

## SCHUR MULTIPLICATIVE MAPS ON MATRICES

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### Abstract

The structure of Schur multiplicative maps on matrices over a field is studied. The result is then used to characterize Schur multiplicative maps  $f$  satisfying  $f(S) \subseteq S$  for different subsets  $S$  of matrices including the set of rank  $k$  matrices, the set of singular matrices, and the set of invertible matrices. Characterizations are also obtained for maps on matrices such that  $\Gamma(f(A)) = \Gamma(A)$  for various functions  $\Gamma$  including the rank function, the determinant function, and the elementary symmetric functions of the eigenvalues. These results include analogs of the theorems of Frobenius and Dieudonné on linear maps preserving the determinant functions and linear maps preserving the set of singular matrices, respectively.

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### 1. Introduction

Let  $M_{m,n}$  be the set of  $m \times n$  matrices over a field  $\mathbf{F}$  with at least three elements. Define the *Schur product* (also known as the *Hadamard product* or *entrywise product*) of  $A = [a_{ij}]$ ,  $B = [b_{ij}] \in M_{m,n}$  by  $A \circ B = [a_{ij}b_{ij}]$ . A map  $f : M_{m,n} \rightarrow M_{m,n}$  is *Schur multiplicative* if

$$f(A \circ B) = f(A) \circ f(B) \quad \text{for all } A, B \in M_{m,n}.$$

The study of the Schur product is related to many pure and applied areas; see [9]. There has been considerable interest in studying linear maps, additive maps, and multiplicative maps  $f$  on matrices with some special properties such as  $f(S) \subseteq S$  for a certain subset of matrices, or  $\Gamma(f(A)) = \Gamma(A)$  for a given function  $\Gamma$  on matrices; for example, see [3, 8, 11, 12, 15] and the references therein. In this paper, we study Schur multiplicative maps on matrices with some of these special properties.

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In Section 2, we consider general Schur multiplicative maps  $f : M_{m,n} \rightarrow M_{m,n}$ . In particular, it is shown that under some mild assumptions the Schur multiplicative map  $f$  has the form:

(†)  $[a_{ij}] \mapsto \mathcal{P}[f_{ij}(a_{ij})]$ , where  $f_{ij} : \mathbf{F} \rightarrow \mathbf{F}$  satisfies  $f_{ij}(0) = 0$  for each  $(i, j)$  pair, and  $\mathcal{P}(X) \in M_{m,n}$  is obtained from  $X$  by permuting its entries in a fixed pattern.

The result is then used in Section 3 to study Schur multiplicative maps which map rank- $k$  matrices to rank- $k$  matrices for a given value  $k$ . In particular, our results include the characterization of those Schur multiplicative maps that preserve the rank function, and those Schur multiplicative maps that map the set of singular (respectively, invertible) square matrices to itself. In Section 4, we study Schur multiplicative maps on square matrices which preserve functions related to eigenvalues including the determinant function and the spectrum. We also characterize maps on matrices of the form (†) that preserve some special sets and functions. These results include analogs of the theorems of Frobenius and Dieudonné on linear maps preserving the determinant functions and linear maps preserving the set of singular matrices, respectively.

In our discussion, let  $J_{m,n}$  denote the  $m \times n$  matrix with all entries equal to 1, and let  $0_{m,n}$  be the  $m \times n$  matrix with all entries equal to 0. Denote by  $\mathcal{B} = \{E_{11}, E_{12}, \dots, E_{m,n}\}$  the standard basis for  $M_{m,n}$ . When  $m = n$ , we use the notation  $M_n, J_n, 0_n$ , etc. The set of non-zero elements in  $\mathbf{F}$  is denoted by  $\mathbf{F}^*$ .

A square matrix is a *monomial matrix* if each row and each column has exactly one non-zero entry. A monomial matrix is a *permutation matrix* if all the non-zero entries equal to the unity in  $\mathbf{F}$ .

## 2. Schur multiplicative maps

The structure of a Schur multiplicative map  $f : M_{m,n} \rightarrow M_{m,n}$  can be quite arbitrary if one does not impose any additional assumptions on  $f$ . In general, one can define  $f(A) = [f_{ij}(A)]$ , where  $f_{ij} : M_{m,n} \rightarrow \mathbf{F}$  is any Schur multiplicative map. For example, one can define  $f(A) = B$  for a fixed matrix  $B$  satisfying  $B \circ B = B$ ; another example is to define  $f(A) = J_{m,n}$  if  $a_{11} \neq 0$  and  $f(A) = E_{11}$  otherwise. On the other hand, if one imposes some mild conditions on a Schur multiplicative map, then its structure will be more tractable as shown in the following theorem.

**THEOREM 2.1.** *Let  $f : M_{m,n} \rightarrow M_{m,n}$ . The following conditions are equivalent.*

- (A1) *The map  $f$  is Schur multiplicative,  $f(0_{m,n}) = 0_{m,n}$ , and  $f(E_{ij}) \neq 0_{m,n}$  for each  $(i, j)$  pair.*
- (A2) *The map  $f$  is Schur multiplicative and  $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$ .*
- (A3) *There is a mapping  $\mathcal{P} : M_{m,n} \rightarrow M_{m,n}$  such that  $\mathcal{P}(A)$  is obtained from  $A$  by permuting its entries in a fixed pattern, and a family of multiplicative maps  $f_{ij} : \mathbf{F} \rightarrow \mathbf{F}$  satisfying  $f_{ij}^{-1}[\{0\}] = \{0\}$  such that*

$$f([a_{ij}]) = \mathcal{P}([f_{ij}(a_{ij})]).$$

**PROOF.** Note that a matrix  $X \in M_{m,n}$  satisfies  $X \circ X = X$  if and only if all of the entries of  $X$  belong to  $\{0, 1\}$ .

Assume that (A1) holds. Suppose that there exists an  $X$  with non-zero  $(i, j)$  entry such that  $f(X) = 0_{m,n}$ . Then  $f(E_{ij}) = f(E_{ij} \circ X/x_{ij}) = f(E_{ij}/x_{ij}) \circ f(X) = 0_{m,n}$ , which is a contradiction. Thus,  $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$ . We see that (A2) holds.

Suppose that (A2) holds. Consider  $X \in \mathcal{B} = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ . Then  $f(X) = f(X) \circ f(X)$ . So, all entries of  $f(X)$  lie in  $\{0, 1\}$ , and  $f(X) \neq 0$  by assumption (A2). For any  $X, Y \in \mathcal{B}$  with  $X \neq Y$ , we have  $f(X) \circ f(Y) = f(0_{m,n}) = 0_{m,n}$ . Thus,  $f(X)$  and  $f(Y)$  have non-zero entries in different positions. As a result, for each  $X \in \mathcal{B}$ ,  $f(X)$  has exactly one non-zero entry. Thus,  $f(\mathcal{B}) = \mathcal{B}$ .

We can apply a map  $\mathcal{P} : M_{m,n} \rightarrow M_{m,n}$  such that  $\mathcal{P}(A)$  is obtained from  $A$  by a fixed permutation of the entries of  $A$  so that  $\mathcal{P}(f(E_{ij})) = E_{ij}$  for all  $(i, j)$ . It remains to show that there are  $f_{ij} : \mathbf{F} \rightarrow \mathbf{F}$  such that  $\mathcal{P}(f(A)) = [f_{ij}(a_{ij})]$  for any  $A = [a_{ij}]$ .

Replace  $f$  by the map  $A \mapsto \mathcal{P}^{-1}(f(A))$ , where  $\mathcal{P}^{-1}(\mathcal{P}(X)) = X$  for all matrices  $X$ . If we can prove the conclusion for the modified map, then the same conclusion will be valid for the original map. So, we assume that  $\mathcal{P}$  is the identity map, that is,  $f(E_{ij}) = E_{ij}$  for all  $(i, j)$  pairs. Now, fix an  $(i, j)$  pair. For any  $a \in \mathbf{F}$ ,  $f(aE_{ij}) = f(aE_{ij}) \circ f(E_{ij}) = bE_{ij}$  for some  $b \in \mathbf{F}$ . Define  $f_{ij} : \mathbf{F} \rightarrow \mathbf{F}$  such that  $f(aE_{ij}) = f_{ij}(a)E_{ij}$ . Since  $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$ ,  $f_{ij}(x) = 0$  if and only if  $x = 0$ . Also, for any  $a, b \in \mathbf{F}$ ,

$$f_{ij}(ab)E_{ij} = f(abE_{ij}) = f(aE_{ij}) \circ f(bE_{ij}) = f_{ij}(a)f_{ij}(b)E_{ij}.$$

Suppose that  $A = [a_{ij}]$  and  $f(A) = [b_{ij}]$ . Then

$$b_{ij}E_{ij} = E_{ij} \circ f(A) = f(E_{ij} \circ A) = f_{ij}(a_{ij})E_{ij}.$$

Thus, we see that  $f(A) = [f_{ij}(a_{ij})]$ , and the conclusion holds.

The implication (A3) implies (A1) is clear. □

**COROLLARY 2.2.** *Let  $f : M_{m,n} \rightarrow M_{m,n}$ . The following are equivalent.*

(A4) *The map  $f$  is Schur multiplicative and injective.*

(A5) *Condition (A3) in Theorem 2.1 holds with the additional assumption that  $f_{ij}$  is injective for each  $(i, j)$  pair.*

**PROOF.** Suppose that  $f$  is Schur multiplicative and injective. Since  $f(0_{m,n}) = f(0_{m,n}) \circ f(0_{m,n})$ , all entries of  $f(0_{m,n})$  lie in  $\{0, 1\}$ . Let  $S$  be the set of  $(i, j)$  pairs such that the  $(i, j)$  entry of  $f(0_{m,n})$  equals 1. Then, for any  $X \in M_{m,n}$ , we have

$$f(0_{m,n}) = f(X \circ 0_{m,n}) = f(X) \circ f(0_{m,n}).$$

Hence the  $(i, j)$  entry of  $f(X)$  equals 1 for each  $(i, j) \in S$ .

For  $(i, j) \neq (p, q)$ , we have  $f(E_{ij}) \neq f(E_{pq})$  and  $f(E_{ij} \circ E_{pq}) = f(0_{m,n})$ . Thus,  $f(E_{ij})$  and  $f(E_{pq})$  cannot have a common non-zero entry at the  $(r, s)$  position if  $(r, s) \notin S$ . Because  $f$  is injective,  $f(E_{ij}) \neq f(0_{m,n})$ . Thus, every  $f(E_{ij})$  has at

least one non-zero entry at a position  $(r, s) \notin S$ . Since  $E_{ij}$  and  $E_{p,q}$  cannot have a common non-zero entry at any  $(r, s)$  position with  $(r, s) \notin S$ , we need at least  $mn$  pairs of  $(r, s) \notin S$  to accommodate the non-zero entries of  $f(E_{ij})$ . Hence, we conclude that  $S = \emptyset$ , that is,  $f(0_{m,n}) = 0_{m,n}$ , and each  $f(E_{ij})$  has exactly one non-zero entry equal to 1. So, condition (A1) of Theorem 2.1 holds and  $f$  has the form described in (A3). Since  $f$  is injective, for  $x \neq y$  in  $\mathbf{F}$  we have seen that  $f(xE_{ij}) \neq f(yE_{ij})$  and hence  $f_{ij}(x) \neq f_{ij}(y)$ . So,  $f_{ij}$  is injective for each  $(i, j)$  pair.

The implication (A5) implies (A4) is clear.  $\square$

**REMARK 2.3.** We shall see in the subsequent discussion that, in the study of preserver problems, we can sometimes assume only that:

(A0)  $f : M_{m,n} \rightarrow M_{m,n}$  is Schur multiplicative and  $f(0_{m,n}) = 0_{m,n}$ ,

together with some preserving property to conclude that  $f$  has the form  $(\dagger)$  with some additional nice structure. In some problems, we believe that one can even remove the assumption that  $f(0_{m,n}) = 0_{m,n}$  in (A0). On the other hand, we shall see that the assumption (A1) or (A0) are indispensable in certain problems.

Note also that our result and proof are valid if  $\mathbf{F}$  is replaced by an integral domain  $\mathbf{D}$ .

### 3. Rank preservers

Linear maps, additive maps, and multiplicative maps on matrices mapping the set of rank- $k$  matrices to itself have been studied by many researchers; for example, see [2–4, 15] and the references therein. In this section, we characterize Schur multiplicative maps that map the set of rank- $k$  matrices to itself. We begin with rank-one preservers.

**THEOREM 3.1.** *Suppose that  $f : M_{m,n} \rightarrow M_{m,n}$  is a Schur multiplicative map. Then  $f(0_{m,n}) = 0_{m,n}$  and  $f$  maps rank-one matrices to rank-one matrices if and only if there exist permutation matrices  $P \in M_m$  and  $Q \in M_n$ , and a multiplicative map  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  satisfying  $\tau(\mathbf{F}^*) \subseteq \mathbf{F}^*$  such that:*

- (a)  $f$  has the form  $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$ ; or
- (b)  $m = n$  and  $f$  has the form  $[a_{ij}] \mapsto P[\tau(a_{ij})]^t Q$ .

**PROOF.** First we consider the ‘only if’ implication. Note that  $A \in M_{m,n}$  has rank one if and only if there are  $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbf{F}$  such that  $A = [a_{ij}] = [x_1, \dots, x_m]^t [y_1, \dots, y_n]$ . Thus, for any injective multiplicative map  $\tau : \mathbf{F} \rightarrow \mathbf{F}$ , we have

$$[\tau(a_{ij})] = [\tau(x_i)\tau(y_j)] = [\tau(x_1), \dots, \tau(x_m)]^t [\tau(y_1), \dots, \tau(y_n)]$$

with rank one. By this observation, the ‘only if’ implication is clear.

Next, we consider the converse. By the given assumption,  $f$  satisfies condition (A1) in Theorem 2.1 and hence its conclusion. Thus,  $f$  has the form  $(\dagger)$ . Without loss

of generality, we may assume that  $m \leq n$ . The case  $n < m$  can be proved by similar arguments. Since  $f(X)$  has rank one for  $X = \sum_{j=1}^n E_{1j}$ , we see that the non-zero entries of  $f(X)$  lie in the same row, or in the same column if  $m = n$ . We may assume that the former case holds. Otherwise, replace  $f$  by a map of the form  $A \mapsto f(A)^t$ . Note that, if we can prove the result for the modified map, the conclusion will be valid for the original map. Then there exist permutation matrices  $P \in M_m$  and  $Q \in M_n$  so that  $f(E_{1j}) = PE_{1j}Q$  for  $j = 1, \dots, n$ . Replace  $f$  by the map  $A \mapsto P^t f(A)Q^t$  so that we have  $f(E_{1j}) = E_{1j}$  for  $j = 1, \dots, n$ . Now, consider  $f(X)$  for  $X = \sum_{i=1}^m E_{i1}$ . Since  $f(E_{11}) = E_{11}$  and  $f$  maps rank-one matrices to rank-one matrices, we see that  $f(X) = X$ . There exists a permutation matrix  $R \in M_m$  such that  $f(E_{i1}) = RE_{i1}$  for  $i = 1, \dots, m$ . We may replace  $f$  by the map  $A \mapsto R^t f(A)$ , and assume that  $f(E_{i1}) = E_{i1}$  for  $i = 1, \dots, m$ . For any  $(i, j)$  with  $i \neq 1$  and  $j \neq 1$ , since  $f(X)$  has rank one for  $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$ , we see that  $f(E_{ij}) = E_{ij}$ . Furthermore, for any  $1 \leq j \leq n$  and  $a \in \mathbf{F}$  the matrix  $f(X)$  has rank one for  $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$ , so we see that  $f_{11}(a) = f_{1j}(a)$ . Similarly, we can show that  $f_{i1}(a) = f_{i1}(a)$  for all  $a \in \mathbf{F}$ . Finally, for any  $(i, j)$  with  $i \neq 1$  and  $j \neq 1$ , since  $f(X)$  has rank one for  $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$  with  $a \in \mathbf{F}$ ,  $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$  for all  $a \in \mathbf{F}$ . Our conclusion follows.  $\square$

The conclusion of Theorem 3.1 may fail if the Schur multiplicative map does not map  $0_{m,n}$  to itself. For example, we can choose a fixed rank-one matrix  $B$  satisfying  $B \circ B = B$  and define  $f(A) = B$  for all  $A \in M_{m,n}$ . Then  $f$  is Schur multiplicative and maps rank-one matrices to rank-one matrices.

Next, we show that one can get a similar conclusion for maps on matrices of the form  $(\dagger)$  even though  $f_{ij}$  is not assumed to be multiplicative *a priori*.

**THEOREM 3.2.** *Suppose that  $f : M_{m,n} \rightarrow M_{m,n}$  has the form  $(\dagger)$ . Then  $f$  maps rank-one matrices to rank-one matrices if and only if there exist invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$  and a multiplicative map  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  satisfying  $\tau(\mathbf{F}^*) \subseteq \mathbf{F}^*$  such that:*

- (a)  $f$  has the form  $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$ ; or
- (b)  $m = n$  and  $f$  has the form  $[a_{ij}] \mapsto P[\tau(a_{ij})]^t Q$ .

**PROOF.** The ‘only if’ implication can be proved as in the proof of Theorem 3.1.

We consider the converse. Assume that  $f$  has the form  $(\dagger)$  and maps rank-one matrices to rank-one matrices. Without loss of generality, we may assume that  $m \leq n$ . Since  $f(X)$  has rank one for  $X = \sum_{j=1}^n E_{1j}$ , we see that the non-zero entries of  $f(X)$  lie in the same row, or in the same column if  $m = n$ . We may assume that the former case holds. Otherwise, replace  $f$  by a map of the form  $A \mapsto f(A)^t$ . Then there exist permutation matrices  $P \in M_m$  and  $Q \in M_n$  so that  $f(E_{1j}) = Pf_{1j}(1)E_{1j}Q$  for  $j = 1, \dots, n$ . Let  $D = \text{diag}(f_{11}(1), f_{12}(1), \dots, f_{1n}(1))$ . Since  $f(E_{1j})$  has rank one, we see that  $f_{1j}(1) \neq 0$  for  $j = 1, \dots, n$ . Replace  $f$  by the map  $A \mapsto P^{-1}f(A)Q^{-1}D^{-1}$  so that we have  $f(E_{1j}) = E_{1j}$  for  $j = 1, \dots, n$ . Now, consider  $f(X)$  for  $X = \sum_{i=1}^m E_{i1}$ . Since  $f(E_{11}) = E_{11}$  and  $f$  maps rank-one matrices to

rank-one matrices, there exists an invertible monomial matrix  $R \in M_m$  such that  $f(E_{i1}) = RE_{i1}$  for  $i = 1, \dots, m$ . We may replace  $f$  by the map  $A \mapsto R^{-1}f(A)$ , and assume that  $f(E_{i1}) = E_{i1}$  for  $i = 1, \dots, m$ . For any  $(i, j)$  with  $i \neq 1$  and  $j \neq 1$ , since  $f(X)$  has rank one for  $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$ , we see that  $f(E_{ij}) = E_{ij}$ .

Note that for any  $(i, j)$  pair and any non-zero  $a \in \mathbf{F}$ ,  $f(aE_{ij}) = f_{ij}(a)E_{ij}$  has rank one, and thus  $f_{ij}(a) \neq 0$ . Furthermore, for any  $1 \leq j \leq n$  and any  $a \in \mathbf{F}$  the matrix  $f(X)$  has rank one for  $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$ , we see that  $f_{11}(a) = f_{1j}(a)$ . Similarly, we can show that  $f_{i1}(a) = f_{i1}(a)$  for all  $a \in \mathbf{F}$ . Finally, for any  $(i, j)$  with  $i \neq 1$  and  $j \neq 1$ , since  $f(X)$  has rank one for  $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$  with  $a \in \mathbf{F}$ ,  $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$  for all  $a \in \mathbf{F}$ .

Let  $f_{11} = \tau$ . For any  $a, b \in \mathbf{F}$ , let  $X = E_{11} + aE_{12} + bE_{21} + abE_{22}$ . Since  $f(X)$  has rank one, we see that  $\tau(ab) = \tau(a)\tau(b)$ . So,  $\tau$  is multiplicative.  $\square$

Next, we characterize maps  $f : M_{m,n} \rightarrow M_{m,n}$  of the form  $(\dagger)$  which map the set of rank- $k$  matrices to itself for  $1 < k < \min\{m, n\}$ . It turns out that such maps will preserve the ranks of all matrices, and have very nice structure. The result will be used to characterize Schur multiplicative maps which preserve rank- $k$  matrices in Corollary 3.4.

**THEOREM 3.3.** *Let  $1 < k < \min\{m, n\}$ . Suppose that  $f : M_{m,n} \rightarrow M_{m,n}$  has the form  $(\dagger)$ . The following are equivalent.*

- (a) We have  $\text{rank}(f(A)) = \text{rank}(A)$  for all  $A \in M_{m,n}$ .
- (b) The map  $f$  maps rank- $k$  matrices to rank- $k$  matrices.
- (c) There are invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$ , and a field monomorphism  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  such that one of the following holds:
  - (c.i)  $f$  has the form  $A \mapsto P[\tau(a_{ij})]Q$ ;
  - (c.ii)  $m = n$  and  $f$  has the form  $A \mapsto P[\tau(a_{ij})]^t Q$ .

Note that for rank preservers  $f$ , we have  $f(0_{m,n}) = 0_{m,n}$ . Thus, one may further relax the assumption that  $f_{ij}(0) = 0$  for all  $(i, j)$  pairs in  $(\dagger)$ , and conclude that conditions (b) and (c) are equivalent.

**PROOF.** The implications (c) implies (a) implies (b) are clear. We focus on the proof of (b) implies (c). Without loss of generality, we may assume that  $m \leq n$ . The proof for the case  $n < m$  is similar. We divide the proof into several assertions.

**ASSERTION 1.** There is a diagonal matrix  $D \in M_m$  and permutation matrices  $P \in M_m$  and  $Q \in M_n$  such that  $f(E_{jj}) = PDE_{jj}Q$  for  $j = 1, \dots, m$ .

Consider  $\mathcal{D} = \{E_{jj} : 1 \leq j \leq m\}$ . If  $X$  is a sum of  $k$  matrices in  $\mathcal{D}$ , then  $k = \text{rank}(X) = \text{rank}(f(X))$ . So,  $f(X)$  must have  $k$  non-zero entries lying on  $k$  distinct rows and  $k$  distinct columns. Thus, the  $m$  non-zero entries of  $f(\sum_{j=1}^m E_{jj})$  lie on  $m$  different rows and  $m$  different columns. Hence, there are permutation matrices  $P \in M_m$  and  $Q \in M_n$  such that  $f(E_{jj}) = Pf_{jj}(1)E_{jj}Q$  for  $j = 1, \dots, m$ . Let  $D = \text{diag}(f_{11}(1), \dots, f_{mm}(1))$ . Then we get the desired conclusion.

By Assertion 1, we may replace  $f$  by the map  $A \mapsto D^{-1}P^t f(A)Q^t$  and assume that  $f(E_{jj}) = E_{jj}$  for  $j = 1, \dots, m$ . We shall make this assumption in the rest of the proof.

**ASSERTION 2.** For any  $(i, j)$  pair,  $f_{ij}(\mathbf{F}^*) \subseteq \mathbf{F}^*$ .

Let  $a \in \mathbf{F}^*$ , and let  $X = aE_{ij} + \sum_{s \in S} E_{ss}$  for a subset  $S$  of  $\{1, \dots, m\} \setminus \{i, j\}$  with  $k - 1$  elements. Since  $f(X)$  has rank  $k$ , we see that  $f_{ij}(a) \neq 0$ .

**ASSERTION 3.** For any  $1 \leq i < j \leq m$ , we have  $f(E_{ij} + E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ji}$  for some  $b_{ij} \in \mathbf{F}^*$ .

For simplicity, assume that  $(i, j) = (1, 2)$ , and  $X = E_{12} + E_{21}$ . If  $Y = X + \sum_{j=3}^k E_{jj}$ , then  $f(Y)$  has rank  $k$ . So,  $f(X) = f_{12}(1)E_{pq} + f_{21}(1)E_{rs}$  for some  $p \neq q$  and  $r \neq s$ . If  $Y = \sum_{j=1}^{k+1} E_{jj} + X$ , then  $f(Y)$  has rank  $k$ . Thus,  $p, q, r, s, \in \{1, \dots, k + 1\}$ ; otherwise, the leading  $(k + 1) \times (k + 1)$  matrix of  $f(Y)$  will be invertible so that  $f(Y)$  has rank larger than  $k$ . Furthermore, we must have  $(p, q) = (s, r)$  and  $f(X) = bE_{pq} + b^{-1}E_{qp}$  for some  $b \in \mathbf{F}$  with  $1 \leq p < q \leq k + 1$ ; otherwise,  $f(Y)$  has rank larger than  $k$ . Now, for any  $s \in \{3, \dots, k + 1\}$ , we have  $k = \text{rank}(Z) = \text{rank}(f(Z))$  for any  $Z \in \{Y - E_{ss} - E_{11}, Y - E_{ss} - E_{22}\}$ . It follows that  $p, q \notin \{3, \dots, k + 1\}$ , that is,  $\{p, q\} = \{1, 2\}$ . So,  $f(X) = bE_{12} + b^{-1}E_{21}$  as asserted.

**ASSERTION 4.** There is an invertible diagonal matrix  $D \in M_m$  such that one of the following holds:

- (i)  $f(E_{ij}) = D^{-1}E_{ij}(D \oplus I_{n-m})$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq m$ ;
- (ii)  $f(E_{ij}) = D^{-1}E_{ji}(D \oplus I_{n-m})$  for all  $1 \leq i, j \leq m$ .

By Assertion 3,  $f(E_{ij} + E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ji}$  for all  $(i, j)$  pairs with  $1 \leq i, j \leq m$ . Let  $D^{-1} = \text{diag}(1, b_{21}, b_{31}, \dots, b_{m1})$ . Then  $f(X) = D^{-1}X(D \oplus I_{n-m})$  for  $X = E_{1j} + E_{j1}$  for  $j = 2, \dots, m$ . Replace  $f$  by the map  $A \mapsto Df(A)(D^{-1} \oplus I_{n-m})$ . Then:

- (i)'  $f(E_{12}) = E_{12}$ ; or
- (ii)'  $f(E_{12}) = E_{21}$ .

Assume that (i)' holds. We prove that (i) holds accordingly as follows. First, consider  $f(E_{1j})$  for  $j = 3, \dots, m$ . Consider

$$A = E_{11} + E_{12} + E_{1j} + E_{22} + E_{2j} + E_{j2} + E_{jj} + \sum_{s \in S} E_{ss},$$

where  $S$  is a subset of  $\{3, \dots, n\} \setminus \{j\}$  with  $k - 2$  elements. Then  $k = \text{rank}(A) = \text{rank}(f(A))$ . If  $f(E_{1j}) = E_{j1}$ , then  $f(A)$  have rank  $k + 1$ , which is a contradiction. Thus,  $f(X) = X$  for  $X \in \{E_{1j}, E_{j1}\}$ .

Now, suppose that  $1 \notin \{i, j\}$ . Let

$$A = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij} + E_{jj} + \sum_{s \in S} E_{ss},$$

where  $S$  is a subset of  $\{2, \dots, n\} \setminus \{i, j\}$  with  $k - 2$  elements. Then  $k = \text{rank}(A)$ . If  $f(E_{ij}) = b_{ij}^{-1} E_{ji}$ , then  $\text{rank}(f(A)) = k + 1$ , which is a contradiction. So,  $f(E_{ij}) = b_{ij} E_{ij}$ . If  $b_{ij} \neq 1$ , consider

$$B = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij} + E_{j1} + \sum_{s \in S} E_{ss},$$

where  $S$  is a subset of  $\{2, \dots, n\} \setminus \{i, j\}$  with  $k - 2$  elements. Then  $\text{rank}(B) = k < k + 1 = \text{rank}(f(B))$ , which is a contradiction. So, we conclude that  $f(X) = X$  for  $X \in \{E_{ij}, E_{ji}\}$ . Our proof of (i) is complete.

If condition (ii)' holds, we can prove (ii) by a similar argument.

**ASSERTION 5.** Suppose that  $m < n$ . Then condition (ii) in Assertion 4 cannot hold, and there is an invertible monomial matrix  $Q \in M_n$  such that  $f(E_{ij}) = E_{ij}Q$  for any  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .

To prove the above assertion, note that if  $r > m$  then  $f(E_{1r}) = E_{pq}$  for some  $q > m$  because  $f(E_{ij}) = E_{ij}$  for  $1 \leq i, j \leq m$ . If  $p \neq 1$ , then for  $A = E_{1r} + E_{11} + \sum_{s \in S} E_{ss}$ , where  $S$  is a subset of  $\{2, \dots, n\} \setminus \{p\}$  with  $k - 1$  elements, we see that  $f(A)$  has  $k + 1$  linear independent rows and thus  $\text{rank}(f(A)) = k + 1$  and  $\text{rank}(A) = k$ , which is a contradiction. So, there are  $b_{1j} \in \mathbf{F}^*$  for  $j = m + 1, \dots, n$  such that

$$\{f(E_{1r}) : m < r \leq n\} = \{b_{1r} E_{1r} : m < r \leq n\}.$$

We may assume that  $f(E_{1r}) = E_{1r}$  for all  $m < r \leq n$ . Otherwise, replace  $f$  by a map of the form  $A \mapsto f(A)Q$ , where  $Q \in M_n$  is a monomial matrix of the form  $I_m \oplus \tilde{Q}$  with  $\tilde{Q} \in M_{n-m}$  is an invertible monomial matrix.

To see that condition (ii) cannot hold, consider

$$A = E_{1,m+1} + E_{12} + E_{23} + \dots + E_{k,k+1}.$$

Then there is  $b \in \mathbf{F}^*$  such that

$$f(A) = bE_{1,m+1} + E_{21} + E_{32} + \dots + E_{k+1,k}$$

has rank  $k + 1$  while  $\text{rank}(A) = k$ , which is a contradiction. So, at this point, we have  $f(X) = X$  for  $X = E_{ij}$  for  $1 \leq i, j \leq m$  and  $X \in \{E_{1r} : m < r \leq n\}$ .

Now, for  $E_{ij}$  with  $i > 1$  and  $j > m$ , consider

$$A = E_{1i} + E_{1j} + E_{i1} + E_{ij} + \sum_{s \in S} E_{ss},$$

where  $S$  is a subset of  $\{2, \dots, n\} \setminus \{i\}$  with  $k - 2$  elements. Since  $k = \text{rank}(A) = \text{rank}(f(A))$ , we conclude that  $f(E_{ij}) = E_{ij}$ .

By the above discussion, we may further replace  $f$  by a map of the form  $A \mapsto Pf(A)Q$  for some suitable invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$  so that the resulting map satisfies:

- (1)  $f(E_{ij}) = E_{ij}$  for all  $(i, j)$  pairs; or
- (2)  $m = n$  and  $f(E_{ij}) = E_{ji}$  for all  $(i, j)$  pairs.

**ASSERTION 6.** There is a field monomorphism  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  such that  $f_{ij} = \tau$  for every  $(i, j)$  pair.

First, for any  $a \in \mathbf{F}$ , consider  $A = aE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{s \in S} E_{ss}$ , where  $S$  is a subset of  $\{3, \dots, m\}$  with  $k - 1$  elements. Since  $k = \text{rank}(A) = \text{rank}(f(A))$ , we have  $f_{11}(a) = f_{1j}(a)$ . Hence  $f_{1j} = f_{11}$  for  $j = 2, \dots, n$ . Similarly, we can show that  $f_{ii} = f_{ij}$  for any  $j \in \{1, \dots, n\} \setminus \{i\}$ .

Next, for any  $a \in \mathbf{F}$  consider  $A = aE_{11} + aE_{j1} + E_{21} + E_{j2} + \sum_{s \in S} E_{ss}$ , where  $S$  is a subset of  $\{3, \dots, m\}$  with  $k - 1$  elements. Since  $k = \text{rank}(A) = \text{rank}(f(A))$ , we have  $f_{11}(a) = f_{j1}(a)$ . Hence  $f_{j1} = f_{11}$  for  $j = 2, \dots, m$ . Similarly, we can show that  $f_{ii} = f_{ri}$  for any  $r \in \{1, \dots, m\} \setminus \{i\}$ .

By the arguments in the above two paragraphs, we conclude that there is  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  such that  $f_{ij} = \tau$  for all  $(i, j)$  pairs.

Suppose that  $\tau(a) = \tau(b)$  for some  $a \neq b$  in  $\mathbf{F}$ . Let

$$A = E_{11} + aE_{12} + E_{21} + bE_{22} + \sum_{j=3}^{k+1} E_{jj}.$$

Then  $\text{rank}(A) = k > k - 1 = \text{rank}(f(A))$ , which is a contradiction. So,  $\tau$  is injective. Now, let

$$A = E_{11} + aE_{12} + bE_{21} + abE_{22} + \sum_{j=3}^{k+1} E_{jj}.$$

Then  $k = \text{rank}(A) = \text{rank}(f(A))$  implies that  $\tau(ab) = \tau(a)\tau(b)$  for all  $a, b \in \mathbf{F}$ .

Finally, let

$$A = E_{11} + aE_{12} + (a + b)E_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s \in S} E_{ss},$$

for some subset  $S$  of  $\{4, \dots, n\}$  with  $k - 2$  elements. Then  $k = \text{rank}(A) = \text{rank}(f(A))$ . Since

$$f(A) = E_{11} + \tau(a)E_{12} + \tau(a + b)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s \in S} E_{ss},$$

this implies  $\tau(a + b) - \tau(b) = \tau(a)$ , or equivalently  $\tau(a + b) = \tau(a) + \tau(b)$ . Thus,  $\tau$  is also additive, and the result follows. □

**COROLLARY 3.4.** Let  $2 < k < \min\{m, n\}$  and  $f : M_{m,n} \rightarrow M_{m,n}$ .

- (1) If  $f$  is Schur multiplicative, then (b) and (c) in Theorem 3.3 are equivalent with the additional requirement in condition (c) that  $P$  and  $Q$  are permutation matrices.
- (2) If  $f$  is Schur multiplicative and has the form  $(\dagger)$  (or satisfies any of the conditions (A1)–(A3) in Theorem 2.1), then conditions (a)–(c) in Theorem 3.3 are equivalent with the additional requirement in condition (c) that  $P$  and  $Q$  are permutation matrices.

**PROOF.** Suppose that  $f$  is Schur multiplicative. Clearly, (c) implies (b) implies (a).

If (b) holds, then condition (A1) in Theorem 2.1 holds, and hence  $f$  has the form ( $\dagger$ ). We can then apply Theorem 3.3 to obtain condition (c) for some invertible monomial matrices  $P$  and  $Q$ . Now, if  $X \circ X = X$ , that is,  $X$  has entries in  $\{0, 1\}$ , then so is  $f(X)$ . Thus, we see that  $P$  and  $Q$  can be chosen to be permutation matrices in condition (c).

If  $f$  is Schur multiplicative and has the form ( $\dagger$ ), we can apply Theorem 3.3 and the argument in the last paragraph to obtain the conclusion.  $\square$

The conclusion in Corollary 3.4(2) is not valid if we just assume that  $f$  is Schur multiplicative and  $f(0_{m,n}) = 0_{m,n}$ . For instance, one can define  $f$  by  $f(0_{m,n}) = 0_{m,n}$  and  $f(A) = B$  for all other  $A$ , where  $B \in M_{m,n}$  is any rank  $k$  satisfying  $B \circ B = B$ . Then  $f$  maps all rank- $k$  matrices to a rank- $k$  matrix, but  $f$  does not have the structure described in Theorem 3.3(c).

One can examine the proof and see that condition (b) in Theorem 3.3 (and also Corollary 3.4) can be replaced by any one of the following conditions:

- (b.i)  $f(A)$  has rank at most  $k$  whenever  $A \in M_{m,n}$  has rank  $k$ ;
- (b.ii)  $f(A)$  has rank at most  $k$  whenever  $A \in M_{m,n}$  has rank at most  $k$ .

In particular, the conclusion holds for those functions which map singular matrices to singular matrices when  $m = n$ . This can be viewed as an analog of the linear preserver result of Dieudonné [5].

Next, we consider preservers of full rank matrices.

**THEOREM 3.5.** *Suppose that  $2 \leq m \leq n$  and  $f : M_{m,n} \rightarrow M_{m,n}$  has the form ( $\dagger$ ). If  $f$  maps rank- $m$  matrices to rank- $m$  matrices, then there exist invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$ , and maps  $f_{ij} : \mathbf{F} \rightarrow \mathbf{F}$  such that  $f_{ij}(\mathbf{F}^*) \subseteq \mathbf{F}^*$  for all  $(i, j)$  pairs and one of the following holds:*

- (a)  $f$  has the form  $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$ ;
- (b)  $m = n$  and  $f$  has the form  $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^t Q$ .

*If one of the  $f_{ij}$  is surjective, then there is an injective multiplicative map  $\tau : \mathbf{F} \rightarrow \mathbf{F}$  such that  $f_{ij} = \tau$  for all  $(i, j)$  pairs; furthermore, if  $m \geq 3$ , then  $\tau$  is a field automorphism.*

**PROOF.** We divide the proof into several assertions.

**ASSERTION 1.** There are invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$  such that  $f(X) = PXQ$  for  $X \in \{E_{jj} : 1 \leq j \leq m\}$ .

To prove the assertion, let  $X = \sum_{j=1}^m E_{jj}$ . Since  $\text{rank}(f(X)) = \text{rank}(X) = m$ , we see that  $f(X)$  has non-zero entries on  $m$  distinct rows and columns. So, there are permutation matrices  $P \in M_m$  and  $Q \in M_n$  such that  $f(E_{jj}) = Pf_{jj}(1)E_{jj}Q$  for  $j = 1, \dots, m$ . We may replace  $f$  by the map  $A \mapsto P^{-1}f(A)Q^{-1}$  for suitable invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$  so that  $f(E_{jj}) = E_{jj}$  for  $j = 1, \dots, m$ .

**ASSERTION 2.** Assume  $m < n$ . There are invertible monomial matrices  $P \in M_m$  and  $Q \in M_n$  such that  $f(X) = PXQ$  for  $X \in \{E_{jj} : 1 \leq j \leq m\} \cup \{E_{1j} : m < j \leq n\}$ . Moreover,  $f(E_{ij}) = f_{ij}(1)PE_{ij}Q$ .

By Assertion 1, we may assume that  $f(E_{jj}) = E_{jj}$  for  $j = 1, \dots, m$ . For any  $r > m$ , consider  $X = E_{1r} + \sum_{s=2}^m E_{ss}$ . Assume that  $f(E_{1r}) = f_{1r}(1)E_{pq}$ . Since  $\text{rank}(f(X)) = \text{rank}(X) = m$  and  $f(E_{jj}) = E_{jj}$ , it is impossible to have  $p > 1$  or  $q \leq m$ . It follows that  $f(E_{1r}) = f_{1r}(1)E_{1q}$  for some  $q > m$ . Thus, we may further modify  $f$  by a map of the form  $A \mapsto f(A)(I_m \oplus Q)$  for a suitable invertible monomial matrix  $Q \in M_{n-m}$  so that

$$f(X) = X \quad \text{for } X \in \{E_{jj} : 1 \leq j \leq m\} \cup \{E_{1j} : m < j \leq n\}. \tag{3.1}$$

If  $n = m + 1$ , consider  $X = E_{i,m+1} + \sum_{j \neq i} E_{jj}$ . Since  $\text{rank}(f(X)) = \text{rank}(X) = m$ , we see that  $f(E_{i,m+1}) = f_{i,m+1}(1)E_{i,m+1}$  for all  $i > 1$ . Now, suppose that  $n > m + 1$ . Consider  $f(E_{2j})$  for  $j > m$ . Let  $X = E_{1r} + E_{2j} + \sum_{s=3}^m E_{ss}$  with  $r > m$  and  $r \neq j$ . Assume that  $f(E_{2j}) = f_{2j}(1)E_{pq}$ . Since  $\text{rank}(f(X)) = \text{rank}(X) = m$  and condition (3.1) holds, we see that  $p = 2, q \neq r$ , and  $q > m$ . Because the argument holds for all  $r > m$  with  $r \neq j$ , we conclude that  $f(E_{2j}) = f_{2j}(1)E_{2j}$ . Using the same argument, we can prove that  $f(E_{ij}) = f_{ij}(1)E_{ij}$  for all  $i > 1$  and  $j > m$  as asserted.

Next, we turn to  $f(E_{ij})$  for  $1 < i \leq m, 1 \leq j \leq m$  with  $i \neq j$ . The result is clear if  $m = 2$ . Assume  $m \geq 3$  and  $f(E_{ij}) = f_{ij}(1)E_{pq}$ . Let

$$X = E_{ij} + E_{j,m+1} + \sum_{s \in S} E_{ss},$$

where  $S = \{1, 2, \dots, m\} \setminus \{i, j\}$ . By the conclusion above  $f(E_{j,m+1}) = f_{j,m+1}(1)E_{j,m+1}$ . Since  $m = \text{rank}(f(X)) = \text{rank}(X)$  and (3.1) holds, we see that  $(p, q) = (i, j)$ .

**ASSERTION 3.** Assume that  $m = n$ . Then there are invertible monomial matrices  $P, Q \in M_n$  such that:

- (i)  $f(E_{ij}) = Pf_{ij}(1)E_{ij}Q$  for all  $(i, j)$  pairs; or
- (ii)  $f(E_{ij}) = Pf_{ij}(1)E_{ji}Q$  for all  $(i, j)$  pairs.

By Assertion 1, we may assume that  $f(E_{jj}) = E_{jj}$  for all  $j$ . Consider  $X = E_{ij} + E_{ji} + \sum_{s \notin \{i,j\}} E_{ss}$ . Since  $m = \text{rank}(X) = \text{rank}(f(X))$ , we see that either:

- (i)'  $f(E_{ij}) = f_{ij}(1)E_{ij}$  and  $f(E_{ji}) = f_{ji}(1)E_{ji}$ ; or
- (ii)'  $f(E_{ij}) = f_{ij}(1)E_{ji}$  and  $f(E_{ji}) = f_{ji}(1)E_{ij}$ .

Assume  $f(E_{12}) = f_{12}(1)E_{12}$ ; otherwise replace  $f$  by the map  $A \mapsto f(A)^t$ . We shall prove that conclusion (i) holds. To this end, let

$$X = E_{12} + E_{2j} + E_{j1} + \sum_{s \in S} E_{ss},$$

where  $S = \{3, \dots, m\} \setminus \{j\}$ . Then  $m = \text{rank}(A) = \text{rank}(f(A))$ . If  $f(E_{j1}) = f_{j1}(1)E_{1j}$ , then  $f(A)$  will only have  $m - 1$  non-zero columns so  $\text{rank}(f(A)) < m$ , which is a contradiction. Thus, condition (i) holds for  $(i, j)$  pairs with  $i = 1$  or  $j = 1$ . Now, for  $X = E_{ij}$  with  $1 \notin \{i, j\}$  and  $i \neq j$ , consider

$$X = E_{1i} + E_{ij} + E_{j1} + \sum_{s \in S} E_{ss},$$

where  $S = \{2, \dots, m\} \setminus \{i, j\}$  with  $m - 3$  elements. Since  $m = \text{rank}(A) = \text{rank}(f(A))$ , we see that condition (i) holds.

By Assertions 2 and 3, we get the first conclusion of the theorem, namely,  $f$  has the form  $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$  or  $m = n$  and  $f$  has the form  $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^t Q$ . We finish the proof by establishing the following assertion.

**ASSERTION 4.** Suppose that there is a  $(p, q)$  such that  $f_{pq}$  is surjective. Then  $f_{ij} = f_{pq}$  for each  $(i, j)$  pair, and  $f_{pq}$  is injective multiplicative. Furthermore, if  $m \geq 3$  then  $f_{pq}$  is a field isomorphism.

Assume condition (a) holds. (If (b) holds, replace  $f$  by the map  $A \mapsto f(A)^t$  and apply a similar argument.) We may further assume that  $P = I_m$  and  $Q = I_n$  in condition (a); otherwise, replace  $f$  by the map  $A \mapsto P^{-1}f(A)Q^{-1}$ . Moreover, we assume that  $(p, q) = (1, 1)$ , that is,  $f_{11}$  is a surjective map. Otherwise, we may find a pair of permutation matrices  $R \in M_m$  and  $S \in M_n$  such that  $RE_{11}S = E_{pq}$ , and replace the map  $f$  by the map  $A \mapsto R^t f(RAS)S^t$ . Furthermore, we may replace  $f$  by the map  $A \mapsto f(A)/f_{22}(1)$  and assume that  $f_{22}(1) = 1$ . Let

$$D_1 = \text{diag}(f_{12}(1), 1, f_{32}(1), \dots, f_{m2}(1))$$

and

$$D_2 = \text{diag}(f_{21}(1), 1, f_{23}(1), \dots, f_{2n}(1)).$$

We may replace  $f$  by the map  $A \mapsto D_1^{-1}f(A)D_2^{-1}$  and assume that

$$f(X) = X \quad \text{for } X \in \{E_{i2} : 1 \leq i \leq m\} \cup \{E_{2j} : 1 \leq j \leq n\}.$$

We claim that  $f_{i1} = f_{11}$  for all  $i > 1$ . To see this, let  $a \in \mathbf{F}$  and let

$$\{s_3, \dots, s_m\} = \{1, \dots, m\} \setminus \{1, i\}.$$

If  $b \neq a$ , then  $Y = bE_{11} + aE_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m E_{s_k,k}$  has rank  $m$  and so has

$$f(Y) = f_{11}(b)E_{11} + f_{i1}(a)E_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m f_{s_k,k}(1)E_{s_k,k}.$$

It follows that  $f_{11}(b) \neq f_{i1}(a)$  whenever  $b \neq a$ . Since  $f_{11}$  is surjective,  $f_{i1}(a)$  is in the range of  $f_{11}$ . Thus,  $f_{i1}(a) = f_{11}(a)$ .

Next, we show that  $f_{1j} = f_{11}$  for all  $j > 1$ . To see this, let  $a \in \mathbf{F}$  and let  $\{s_3, \dots, s_m\}$  be an  $m - 2$  element subset of  $\{1, \dots, n\} \setminus \{1, j\}$ . If  $b \neq a$ , then

$$Y = bE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{j=3}^m E_{j,s_j}$$

has rank  $m$  and so has

$$f(Y) = f_{11}(b)E_{11} + f_{1j}(a)E_{1j} + E_{21} + E_{2j} + \sum_{k=3}^m f_{k,s_k}(1)E_{k,s_k}.$$

It follows that  $f_{11}(b) \neq f_{1j}(a)$  whenever  $b \neq a$ . Since  $f_{11}$  is surjective,  $f_{1j}(a)$  is in the range of  $f_{11}$ . Thus,  $f_{11}(a) = f_{1j}(a)$ .

Now, consider  $f_{ij}$  with  $i, j > 1$ . Let  $a \in \mathbf{F}$ ,  $\{r_3, \dots, r_m\} = \{1, \dots, m\} \setminus \{1, i\}$ , and  $\{s_3, \dots, s_m\}$  be an  $m - 2$  element subset of  $\{1, \dots, n\} \setminus \{1, j\}$ . If  $b \neq a$ , then  $Z = bE_{11} + bE_{i1} + bE_{1j} + aE_{ij} + \sum_{k=3}^m E_{r_k,s_k}$  has rank  $m$  and so has

$$f(Z) = f_{11}(b)E_{11} + f_{11}(b)E_{i1} + f_{11}(b)E_{1j} + f_{ij}(a)E_{ij} + \sum_{k=3}^m f_{r_k,s_k}(1)E_{r_k,s_k}.$$

It follows that

$$f_{11}(b) \neq f_{ij}(a) \quad \text{whenever } b \neq a. \tag{3.2}$$

Since  $f_{11}$  is surjective,  $f_{ij}(a)$  is in the range of  $f_{11}$ . Thus,  $f_{11}(a) = f_{ij}(a)$ .

At this point, we may assume that  $f_{ij} = f_{11} = \tau$  for all  $(i, j)$  pairs, with  $\tau(0) = 0$  and  $\tau(1) = f_{11}(1) = 1$ .

Now, we show that  $\tau$  is multiplicative. Let  $a, b \in \mathbf{F}$ . If  $a = 0$  or  $b = 0$ , then  $\tau(ab) = 0 = \tau(a)\tau(b)$ . If  $ab \neq 0$ , then for any  $c \neq ab$ , the matrix

$$X = cE_{11} + aE_{12} + bE_{21} + \sum_{j=2}^m E_{jj},$$

has rank  $m$ , and so is

$$f(X) = \tau(c)E_{11} + \tau(a)E_{12} + \tau(b)E_{21} + \sum_{j=2}^n E_{jj}.$$

Thus,  $\tau(c) \neq \tau(a)\tau(b)$ . Since  $\tau$  is surjective,  $\tau(a)\tau(b)$  is in the range of  $\tau$ . Thus,  $\tau(ab) = \tau(a)\tau(b)$ . Note that for  $b \neq a$ , we have  $\tau(b) = f_{11}(b) \neq f_{ij}(a) = \tau(a)$  by (3.2). Thus,  $\tau$  is injective.

Finally, suppose that  $m \geq 3$ . Let  $a, b \in \mathbf{F}$ . If  $a = 0$  or  $b = 0$ , then  $\tau(a + b) = \tau(a) + \tau(b)$ . Suppose that  $ab \neq 0$ . Let  $c \neq a + b$ , and

$$X = E_{11} + aE_{12} + cE_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s=4}^m E_{ss}.$$

Then  $X$  has rank  $m$  and so is

$$f(X) = E_{11} + \tau(a)E_{12} + \tau(c)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s=4}^m E_{ss}.$$

Thus,  $\tau(c) - \tau(b) \neq \tau(a)$ , or equivalently  $\tau(c) \neq \tau(a) + \tau(b)$ . Since  $\tau$  is surjective,  $\tau(a) + \tau(b)$  is in the range of  $\tau$ . Thus,  $\tau(a + b) = \tau(a) + \tau(b)$ .  $\square$

**COROLLARY 3.6.** *Suppose that  $f$  satisfies the hypothesis of Theorem 3.5 and that  $f$  is Schur multiplicative. Then the conclusion of Theorem 3.5 holds with the additional restriction that  $P$  and  $Q$  are permutation matrices and  $f_{ij}$  is multiplicative for each  $(i, j)$  pair.*

Clearly, the conclusion of Corollary 3.6 holds if  $f$  is Schur multiplicative and satisfies any of the conditions (A1)–(A3) in Theorem 2.1. However, the conclusion is no longer valid if we just assume that  $f$  is Schur multiplicative. For instance, one can define  $f$  such that  $f(0_{m,n}) = 0_{m,n}$  and  $f(A) = B$  for all other  $A$ , where  $B$  is any rank  $m$  matrix satisfying  $B \circ B = B$ .

#### 4. Preservers of determinant and other functions of eigenvalues

Linear maps, additive maps, and multiplicative maps of determinants and functions of eigenvalues have been studied by researchers; see [6, 7, 16]. In this section, we study Schur multiplicative maps preserving determinant and related functions. In most cases, we obtain results for maps of the form  $(\dagger)$ , and then use them to study Schur multiplicative maps. We begin with an analog of the result of Frobenius [7] on linear preservers of the determinant function.

**THEOREM 4.1.** *Suppose that  $f : M_n \rightarrow M_n$  has the form  $(\dagger)$ . Then  $f$  satisfies  $\det(f(A)) = \det(A)$  for all  $A$  if and only if there are monomial matrices  $P$  and  $Q$  satisfying  $\det(PQ) = 1$  such that  $f$  has the form*

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^tQ.$$

**PROOF.** The ‘only if’ implication is clear. We consider the converse. Note that  $f(I_n) = P$  is a monomial matrix with determinant 1. Replacing  $f$  by the mapping  $A \mapsto P^{-1}f(A)$ , we may assume that  $f(I_n) = I_n$ . Furthermore, we may replace  $f$  by a map of the form  $A \mapsto Q^t f(A)Q$  for a suitable permutation matrix  $Q$ , so that  $f(E_{jj}) = E_{jj}$  for  $j = 1, \dots, n$ . Now, for any  $a \in \mathbf{F}$ ,

$$a = \det(I_n + (a - 1)E_{jj}) = \det(f(I_n + (a - 1)E_{jj})) = f_{jj}(a).$$

So,  $f_{jj}(a) = a$  for all  $a \in \mathbf{F}$ .

Next, we show that  $f(E_{ij} + E_{ji}) = \gamma_{ij}E_{ij} + \gamma_{ij}^{-1}E_{ji}$  for all  $1 \leq i < j \leq n$ . For simplicity, assume that  $(i, j) = (1, 2)$ . Then  $-1 = \det(X) = \det(f(X))$  for

$$X = E_{12} + E_{21} + \sum_{j=3}^n E_{jj}.$$

Thus,  $f(E_{12} + E_{21}) = \gamma_{12}E_{12} + \gamma_{12}^{-1}E_{21}$ . Let  $D = \text{diag}(1, \gamma_{12}, \dots, \gamma_{1n})$ . We may replace  $f$  by the map  $A \mapsto D^{-1}f(A)D$ , and assume that  $f(X) = X$  for all  $X = E_{1j} + E_{j1}$  with  $2 \leq j \leq n$ .

Now, for  $X_a = aE_{12} + E_{21} + \sum_{j=3}^n E_{jj}$ , we have  $-a = \det(X_a) = \det(f(X_a))$ . We see that one of the following holds:

- (i)  $f(E_{12}) = E_{12}$ ,  $f(E_{21}) = E_{21}$ , and  $f_{12}(a) = a$  for all  $a \in \mathbf{F}$ ; or
- (ii)  $f(E_{12}) = E_{21}$ ,  $f(E_{21}) = E_{12}$ , and  $f_{21}(a) = a$  for all  $a \in \mathbf{F}$ .

We may assume that (i) holds. Otherwise, replace  $f$  by the map  $A \mapsto f(A)^t$ .

We are done if  $n = 2$ . Assume that  $n \geq 3$ . For  $j \geq 3$ , let  $S = \{3, \dots, n\} \setminus \{j\}$ , and

$$X_j = E_{12} + E_{1j} + E_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}.$$

Then  $X_j$  has determinant 1 and so does

$$f(X_j) = E_{12} + E_{1j} + \gamma_{2j}E_{2j} + E_{j1} + \gamma_{2j}^{-1}E_{j2} + \sum_{s \in S} E_{ss}.$$

It follows that  $\gamma_{2j} = 1$ . Next, note that

$$X_a = E_{12} + E_{1j} + aE_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}$$

has determinant  $a$  and so does  $f(X_a)$ . We conclude that  $f(aE_{2j}) = aE_{2j}$  and  $f(E_{j2}) = E_{j2}$ . Since  $\det(f(Y_a)) = \det(Y_a)$  for

$$Y_a = E_{21} + E_{1j} + E_{2j} + E_{j1} + aE_{j2} + \sum_{s \in S} E_{ss},$$

we see that  $f(aE_{j2}) = aE_{j2}$ . Now, note that

$$Z_a = aE_{1j} + E_{21} + E_{j2} + \sum_{s \in S} E_{ss}$$

has determinant  $a$  and so does  $f(Z_a)$ . We conclude that  $f(aE_{1j}) = aE_{1j}$  and  $f(E_{j1}) = E_{j1}$ . Furthermore,

$$U_a = aE_{j1} + E_{12} + E_{2j} + \sum_{s \in S} E_{ss}$$

has determinant  $a$  and so does  $f(U_a)$ . We conclude that  $f(aE_{j1}) = aE_{j1}$ .

We are done if  $n \leq 3$ . Otherwise, consider  $j \geq 4$  and  $S = \{1, 4, \dots, n\} \setminus \{j\}$ . Let  $X_a = E_{23} + aE_{3j} + E_{j2} + \sum_{s \in S} E_{jj}$ . Using the fact that  $a = \det(X_a) = \det(f(X_a))$ , we conclude that  $f(aE_{3j}) = aE_{3j}$ . Using the matrix  $Y_a = E_{32} + aE_{j3} + E_{2j} + \sum_{s \in S} E_{jj}$ , we see that  $f(aE_{j3}) = aE_{j3}$ .

We can repeat the above argument until we conclude that  $f(aE_{ij}) = aE_{ij}$  for all  $(i, j)$  pairs. □

If  $f$  is Schur multiplicative of the form  $(\dagger)$ , then one can show that conclusion of Theorem 4.1 holds with the additional restriction that  $P$  and  $Q$  are permutation matrices. In the following, we show that one can obtain the same conclusion for Schur multiplicative maps  $f : M_n \rightarrow M_n$  such that  $f(0_n) = 0_n$  and  $\det(f(A)) = \det(A)$  for all  $A \in M_n$ , if  $n \geq 3$ . For  $n = 2$ , one can define  $f$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bc \\ 1 & d \end{pmatrix}.$$

Then  $f$  will be Schur multiplicative and preserves determinant. (In fact, it preserves all eigenvalues.)

**THEOREM 4.2.** *Suppose that  $n \geq 3$  and  $f : M_n \rightarrow M_n$  is Schur multiplicative such that  $f(0_n) = 0_n$  and  $\det(f(A)) = \det(A)$  for all  $A \in M_n$ . Then there are permutation matrices  $P, Q \in M_n$  such that  $f$  has the form*

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^tQ.$$

**PROOF.** Suppose that  $f$  satisfies the hypothesis of the theorem. We shall show that  $f$  also satisfies condition (A1) of Theorem 2.1. So,  $f$  has the form  $(\dagger)$ , and we can apply Theorem 4.1 to get the desired conclusion. We divide our proof into several assertions.

**ASSERTION 1.** The function  $f$  maps the set  $S_n$  of permutation matrices back to itself bijectively.

Suppose that  $R \in S_n$ . Let  $S = E_{12} + E_{23} + \dots + E_{n,n-1} + E_{n1}$  be the basic circulant, and let  $\mathcal{T} = \{R, RS, RS^2, \dots, RS^{n-1}\}$ . For any two distinct  $X, Y \in \mathcal{T}$ :

- (1)  $X \circ X = X$  implies  $f(X) \circ f(X) = f(X)$  so that all entries of  $f(X)$  belongs to  $\{0, 1\}$ ;
- (2)  $X \circ Y = 0_n$  implies  $f(X) \circ f(Y) = f(X \circ Y) = f(0_n) = 0_n$  so that the non-zero entries of  $f(X)$  must be disjoint from the non-zero entries of  $f(Y)$ ;
- (3)  $\pm 1 = \det(X) = \det(f(X))$  so that  $f(X)$  has at least  $n$  non-zero entries on distinct rows and columns.

Consequently,  $f(X)$  must have exactly  $n$  non-zero entries, and therefore each  $f(X)$  is a permutation matrix. In particular,  $f(R)$  is a permutation matrix. For two distinct permutation matrices  $R_1, R_2$ , the condition  $f(R_1) = f(R_2)$  implies that  $1 = \det(f(R_1) \circ f(R_2)) = \det(R_1 \circ R_2) = 0$ , a contradiction. Therefore,  $f(S_n) = S_n$ .

**ASSERTION 2.** Two matrices  $R_1, R_2 \in S_n$  satisfy  $R_1 \circ R_2 = 0_n$  if and only if  $f(R_1) \circ f(R_2) = 0_n$ .

Recall that a permutation matrix is a derangement if none of its diagonal entries are non-zero, and there is a fixed number, say,  $d_n$ , for  $n \times n$  derangements by basic combinatorial theory. Now consider

$$\mathcal{N}(R) = \{X : X \in S_n, R \circ X = 0_n\} = \{X : RX^t \text{ is a derangement}\}.$$

Then  $\mathcal{N}(R)$  has  $d_n$  elements and so has  $\mathcal{N}(f(R))$ . For  $X \in \mathcal{N}(R)$ , we have  $0_n = R \circ X$ , so that  $0_n = f(R \circ X) = f(R) \circ f(X)$ , and then  $f(X) \in \mathcal{N}(f(R))$ . So  $f(\mathcal{N}(R)) \subseteq \mathcal{N}(f(R))$ . Since each set has  $d_n$  elements, the sets must be equal. Then for any permutation matrix  $Z$ ,  $Z \notin \mathcal{N}(R)$  if and only if  $f(Z) \notin \mathcal{N}(f(R))$ . So for any permutations  $R_1, R_2$ , we have  $R_1 \circ R_2 = 0_n$  if and only if  $f(R_1) \circ f(R_2) = 0_n$ .

**ASSERTION 3.** The function  $f$  satisfies condition (A1) in Theorem 3.1.

Note that, for  $n \geq 3$ , each  $E_{ij}$  can be written as  $E_{ij} = R_1 \circ R_2$  for a pair of permutations  $R_1, R_2$ . Therefore,  $f(E_{ij}) = f(R_1) \circ f(R_2) \neq 0_n$ . So  $f$  satisfies (A1), and, by Theorem 2.1,  $f$  is of the form  $(\dagger)$ , so we can apply Theorem 4.1 and conclude that  $f$  has the asserted form.  $\square$

We believe that in the theorem one may even remove the assumption that  $f(0_n) = 0_n$  to obtain the same conclusion. It would be nice to prove or disprove this.

If  $\mathbf{F}$  is algebraically closed, then  $\det(A)$  is the product of the eigenvalues of  $A$ . Researchers have studied linear preservers of other elementary symmetric functions of the eigenvalues; see [1, 10, 13, 14]. Denote by  $E_k(A)$  the  $k$ th elementary symmetric function of the eigenvalues of  $A \in M_n$ . Then  $E_n(A) = \det(A)$ , and  $E_k(A)$  equals the sum of the  $k \times k$  principal minors of  $A$ . Defining  $E_k(A)$  as the sum of the  $k \times k$  minors of  $A$ , we can study  $E_k(A)$  even if  $\mathbf{F}$  is not algebraically closed. We have the following result.

**THEOREM 4.3.** Suppose that  $f : M_n \rightarrow M_n$  has the form  $(\dagger)$  and  $3 \leq k < n$ . Then  $E_k(A) = E_k(f(A))$  for all  $A$  if and only if there is an invertible monomial matrix  $P$  and a scalar  $\gamma \in \mathbf{F}$  satisfying  $\gamma^k = 1$  such that  $f$  has the form

$$A \mapsto \gamma P A P^{-1} \quad \text{or} \quad A \mapsto \gamma P A^t P^{-1}.$$

If  $f$  is Schur multiplicative of the form  $(\dagger)$ , then the above conclusion holds with the additional restriction that the matrix  $P$  is a permutation matrix and  $\gamma = 1$ .

**PROOF.** The ‘only if’ implication is clear. For the converse, consider any  $k$  element subset  $S$  of  $\{1, \dots, n\}$ , and  $X = \sum_{j \in S} E_{jj}$ . Since  $1 = E_k(X) = E_k(f(X))$  we see that  $f(X)$  is the sum of  $k$  non-zero matrices in  $\{\gamma E_{jj} : 1 \leq j \leq n, \gamma \in \mathbf{F}\}$  with distinct  $j$ . Thus, there exists a permutation matrix  $P \in M_n$  such that  $f(E_{jj}) = \gamma_j P E_{jj} P^t$  for  $j = 1, \dots, n$ . Since

$$1 = E_k(X) = E_k(f(X)) = \prod_{s \in S} \gamma_s,$$

for any  $k$  element subset  $S$  of  $\{1, \dots, n\}$ , we see that  $\gamma_1 = \dots = \gamma_n = \gamma$  such that  $\gamma^k = 1$ .

Replace  $f$  by the map  $A \mapsto P^t f(A)P/\gamma$  so that we may assume that  $f(E_{jj}) = E_{jj}$  for  $j = 1, \dots, n$ . Now, for any  $a \in \mathbf{F}$ ,

$$a = E_k \left( aE_{11} + \sum_{j=2}^k E_{jj} \right) = E_k \left( f \left( aE_{11} + \sum_{j=2}^k E_{jj} \right) \right) = f_{11}(a).$$

So,  $f_{11}(x) = x$  for all  $x \in \mathbf{F}$ . Similarly, we can show that  $f_{jj}(x) = x$  for all  $x \in \mathbf{F}$ .

Next, we show that  $f(E_{ij} + E_{ji}) = \gamma_{ij}E_{ij} + \gamma_{ij}^{-1}E_{ji}$  for all  $(i, j)$  pairs. For simplicity, assume that  $(i, j) = (1, 2)$ . For any subset  $S$  of  $\{3, \dots, n\}$  with  $k - 2$  elements, consider  $X_S = E_{12} + E_{21} + \sum_{j \in S} E_{jj}$ . We have  $-1 = E_k(X_S) = E_k(f(X_S))$ . Thus,

$$f(E_{12} + E_{21}) = \gamma_{12}E_{pq} + \gamma_{21}E_{rs},$$

satisfies  $\{p, q, r, s\} \cap S = \emptyset$ . As this is true for all subsets  $S$  of  $\{3, \dots, n\}$  with  $k - 2$  elements, we conclude that  $f(E_{12} + E_{21}) = \gamma_{12}E_{12} + \gamma_{21}E_{21}$ . Since

$$-1 = E_k(X) = E_k(f(X)),$$

for  $X = E_{12} + E_{21} + \sum_{j=3}^k E_{jj}$ , we see that  $\gamma_{12}\gamma_{21} = 1$ .

Let  $D = \text{diag}(1, \gamma_{12}, \gamma_{13}, \dots, \gamma_{1n})$ . Replace  $f$  by the map  $A \mapsto Df(A)D^{-1}$  so that we can assume that  $f(X) = X$  for

$$X \in \{E_{jj} : 1 \leq j \leq n\} \cup \{E_{1j} + E_{j1} : 2 \leq j \leq n\}.$$

Now, consider  $X_a = aE_{12} + E_{21} + \sum_{j=3}^k E_{jj}$  with  $a \in \mathbf{F}$ . Since  $E_k(X_a) = E_k(f(X_a))$ , one of the following holds:

- (i)  $f(E_{12}) = E_{12}$ ,  $f(E_{21}) = E_{21}$ , and  $f_{12}(a) = a$  for all  $a \in \mathbf{F}$ ; or
- (ii)  $f(E_{12}) = E_{21}$ ,  $f(E_{21}) = E_{12}$ , and  $f_{12}(a) = a$  for all  $a \in \mathbf{F}$ .

We may assume that (i) holds. Otherwise, replace  $f$  by the mapping  $A \mapsto f(A)^t$ .

For  $j \geq 3$ , let  $S$  be an  $k - 3$  element subset of  $\{3, \dots, n\} \setminus \{j\}$ , and

$$X_j = E_{12} + E_{1j} + E_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}.$$

Then  $1 = E_k(X_j) = E_k(f(X_j))$  where

$$f(X_j) = E_{12} + E_{1j} + \gamma_{2j}E_{2j} + E_{j1} + \gamma_{2j}^{-1}E_{j2} + \sum_{s \in S} E_{ss}.$$

It follows that  $\gamma_{2j} = 1$ . Next, note that

$$a = E_k(X_a) = E_k(f(X_a)) \quad \text{for } X_a = E_{12} + E_{1j} + aE_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that  $f(aE_{2j}) = aE_{2j}$  and  $f(E_{j2}) = E_{j2}$ . It follows that  $f_{2j}(a) = a$ . Since

$$a = E_k(f(Y_a)) = E_k(Y_a) \quad \text{for } Y_a = E_{21} + E_{1j} + E_{2j} + E_{j1} + aE_{j2} + \sum_{s \in S} E_{ss},$$

we see that  $f(aE_{j2}) = aE_{j2}$ . Now, note that

$$a = E_k(Z_a) = E_k(f(Z_a)) \quad \text{for } Z_a = aE_{1j} + E_{21} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that  $f(aE_{1j}) = aE_{1j}$  and  $f(E_{j1}) = E_{j1}$ . Furthermore,

$$a = E_k(U_a) = E_k(f(U_a)) \quad \text{for } U_a = aE_{j1} + E_{12} + E_{2j} + \sum_{s \in S} E_{ss}.$$

We conclude that  $f(aE_{j1}) = aE_{j1}$ .

We are done if  $n \leq 3$ . Otherwise, consider  $j \geq 4$  and let  $S$  be a  $k - 3$  element subset of  $\{1, 4, \dots, n\} \setminus \{j\}$ . Note that

$$a = E_k(X_a) = E_k(f(X_a)) \quad \text{for } X_a = E_{23} + aE_{3j} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that  $f(aE_{3j}) = aE_{3j}$ . Next note that

$$a = E_k(Y_a) = E_k(f(Y_a)) \quad \text{for } Y_a = E_{32} + aE_{j3} + E_{2j} + \sum_{s \in S} E_{ss},$$

We conclude that  $f(aE_{j3}) = aE_{j3}$ .

We can repeat the above argument until we conclude that  $f(aE_{ij}) = aE_{ij}$  for all  $(i, j)$  pairs.

The proof for Schur multiplicative maps satisfying (A1) is similar and simpler.  $\square$

Note that  $E_1(A)$  is the trace function. It is easy to prove that a map  $f : M_n \rightarrow M_n$  has the form  $(\dagger)$  and satisfies  $E_1(A) = E_1(f(A))$  for all  $A \in M_n$  if and only if

$$\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\},$$

and, for each  $j \in \{1, \dots, n\}$  we have  $f_{jj}(x) = x$  for all  $x \in \mathbf{F}$ . Moreover, a Schur multiplicative map  $f : M_n \rightarrow M_n$  satisfying (A1) also satisfies  $E_1(A) = E_1(f(A))$  for all  $A \in M_n$  if and only if  $f$  has the form described in Theorem 2.1 with the additional condition that

$$\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\},$$

and, for each  $j \in \{1, \dots, n\}$  we have  $f_{jj}(x) = x$  for all  $x \in \mathbf{F}$ .

For  $E_2(A)$ , there are a lot of linear preservers; see [10, 14]. In our case, we have the following result.

**THEOREM 4.4.** *Suppose that  $2 < n$  and  $f : M_n \rightarrow M_n$  has the form  $(\dagger)$ . Then  $E_2(A) = E_2(f(A))$  if and only if:*

(a) *there is  $\mu = \pm 1$  such that  $\mu f_{jj}$  is the identity map on  $\mathbf{F}$  for all  $j$  with*

$$\{f(E_{jj}) : 1 \leq j \leq n\} = \{\mu E_{jj} : 1 \leq j \leq n\}; \quad \text{and}$$

(b)  *$f_{ij}(a)f_{ji}(b) = ab$  for any  $a, b \in \mathbf{F}$  with*

$$\{f(E_{ij} + E_{ji}) : 1 \leq i < j \leq n\} = \{E_{ij} + E_{ji} : 1 \leq i < j \leq n\}.$$

*If  $f$  is a Schur multiplicative map of the form  $(\dagger)$ , then the above conclusion holds with the additional restriction that  $f_{ij}$  is multiplicative in condition (b), and  $\mu = 1$  in condition (a).*

**PROOF.** Note that, for  $A = (a_{ij})$ , we have

$$E_2(A) = E_2(a_{11}, \dots, a_{nn}) - \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}.$$

By this observation, the ‘only if’ implication follows.

For the converse, note that  $a = E_2(aE_{ii} + E_{jj}) = E_2(f(aE_{ii} + E_{jj}))$  for any  $a \in \mathbf{F}$  and  $i \neq j$ . We see that  $\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\}$  and there is  $\mu = \pm 1$  so that  $\mu f_{jj}$  is the identity map on  $\mathbf{F}$  for each  $j \in \{1, \dots, n\}$ .

Now,

$$-ab = E_2(aE_{ij} + bE_{ji}) = E_2(f(aE_{ij} + bE_{ji})) = E_2(f_{ij}(a)E_{pq} + f_{ji}(b)E_{rs}),$$

for some  $(p, q), (r, s)$  pairs with  $p \neq q$  and  $r \neq s$ . We see that  $(p, q) = (s, r)$  and  $f_{ij}(a)f_{ji}(b) = ab$  for all  $x \in \mathbf{F}$ . The result follows.

One can readily verify the last assertion concerning Schur multiplicative maps.  $\square$

By Theorem 4.3, one can easily determine the structures of eigenvalue preservers on  $M_n$  with  $n \geq 4$ . Here we show that they have the same structure as the preservers of the spectrum (not counting multiplicities of the eigenvalues) of matrices. Note that if  $\mathbf{F}$  is not algebraically closed,  $A \in M_n$  may not have eigenvalues in  $\mathbf{F}$ . In such a case, we may assume that the spectrum of  $A$  is the empty set.

**THEOREM 4.5.** *Let  $\Gamma(A)$  denote the set of eigenvalues (counting multiplicities) or the spectrum of  $A \in M_n$ . Suppose that  $f : M_n \rightarrow M_n$  satisfies  $(\dagger)$ . Then  $f$  satisfies  $\Gamma(f(A)) = \Gamma(A)$  for all  $A \in M_n$  if and only if there is an invertible monomial matrix  $Q$  such that  $f$  has the form*

$$A \mapsto Q^{-1}AQ \quad \text{or} \quad A \mapsto Q^{-1}A^tQ.$$

*If  $f$  is a Schur multiplicative map satisfying  $f(0_n) = 0_n$ , then the above conclusion holds with the additional restriction that  $Q$  is a permutation matrix.*

**PROOF.** The ‘only if’ implication is clear. We focus on the converse. If  $f$  preserves the set of eigenvalues, it will preserve the spectrum. Suppose that  $f$  preserves the spectrum. We shall show that  $f$  has the asserted form.

Let  $\text{Spec}(A)$  denote the spectrum of  $A \in M_n$ . Then  $\text{Spec}(E_{ii}) = \{1, 0\}$ . By Theorem 2.1,  $f(E_{ii}) = E_{jk}$ . Since  $f$  preserves the spectrum, we see that  $j = k$ . Thus  $f(E_{ii}) = E_{jj}$ , and we can assume that all  $f(E_{ii}) = E_{ii}$  by replacing the mapping  $f$  with  $A \rightarrow Qf(A)Q^t$  for a suitable permutation matrix  $Q$ . For any  $a$  in the field consider  $aE_{ii}$  with  $\text{Spec}(aE_{ii}) = \{a, 0\}$ . We have  $f(aE_{ii}) = f_{ii}(a)E_{ii}$  with  $\text{Spec}(f_{ii}(a)E_{ii}) = \{f_{ii}(a), 0\}$ , and therefore we conclude that  $f_{ii}(a) = a$ .

Now for  $X = E_{12} + E_{21} + E_{11} + E_{22}$  we have  $\text{Spec}(X) = \{2, 0\}$ . Since we have already concluded that the diagonal entries map to themselves, we see that  $f(E_{12} + E_{21}) = \mu_{12}E_{12} + \mu_{12}^{-1}E_{21}$ . Using a similar argument, we can prove that  $f(E_{ij} + E_{ji}) = \mu_{ij}E_{ij} + \mu_{ij}^{-1}E_{ji}$  with  $\mu_{ij} \in \mathbf{F}^*$  for each  $(i, j)$  pair with  $i < j$ . Let  $D = \text{diag}(1, \mu_{12}, \dots, \mu_{1n})$ . Replace  $f$  by the map  $A \mapsto Df(A)D^{-1}$  so that we have  $f(X) = X$  for  $X \in \{E_{jj} : 1 \leq j \leq n\} \cup \{E_{1j} + E_{j1} : 2 \leq j \leq n\}$ . We can then consider two cases:

- (i)  $f(E_{12}) = E_{12}, f(E_{21}) = E_{21}$ ; or
- (ii)  $f(E_{12}) = E_{21}, f(E_{21}) = E_{12}$

We can assume (i) holds; otherwise, replace  $f$  with the mapping  $A \rightarrow f(A)^t$ . Observe that  $\text{Spec}(X_a) = \{1 + a, 1, 0\}$  for

$$X_a = E_{11} + aE_{12} + E_{21} + aE_{22} + \sum_{i=3}^n E_{ii}.$$

If  $f_{12}(a) \neq a$ , then  $f(X_a)$  is non-singular, and  $0 \notin \text{Spec}(f(X_a))$ , which is a contradiction. Therefore  $f_{12}(a) = a$ .

Note that  $f(E_{1j}) = E_{1j}$  or  $E_{j1}$  for  $j > 2$ . For  $X_j = E_{12} + E_{1j} + E_{j2} + E_{2j}$ ,  $\text{Spec}(X_j) = \{0, -1, 1\} = \text{Spec}(f(X_j))$ . If  $f(E_{1j}) = E_{j1}$ , then  $\text{Spec}(f(X_j))$  is a subset of the set of zeros of the polynomial  $z(z^3 - z - 1)$ , which contains neither 1 nor  $-1$ . Therefore  $f(E_{1j}) = E_{1j}$  and  $f(E_{j1}) = E_{j1}$ . For any  $a \in \mathbf{F}$ ,  $0 \in \text{Spec}(Y_a) = \text{Spec}(f(Y_a))$  for

$$Y_a = aE_{11} + E_{j1} + aE_{1j} + \sum_{i=2}^n E_{ii}.$$

Thus,  $f(aE_{1j}) = aE_{1j}$ . Similarly,  $f(aE_{j1}) = aE_{j1}$  for all  $a \in \mathbf{F}$ .

We are done if  $n = 2$ . Assume  $n \geq 3$ . For  $j \geq 3$ . Since  $0 \notin \text{Spec}(f(X_j)) = \text{Spec}(X_j)$  for  $X_j = E_{12} + E_{2j} + E_{j1} + \sum_{s \in S} E_{ss}$  with  $S = \{1, \dots, n\} \setminus \{1, 2, j\}$ , we see that  $f(E_{2j}) = E_{2j}$ , and thus  $f(E_{j2}) = E_{j2}$ . Assume  $n \geq 4$ . For  $j \geq 4$ , since  $0 \notin \text{Spec}(f(X_j)) = \text{Spec}(X_j)$  for  $X_j = E_{23} + E_{3j} + E_{j2} + \sum_{s \in S} E_{ss}$  with  $S = \{1, \dots, n\} \setminus \{2, 3, j\}$ , we see that  $f(E_{3j}) = E_{3j}$ , and thus  $f(E_{j3}) = E_{j3}$ . We can repeat the argument until we conclude that  $f(E_{ij}) = E_{ij}$  for each  $(i, j)$  pair with

$i \neq j$ . Moreover, for any  $(i, j)$  pair with  $i \neq j$ , since  $0 \in \text{Spec}(f(Y)) = \text{Spec}(Y)$  for  $Y = aE_{ij} + E_{ji} + aE_{ii} + \sum_{s \neq i} E_{ss}$ , we see that  $f(aE_{ij}) = aE_{ij}$  for any  $a \in \mathbf{F}$ . Thus,  $f$  has the asserted form.

Next, we consider Schur multiplicative maps satisfying  $f(0_n) = 0_n$ . Consider  $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$  such that  $\mathcal{T}_1 = \{E_{jj} : 1 \leq j \leq n\}$  and  $\mathcal{T}_2 = \{E_{ij} + E_{ji} : 1 \leq i, j \leq n\}$ . Then

$$f(X) = f(X \circ X) = f(X) \circ f(X), \{1, 0\} = \text{Spec}(X) = \text{Spec}(f(X)),$$

for  $X \in \mathcal{T}_1$  and  $\{1, -1\} \subseteq \text{Spec}(X) = \text{Spec}(f(X))$  for  $X \in \mathcal{T}_2$ . Thus,  $f(X)$  is non-zero with entries in  $\{0, 1\}$ . For any  $X \neq Y$  in  $\mathcal{T}$ ,  $0 = f(X \circ Y) = f(X) \circ f(Y)$ , that is,  $f(X)$  and  $f(Y)$  have non-zero entries in disjoint positions. Clearly, it is impossible to have more than  $n(n - 1)/2$  matrices with more than one non-zero entries in  $f(\mathcal{T})$ . Thus, at least  $n$  matrices in  $\mathcal{T}$  have exactly one non-zero entry. Since none of these matrices have spectrum equal to  $\{0\}$ , we see that there are exactly  $n$  such matrices, which are the matrices in  $\mathcal{T}_1$ . As a result, we see that  $f(\mathcal{T}_1) = \mathcal{T}_1$ . Consequently,  $f(X)$  has at least two non-zero entries for each  $X \in \mathcal{T}$ . Since the non-zero entries can only lie in the off-diagonal positions, we see that  $f(X)$  has exactly two non-zero entries for each  $X \in \mathcal{T}_2$ . Since  $\{1, -1\} \subseteq \text{Spec}(X) = \text{Spec}(f(X))$ , we see that  $f(X) \in \mathcal{T}_2$ . Hence  $f(\mathcal{T}_2) = \mathcal{T}_2$ .

We may assume that  $f(X) = X$  for all  $X \in \mathcal{T}_1$ . Otherwise, we may replace  $f$  by a map of the form  $A \mapsto Pf(A)P^t$  for a suitable permutation  $P$ . Note that for any  $1 \leq j < k \leq n$ , if  $X = E_{jj} + E_{jk} + E_{kj} + E_{kk}$  then  $X \circ X = X$  and  $2 \in \text{Spec}(X)$ ; moreover,  $Y \in \mathcal{T}$  satisfies  $\{1, -1\} \subseteq \text{Spec}(X \circ Y)$  if and only if  $Y = E_{jk} + E_{kj}$ . It follows that  $f(X) \circ f(X) = f(X)$  and  $2 \in \text{Spec}(f(X))$ ; moreover,  $Z \in f(\mathcal{T})$  satisfies  $\{1, -1\} \subseteq (f(X) \circ Z)$  if and only if  $Z = f(E_{jk} + E_{kj})$ . Hence  $f(E_{jk} + E_{kj}) = E_{jk} + E_{kj}$ . As a result, we have  $f(X) = X$  for every  $X \in \mathcal{T}$ .

Next, we show that  $f(E_{jk}) \in \{E_{jk}, E_{kj}\}$  for any  $1 \leq j < k \leq n$ . For simplicity, we assume  $(j, k) = (1, 2)$ . Let  $X = E_{12} + E_{23} + E_{31}$ . Then  $Y \in \mathcal{T}$  satisfies  $X \circ Y = 0$  whenever  $Y \notin \mathcal{T}_0 = \{E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}\}$ . Thus,

$$f(X) \circ f(Y) = f(X \circ Y) = f(0_n) = 0_n,$$

whenever  $Y \in \mathcal{T} \setminus \mathcal{T}_0$ . Since

$$f(X) = f(X \circ X) = f(X) \circ f(X) \quad \text{and} \quad 1 \in \text{Spec}(X) = \text{Spec}(f(X)),$$

we see that  $f(X) \in \{X, X^t\}$ . Now, for  $\widehat{X} = E_{12} + E_{21}$ , we have  $E_{12} = X \circ \widehat{X}$ , and hence

$$f(E_{12}) = f(X) \circ f(\widehat{X}) = f(X) \circ \widehat{X} \in \{E_{12}, E_{21}\}.$$

By the above arguments, we see that  $f(E_{ij}) \neq 0$  for any  $(i, j)$  pair. Thus, condition (A1) of Theorem 2.1 holds. Thus,  $f$  has the form  $(\dagger)$ , and hence satisfies the desired conclusion by the first part of our proof. □

Clearly, the conclusions of most of the results in this section are valid if  $f$  is Schur multiplicative and satisfies any of the conditions (A1)–(A3) of Theorem 2.1. If  $n = 2$ , the conclusions of the theorems are not valid if we just assume that  $f$  is Schur multiplicative. For example, one may define  $f$  by  $f(A) = 0$  if  $A$  is strictly upper or lower triangular, and  $f(A) = A$  otherwise. However, for  $n \geq 3$ , it is unclear whether the conclusions of the theorems hold if we just assume that  $f$  is Schur multiplicative. It would be interesting to prove the results under the weaker assumption, or give examples showing the results are not true.

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