (1{:}0{:}-1{:}0)$,
$(1{:}-1{:}0{:}0), (1{:}1{:}1{:}1)$.



Arrangement no. 86[a] with the same numerical data as arrangement 84 may be defined by the equation

$$(x - t)(x + t)(y - t)(y + t)(z - t)(z + t)(x + y + z - t)(x + y + z - 3t).$$

p_4^0 points: $(1{:}-1{:}-1{:}-1), (1{:}-1{:}1{:}1)$,
$(1{:}1{:}-1{:}1), (1{:}0{:}0{:}0), (0{:}1{:}0{:}0)$,
$(0{:}0{:}1{:}0), (0{:}1{:}-1{:}0), (1{:}0{:}-1{:}0)$,
$(1{:}-1{:}0{:}0), (1{:}1{:}1{:}1)$.



Arrangement no. 87 may be defined by the equation

$$(x - t)(x + t)(y - t)(y + t)(z - t)(z + t)(x + y + z + t)(x + y + z - t).$$

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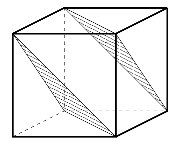
It consists of the faces of a cube and additional two parallel planes through three vertices. The 4-fold points are: six vertices, three points at infinity which are the intersection of parellel edges of the cube, and three points of intersection at infinity of a pair of parallel faces of the cube and the additional two planes.

p_4^0 points: $(1{:}1{:}-1{:}-1), (1{:}-1{:}1{:}-1), (1{:}-1{:}-1{:}1),$
$(1{:}0{:}0{:}0), (0{:}1{:}0{:}0), (0{:}0{:}1{:}0),$
$(0{:}1{:}-1{:}0), (1{:}0{:}-1{:}0), (1{:}-1{:}0{:}0),$
$(1{:}1{:}-1{:}1), (1{:}-1{:}1{:}1), (-1{:}1{:}1{:}1).$



Equivalently this arrangement may be described as a symmetric octahedron. The 4-fold points are now: six vertices of the octahedron and six points at infinity of intersections of parallel edges.

3.2 One-Dimensional Families

The following table lists equations of one–dimensional families containing arrangements with $h^1\mathcal{T}_{\tilde{X}} = 1$.

1	$xyzt(x+y)(y+z)(z+t)(Ax+Bt)$
5	$xy(x-y)(y-z)(y-t)(x-z)(x-t)(Ax+By-Az+(A-B)t)$
10	$xyzt(x+y)(x+t)(z+t)(Ax+(A-B)y+(B-A)z+Bt)$
11	$xyzt(x+y)(x+t)(z+t)(Ay+Bz+(B-A)t)$
14	$xyzt(x+y)(x+z)(y-z+t)(Ay-Az+Bt)$
18	$xyzt(x+y)(x+z)(Ax+By+At)(Ax+Bz+At)$
22	$xyzt(x+y)(x+z)(Ax+Ay+Bz+Bt)(Ay-Az+Bt)$
28	$xyzt(x+y)(x+z)(Ay-Az+Bt)(x+y+z+t)$
32	$xyzt(x+y)(z+t)(Ay+Az+Bt)(Bx+(B-A)y+(B-A)z)$
43	$xyzt(x+y+z+t)(Ax+By+Az+Bt)$ $\times(Ax+Ay+Bz+Bt)(Bx+By+Az+At)$
55	$xyzt(x+y+z+t)(Ay+Bz+Bt)(Ax-Bz+At)(Ax+Ay+(A+B)t)$
61	$xyz(x-t)(y-t)(z-t)(x+y+z-2t)(Ax+By)$
83	$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)$ $\times(Ax+By+Bz-At)(Ax+By+Bz+(A+2B)t)$
85	$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)$ $\times(Ax+By+Bz-At)(Ax+By+Bz+At)$

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4 The L-Series of Rigid Calabi–Yau Manifolds

If $\tilde{X}$ is a Calabi–Yau manifold defined over $\mathbb{Q}$, and p is a good prime (*i.e.,* a prime such that the reduction of $\tilde{X}$ mod p is nonsingular) the map

$$\mathrm{Frob}_p^*\colon H^i_{\text{ét}}(\tilde{X}, \mathbb{Q}_l) \mapsto H^i_{\text{ét}}(\tilde{X}, \mathbb{Q}_l)$$

on l-adic cohomology induced by the geometric Frobenius morphism gives rise to l-adic Galois representations

$$\rho_{l,i}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mapsto \mathrm{GL}_{h^i}(\mathbb{Q}_l).$$

If a Calabi–Yau manifold $\tilde{X}$ is *rigid* (*i.e.,* $h^{1,2}(\tilde{X}) = 0$ or equivalently $h^2(\tilde{X}) = 2$) then $\tilde{X}$ is expected to be *modular* (see [10, 14] for a good account on this conjecture). More precisely, it is conjectured that for any rigid Calabi–Yau $\tilde{X}$ the L-series of $\tilde{X}$ equals the L-series of a cusp form f of weight 4 for $\Gamma_0(N)$.

We shall verify the modularity conjecture for all rigid Calabi–Yau manifolds constructed in the paper.

Lemma 4.1 *The Calabi–Yau manifolds $\tilde{X}_p$ associated to Arrangements no.* 2, 6, 23, 29, 44, 62, 87 *are smooth for all primes $p \geq 3$; the Calabi–Yau manifolds $\tilde{X}_p$ associated to Arrangements no.* 84, 86, 86^a *are smooth for all primes $p \geq 5$.*

Proof Since the singularities of arrangements of planes are defined by ranks of some minors of 8×4 matrices of coefficients, it is enough to verify the lemma for the primes dividing any minor of the matrices. This is easily done with a computer. ■

Lemma 4.2 *All eigenvalues of* Frob_p^* *on $H^2_{\acute{e}t}(\tilde{X})$ are equal to p (for $p \geq 5$).*

Proof The Picard group $\mathrm{Pic}(\tilde{X})$ of $\tilde{X}$ splits into a sum of symmetric part and skew-symmetric part. The symmetric part is naturally isomorphic to $\mathrm{Pic}(Y)$. By Lemma 2.2

$$\operatorname{rank} \mathrm{Pic}(Y) = 29 + p_4^0 + p_4^1 + 6p_5^0 + 7p_5^1 + 8p_5^2 + l_3.$$

Consequently for Arrangements no. 2, 6, 23, 29, 44, 62, 84, we get $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(Y)$, *i.e.,* all the divisors are even and defined over $\mathbb{Q}$. For Arrangements no. 86, 86^a the rank of the skew-symmetric part of the Picard group is one; it is generated by the divisor associated to the contact hyperplane $t = 0$. For Arrangement 87 the rank of the skew-symmetric part of the Picard group is three; it is generated by the divisors associated to the contact hyperplanes $t = 0$, $x + y = 0$ and $x + z = 0$. In all cases the Picard group of $\tilde{X}$ is generated by divisors defined over $\mathbb{Q}$. ■

Denote

$$t_i := \operatorname{tr} \mathrm{Frob}_p^* |H^i_{\text{ét}}(\tilde{X})).$$

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From the above Lemma, Poincaré duality and the Weil conjectures, we get

$$t_0 = 1, \qquad t_1 = 0, \qquad t_2 = p \cdot h^{1,1},$$
$$t_4 = p^2 \cdot h^{1,1}, \qquad t_5 = 0, \qquad t_6 = p^3.$$

The coefficients of the L-series can now be computed from the Lefschetz fixed point formula,

$$a_p := t_3 = 1 + p^3 + h^{1,1}(p + p^2) - \#\tilde{X}(\mathbb{F}_p).$$

For computations of the number of points we used a computer program. We should note that the number does not only depend on the branch divisor, but actually on its equation. Multiplying the equation of the branch divisor by squarefree integers we get new (non-isomorphic over $\mathbb{Q}$) Calabi–Yau manifolds.

The computation was organized as follows: first we computed the number of points on the singular double covering of $\mathbb{P}^3(\mathbb{F}_p)$, *i.e.,* the number of points in $\mathbb{P}^3(\mathbb{F}_p)$ for which the value of the branch divisor equation is a square (in $\mathbb{F}_p$). Then we have to take into account the resolution of singularities.

Blowing up a 5-fold point replaces a point on the double covering by a plane (since the exceptional divisor is contained in the branch locus), but we add five double lines and $0, 1$ or two p_4^1 points (depending on the number of triple lines through this point).

Blowing up a triple line replaces a line on the double covering by $\mathbb{P}^1 \times \mathbb{P}^1$. This introduces new double lines: altogether 3 plus the number of 4-fold points on the triple line.

Blowing up a double line replaces a line on the double covering by a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ which is also $\mathbb{P}^1 \times \mathbb{P}^1$, so we add $p^2 + 2p + 1 - (p + 1) = p^2 + p$ points.

Altogether, blowing up double and triple lines and 5-fold points adds

$$(p_4^1 + 6p_5^0 + 7p_5^1 + 8p_5^2 + l_3 + 28)(p + p^2)$$

points to the double covering.

We cannot write down a similarly simple formula for blowing up a 4-fold point. The reason is that the blow-up of a 4-fold point replaces a point on the double covering by a double covering of a projective plane branched along four lines (projectivisation of the normal cone). So we have to write down the equation for every 4-fold point and compute the number of points on the double covering (in the same way as for the double covering of $\mathbb{P}^3$). We should however take into account the coefficient coming from the planes not passing through the 4-fold point (product of the values of the equations).

All this can be done with a computer, leading to the table of coefficients (Table 2).

Comparing the computed values with the coefficients of certain cusp forms of weight 4 from Stein's table [11] (we use his notation for the classification of newforms) we observe that they agree for all listed primes. Following the guidelines of [14] or [8], we can conclude that they agree for all primes $p \geq 5$ and so the studied Calabi–Yau manifolds are modular. The above method requires that

$$\det(\mathrm{Frob}_p^* | H^3_{\text{ét}}(\tilde{X})) = p^3$$

Table 2: Coefficients

p	5	7	11	13	17	19	23	73
Arrangements 2, 87					8k4A			
a_p	-2	24	-44	22	50	44	-56	154
Arrangement 6					32k4C			
a_p	-10	-16	40	-50	-30	-40	-48	-630
Arrangement 23					64k4A			
a_p	-22	0	0	18	-94	0	0	1098
Arrangements 29, 44					16k4A			
a_p	-2	-24	44	22	50	-44	56	154
Arrangement 62					64k4C			
a_p	2	-24	-44	-22	50	44	56	154
Arrangements 84, 86					6k4A			
a_p	6	-16	12	38	-126	20	168	218
Arrangement 86[a]					12k4A			
a_p	-18	8	36	-10	18	-100	72	26

for the considered finite set of primes. There are simple proofs for this under the assumption that $\operatorname{tr}(\mathrm{Frob}_p^* \,|H^3_{\text{ét}}(\tilde{X})) \neq 0$. This assumption is satisfied for all arrangements except no. 23. Computing the coefficients of the zeta function of this Calabi–Yau, we get

$$\#\tilde{X}(\mathbb{F}_{p^2}) = 2\det(\mathrm{Frob}_p^* \,|H^3_{\text{ét}}(\tilde{X})) + p^6 + 54(p^4 + p^2) + 1$$

for any prime p such that $\operatorname{tr}(\mathrm{Frob}_p^* \,|H^3_{\text{ét}}(\tilde{X})) = 0$. So the required formula follows from counting points over the field $\mathbb{F}_{p^2}$.

Remark 4.3 The Calabi–Yau threefolds constructed from Arrangements 2 and 87 have the same L-series (up to factors at the primes of bad reduction), but different Hodge numbers. We can obtain the same L-series by multiplying the equations of no. 29, resp. 44 and 62, by -1, resp. -1, -2.

Remark 4.4 A finer and more systematic classification of arrangements of eight planes was carried out by C. Meyer in [9].

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