J. Austral. Math. Soc. (*Series A*) **27** (1979), 454–466

APPROXIMATION BY A SUM OF POLYNOMIALS OF DIFFERENT DEGREES INVOLVING PRIMES

MING-CHIT LIU

(Received 20 March 1977; revised 7 March 1978)

Communicated by J. Pitman

Abstract

Let λ_j $(1 \leqslant j \leqslant 4)$ be any nonzero real numbers which are not all of the same sign and not all in rational ratio and let $\mathfrak{p}_j$ be polynomials of degree one or two with integer coefficients and positive leading coefficients. The author proves that if exactly two $\mathfrak{p}_j$ are of degree two then for any real η there are infinitely many solutions in primes p_j of the inequality

$$\left|\eta + \sum_{j=1}^{4} \lambda_j \mathfrak{p}_j(p_j)\right| < (\max p_j)^{-\beta}$$

where $0 < \beta < (\sqrt{(21)} - 1)/5760$.

1980 Mathematics subject classification (*Amer. Math. Soc.*): 10 J 15, 10 F 15, 10 B 45.

1. Introduction

Let $\lambda_1, \ldots, \lambda_s$ $(s \geqslant 3)$ be any nonzero real numbers which are not all of the same sign and not all in rational ratio. Baker (1967), pp. 166–167, introduced a new kind of approximation analogous to Davenport and Heilbronn (1946), p. 186, by proving that if $s = 3$ then for any positive integer N, (1.1) has infinitely many solutions in primes p_j:

$$(1.1) \qquad |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\log \max p_j)^{-N}.$$

Recently, Vaughan (1974a), p. 374, improved (1.1) and a result of Ramachandra's (1973), Theorem 3, by showing that for any real η, (1.1) can be replaced by

$$(1.2) \qquad |\eta + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\max p_j)^{-1/10} (\log \max p_j)^{20}.$$

(1.2) has been extended (Vaughan (1974b), p. 386, and Liu (1978), Theorems 1, 2) to polynomials $\mathfrak{p}_j(x)$ of the same degree $k \geqslant 2$ with integer coefficients and positive leading coefficients, namely if $s \geqslant s_0(k)$, $0<\gamma<\gamma_0(k)$ then (1.3) has infinitely many solutions in primes p_j, where $s_0(k)$ and $\gamma_0(k)$ depend on k only (in particular, $s_0(2)=5$):

$$\left|\eta+\sum_1^s \lambda_j \mathfrak{p}_j(p_j)\right|<(\max p_j)^{-\gamma}. \tag{1.3}$$

In this paper we shall modify the methods of Schwarz (1963) and Vaughan (1974) and prove

THEOREM 1. *Let* λ_j $(1 \leqslant j \leqslant 4)$ *be any nonzero real numbers which are not all of the same sign and not all in rational ratio. Let* $\mathfrak{p}_j$ *be polynomials of degree one or two with integer coefficients and positive leading coefficients. If exactly two* $\mathfrak{p}_j$ *are of degree two then for any real* η *there are infinitely many solutions in primes* p_j *of the inequality*

$$\left|\eta+\sum_{j=1}^4 \lambda_j \mathfrak{p}_j(p_j)\right|<(\max p_j)^{-\beta},$$

where $0<\beta<(\sqrt{(21)}-1)/5760$.

REMARK. Since all preliminary lemmas in Section 3 are valid for $\mathfrak{p}_j$ of degrees $k_j>2$, the above theorem can be extended with no difficulty to $s>4$ polynomials $\mathfrak{p}_j$ of different degrees k_j with $\max k_j>2$. This kind of generalization will certainly lead to a complete improvement of the results in Liu (1977), p. 199. For polynomials of higher different degrees, a more interesting problem is to obtain a better (or smaller) value of $s_0(k)$ where $k=\max k_j$, for which (1.3) has infinitely many solutions in primes p_j. This problem seems to require a new idea.

In the following proof we shall see that the hypothesis in Theorem 1 that exactly two $\mathfrak{p}_j$ are of degree two is needed only in the proof of Lemma 9. So by the same proof we can extend Theorem 1 to the case that exactly three $\mathfrak{p}_j$ are of degree two provided that λ_i/λ_j is irrational for at least one pair $\mathfrak{p}_i, \mathfrak{p}_j$ which are both of degree two. That is

THEOREM 2. *Let* λ_j $(1 \leqslant j \leqslant 4)$ *be any nonzero real numbers which are not all of the same sign and let* λ_1/λ_2 *be irrational. Let* $\mathfrak{p}_j$ *be polynomials of degree one or two with integer coefficients and positive leading coefficients. If* $\mathfrak{p}_1, \mathfrak{p}_2$ *and exactly one of* $\mathfrak{p}_3, \mathfrak{p}_4$ *are of degree two then for any real* η *there are infinitely many solutions in primes* p_j *of the inequality*

$$\left|\eta+\sum_{j=1}^4 \lambda_j \mathfrak{p}_j(p_j)\right|<(\max p_j)^{-\beta},$$

where $0<\beta<(\sqrt{(21)}-1)/5760$.

The author wishes to thank the referee for his valuable comments and suggestions which brought improvement in the presentation of the paper and for pointing out that Theorem 2 can be obtained simultaneously.

2. Notation

We shall only give a proof for Theorem 1. Throughout, n and p with or without suffices denote positive integers and primes respectively; x is a real variable and $[x]$ is its integral part. We write $e(x) = \exp(i2\pi x)$. k_j and α_j ($\geqslant 1$) are the degree and the leading coefficient of $\mathfrak{p}_j$ respectively. For the given β, let α be some positive constant satisfying

(2.1) $$192\beta < \alpha < (\sqrt{(21)} - 1)/30.$$

Without loss of generality let λ_1/λ_2 be irrational and $|\lambda_1| \leqslant |\lambda_2|$. Then it is known (Hardy and Wright (1960), Theorem 183) that there are infinitely many convergents a/q with $(a, q) = 1$, $1 \leqslant q$ such that

(2.2) $$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| < \frac{1}{2q^2}.$$

Put

(2.3) $$P = q^{1/(1-2\alpha)}, \quad L = \log P,$$

(2.4) $$Q_j = P^{1/k_j}, \qquad L_j = \log Q_j.$$

We always choose P (that is, q) to be large and ε small so that all inequalities in Sections 3–5 hold. If $X > 0$ we use $Y \ll X$ (or $X \gg Y$) to denote $|Y| < KX$, where K is some positive constant which may depend on the given constants $\alpha_j, \lambda_j, \varepsilon$ only. Let

(2.5) $$\tau = P^{-\beta},$$
$$K_\tau = K_\tau(x) = \begin{cases} \tau^2 & \text{if } x = 0, \\ (\sin \pi\tau x)^2/(\pi x)^2 & \text{otherwise.} \end{cases}$$

Obviously, we have

(2.6) $$K_\tau \leqslant \tau^2.$$

Let

(2.7) $$\begin{cases} g_j = g_j(x) = \displaystyle\sum_{\varepsilon Q_j \leqslant p \leqslant Q_j} e(x\mathfrak{p}_j(p)), \\ I_j = I_j(x) = \displaystyle\int_{\varepsilon Q_j}^{Q_j} e(x\mathfrak{p}_j(y))/\log y \, dy, \end{cases}$$

(2.8) $$A = (\sqrt{(21)} - 1)/10, \quad \sigma_0 = 1 - A.$$

We use $\rho = \sigma + it$ to denote a typical zero of the Riemann zeta function $\zeta(s)$ and $\sum_j^*$ (or $\sum_\rho$) to denote the summation over all those zeros ρ with $|t| \leqslant Q_j^A$ and $\sigma \geqslant \sigma_0$. It is known (Ingham (1940) that

(2.9) $$\sum_j^* 1 \ll Q_j^{A3(1-\sigma_0)/(2-\sigma_0)} L_j^5 \ll Q_j^A.$$

Let

(2.10) $$G_j(x,\rho) = \sum n^{-1+(\rho/k_j)} e(x[\mathrm{p}_j(n^{1/k_j})])/\log n$$

where summation is over all n such that $(\varepsilon Q_j)^{k_j} \leqslant n \leqslant P$;

(2.11) $$J_j = J_j(x) = \sum_j^* G_j(x,\rho),$$

(2.12) $$\Delta_j = \Delta_j(x) = g_j + J_j - I_j.$$

3. Preliminary lemmas

The proof of Lemmas 4, 5, 8 is similar to that of Lemmas 9, 10, 13 in Liu (1978).

LEMMA 1. *For any real y we have*

$$\int_{-\infty}^{\infty} e(xy) K_\tau(x)\,dx = \max(0, \tau - |y|).$$

PROOF. This follows from Lemma 4 in Davenport and Heilbronn (1946).

LEMMA 2. *Let $k = \max_{1 \leqslant j \leqslant m} k_j$. If $m \geqslant 2^{k-1}$, then*

$$\int_{-\infty}^{\infty} \prod_{j=1}^{m} \Big| \sum_{\varepsilon Q_j \leqslant p \leqslant Q_j} e(x\lambda_j \mathrm{p}_j(p))\Big|^2 K_\tau(x)\,dx \ll \tau(\log \max Q_j)^C \prod_{j=1}^{m} Q_j^{\{2-(k_j/m)\}},$$

where C is a positive constant depending on k only.

PROOF. This can be proved by the same argument as Lemma 4 in Liu (1977), since Theorem 4 in Hua (1965) (that is Lemma 3 in Liu (1977)) is valid for polynomials with integer coefficients.

LEMMA 3. (a) *Suppose that $2 \leqslant Y \leqslant Q_j$. Then*

$$\sum_{p \leqslant Y} \log p + \sum_j^* Y^\rho \rho^{-1} - Y \ll Q_j^{\sigma_0} L_j^2.$$

where D is some large positive constant.

(b) $$\sum_j^* Q_j^\sigma \ll Q_j \exp(-L_j^{1/5}).$$

PROOF. (a) can be proved by the same argument as that of Lemma 3 in Vaughan (1974a), p. 376. (b) can be shown by the same proof as that of Lemma 8 in Vaughan (1974a), p. 379.

LEMMA 4. *We have*

$$\Delta_j(x) \ll Q_j^{\sigma_0} L_j^6 (1+|x|P),$$

where D is the same positive constant in Lemma 3(a).

PROOF. For simplicity, in the following proof we shall drop all suffices j whenever there is no ambiguity. Without loss of generality we replace εQ_j and $(\varepsilon Q_j)^{k_j}$ in (2.7), (2.10) simply by 2. Let

$$\text{(3.1)} \qquad a_n = \begin{cases} \log n + \sum^* n^{-1+(\rho/k)} & \text{if } n = p^k \text{ for some } p \leqslant Q, \\ \sum^* n^{-1+(\rho/k)} & \text{otherwise;} \end{cases}$$

$$b_n = e(x[\mathfrak{p}(n^{1/k})])/\log n \quad \text{and} \quad b_n' = e(x\mathfrak{p}(n^{1/k}))/\log n.$$

Then by (2.7), (2.11) we have

$$\text{(3.2)} \qquad g(x) + J(x) = \sum_{2 \leqslant n \leqslant P} a_n(b_n - b_n') + a_n b_n' = S_1 + S_2, \quad \text{say.}$$

As for any real y

$$e(x[y]) - e(xy) \ll |x|$$

and $\mathfrak{p}(n)$ is integral valued, we have

$$\text{(3.3)} \qquad S_1 = \sum^* \sum_{2 \leqslant n \leqslant P} n^{-1+(\rho/k)}(b_n - b_n')$$

$$\ll |x| \sum^* Q^\sigma \ll |x|\, Q \exp(-L^{1/5}).$$

The last inequality follows from Lemma 3(b).

We come now to consider S_2. Note that by Abel's partial summation,

$$\sum_{n \leqslant z} n^{(\rho/k)-1} = [z]^{\rho/k} - \sum_{n \leqslant z-1} n\{(n+1)^{(\rho/k)-1} - n^{(\rho/k)-1}\}$$

$$= [z]\, z^{(\rho/k)-1} + \int_1^z (1-\rho/k)\,[y]\, y^{(\rho/k)-2}\, dy.$$

But if $z \leqslant Q^k$, $\sigma_0 \leqslant \sigma < 1$, $|t| \leqslant Q^A$, then

$$\left| \int_1^z (1-\rho/k)\, y^{(\rho/k)-2}([y]-y)\, dy \right| \leqslant (1+(\sigma+|t|)/k) \int_1^z y^{(\sigma/k)-1} y^{-1}\, dy \ll Q^A L.$$

Hence

$$\text{(3.4)} \qquad \sum_{n \leqslant z} n^{(\rho/k)-1} - z^{\rho/k}(k/\rho) \ll Q^A L.$$

It follows from (3.1), (3.4), (2.9) and Lemma 3(a) that for any $z \leqslant Q^k$

(3.5) $$\sum_{n \leqslant z} \frac{a_n}{k} - z^{1/k} = \sum_{p \leqslant z^{1/k}} \log p + \sum\nolimits^* z^{\rho/k} \rho^{-1} - z^{1/k} + O(Q^A L) \sum\nolimits^* 1$$
$$\ll Q^{\sigma_0} L^2 + Q^A L^6 Q^{A3(1-\sigma_0)/(2-\sigma_0)} \ll Q^{\sigma_0} L^6.$$

The last inequality follows from (2.8). Putting $A(z) = \sum_{n \leqslant z} a_n/k$ and using Abel's partial summation (Theorem 421 in Hardy and Wright (1960)) we have

$$S_2 = kA(P) \frac{e(xp(P^{1/k}))}{\log P} - \int_2^P kA(z) \frac{d}{dz} \left\{ \frac{e(xp(z^{1/k}))}{\log z} \right\} dz - a_1 b_2'$$
$$= \frac{kP^{1/k}}{\log P} e(xp(P^{1/k})) - \int_2^P kz^{1/k} \frac{d}{dz} \left\{ \frac{e(xp(z^{1/k}))}{\log z} \right\} dz + O(Q^{\sigma_0} L^6 (1 + |x| P)).$$

The last equality follows from (3.5) and (2.9) by which $a_1 b_2' \ll \sum^* 1 \ll Q^A$. Then

(3.6) $$S_2 = I(x) + O(Q^{\sigma_0} L^6 (1 + |x| P))$$

on integrating by parts and changing the variable to $y = z^{1/k}$. Lemma 4 follows from (3.2), (3.3) and (3.6).

LEMMA 5. *Let*

(3.7) $$\delta = P^{-1+\alpha}.$$

We have

(3.8) $$I_j(x) \ll Q_j \min(1, (|x| P)^{-1}),$$

(3.9) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J_j(x)|^2 \, dx \ll Q_j^{2-k_j} \exp(-2L_j^{1/5}),$$

(3.10) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I_j(x)|^2 \, dx \ll Q_j^{2-k_j},$$

(3.11) $$\int_{-\delta}^{\delta} |\Delta_j(x)|^2 \, dx \ll Q_j^{2-k_j} \exp(-2L_j^{1/5}),$$

(3.12) $$\int_{-\delta}^{\delta} |g_j(x)|^2 \, dx \ll Q_j^{2-k_j}.$$

PROOF. In the proof we shall drop all suffices j. (3.8) follows from (2.7) by partial integration. By (2.11) and Hölder's inequality,

(3.13) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J(x)|^2 \, dx \leqslant \sum_{\rho_1} \sum_{\rho_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho_1) G(x, \rho_2)| \, dx$$
$$\leqslant \sum_{\rho_1} \sum_{\rho_2} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho_1)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho_2)|^2 \, dx \right)^{\frac{1}{2}}$$
$$= \left(\sum_{\rho} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho)|^2 \, dx \right)^{\frac{1}{2}} \right)^2.$$

Note that for any large positive integers m, n with $|m-n| \geqslant 2$, we have

$$[\mathfrak{p}(m^{1/k})] \neq [\mathfrak{p}(n^{1/k})]$$

since when y tends to infinity, $(d/dy)\mathfrak{p}(y^{1/k})$ tends to the value of the leading coefficient of $\mathfrak{p}$ which is not less than one. Let $H(n) = n^{-1+(\sigma/k)}(\log n)^{-1}$. Then by (2.10), Parseval's identity and $\sigma < 1$

$$\text{(3.14)} \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x,\rho)|^2\,dx \ll \sum_{(\varepsilon Q)^k \leqslant n \leqslant P} \{H(n)^2 + H(n)\,H(n-1) + H(n)\,H(n+1)\} \ll Q^{-k+2\sigma} L^{-2}.$$

Then (3.9) follows from (3.13), (3.14) and Lemma 3(b).

(3.10) follows from (3.8) and the partition of the interval $|x| \leqslant 1/2$ at $\pm P^{-1}$.

By Lemma 4, (3.7), (2.4) we have

$$\int_{-\delta}^{\delta} |\Delta(x)|^2\,dx \ll Q^{2\sigma_0} L^{12} \delta^3 Q^{2k} \ll Q^{2\sigma_0+3\alpha k-k} L^{12}.$$

Then (3.11) follows since by $k \leqslant 2$, (2.1) and (2.8) we have

$$2\sigma_0 + 3\alpha k < 2\sigma_0 + 2\mathrm{A} = 2.$$

(3.12) follows from (2.12), (3.9), (3.10), (3.11) easily. This proves Lemma 5.

4. Contribution of the integrals over E_1, E_2, E_3

Let

$$\text{(4.1)} \qquad \Psi = \Psi(x) = \prod_1^4 g_j(\lambda_j x), \quad \Psi^* = \Psi^*(x) = \prod_1^4 I_j(\lambda_j x);$$

$$\text{(4.2)} \quad E_1 = \{x \mid |x| \leqslant P^{-1+\alpha}\}, \quad E_2 = \{x \mid P^{-1+\alpha} < |x| \leqslant P^{\alpha}\}, \quad E_3 = \{x \mid |x| > P^{\alpha}\};$$

$$\text{(4.3)} \qquad S = \left(\sum_{j=1}^{4} 1/k_j\right) - 1.$$

LEMMA 6. *We have*

$$\int_{E_1} |\Psi(x) - \Psi^*(x)|\, K_\tau(x)\,dx \ll \tau^2 P^S \exp(-L^{1/5}).$$

PROOF. By (4.1), (2.12)

$$\text{(4.4)} \qquad \Psi - \Psi^* = \sum_{j=1}^{4} (\Delta_j(\lambda_j x) - J_j(\lambda_j x)) \prod_1^{j-1} g_h(\lambda_h x) \prod_{j+1}^{4} I_h(\lambda_h x),$$

where $\prod_1^0 g_h = \prod_5^4 I_h = 1$. It follows from (4.4), (2.6) and $|I_j|$, $|g_j| \leqslant Q_j$ that

$$(4.5)\qquad \int_{E_1} |\Psi-\Psi^*|\,K_\tau\,dx \ll \tau^2\left\{\int_{E_1} (|\Delta_1|+|J_1|)(|I_4|\,Q_2\,Q_3)\,dx + \sum_{j=2}^{4}\int_{E_1} (|\Delta_j|+|J_j|)(|g_1|\prod_{h\neq 1,j} Q_h)\,dx\right\}.$$

Then Lemma 6 follows from (4.5), Hölder's inequality and Lemma 5.

LEMMA 7. *Suppose that a and q are integers such that $q\geqslant 1$, $(a,q)=1$ and $|x-a/q|\leqslant q^{-2}$. If*

$$\log V > 2^{(6k_j-2)}(2k_j+1)\log\log Q_j,$$

where

$$(4.6)\qquad V=\min(Q_j^{1/3}, q, P/q),$$

then

$$\textstyle\sum_{p\leqslant Q_j} e(x\mathfrak{p}_j(p)) \ll Q_j\,V^{-\mu_j},$$

where $\mu_j=((k_j+1)\,2^{2(k_j+1)})^{-1}$.

PROOF. This lemma is a direct consequence of the theorem in Vinogradov (1938), p. 5.

LEMMA 8. *Let $j=1,2$, and $x\in E_2$. If there are integers a_j, q_j with $(a_j,q_j)=1$ and $q_j\geqslant 1$ such that*

$$(4.7)\qquad |\lambda_j x-a_j/q_j|\leqslant \varepsilon q_j^{-1}P^{-1+\alpha}$$

then either $q_1>P^\alpha$ or $q_2>P^\alpha$.

PROOF. We first show that $a_2\neq 0$. For if $a_2=0$ then by (4.7), we have $x\notin E_2$. This is impossible.

Next, suppose that both

$$(4.8)\qquad q_1\leqslant P^\alpha \quad\text{and}\quad q_2\leqslant P^\alpha.$$

By (4.7), (4.8) and $x\in E_2$

$$(4.9)\qquad \left|\frac{a_2}{q_2}\frac{1}{\lambda_2 x}\right| q_1 q_2 \left|\lambda_1 x-\frac{a_1}{q_1}\right| \leqslant (|\lambda_2 x|+\varepsilon q_2^{-1}P^{-1+\alpha})\,|\lambda_2 x|^{-1} q_2\,\varepsilon P^{-1+\alpha} \leqslant (P^\alpha+\varepsilon|\lambda_2|^{-1})\,\varepsilon P^{-1+\alpha}\leqslant 2\varepsilon P^{-1+2\alpha}.$$

Similarly since $|\lambda_1|\leqslant|\lambda_2|$ we have

$$(4.10)\qquad \left|\frac{a_1}{q_1}\frac{1}{\lambda_2 x} q_1 q_2\left(\lambda_2 x-\frac{a_2}{q_2}\right)\right|\leqslant 2\varepsilon P^{-1+2\alpha}.$$

It follows from (4.9), (4.10), (2.3) that

$$(4.11)\qquad |a_2 q_1\lambda_1/\lambda_2-a_1 q_2|\leqslant 4\varepsilon P^{-1+2\alpha}<\tfrac{1}{2}q^{-1}.$$

https://doi.org/10.1017/S1446788700013458 Published online by Cambridge University Press

By (2.2) for any integers a', q' with $1 \leqslant q' < q$ we have

$$\left| q' \frac{\lambda_1}{\lambda_2} - a' \right| \geqslant q' \left(\frac{|aq' - a'q|}{qq'} - \left| \frac{\lambda_1}{\lambda_2} - \frac{a}{q} \right| \right) > \frac{1}{q} - \frac{q'}{2q^2} > \frac{1}{2q}. \tag{4.12}$$

By (4.11), (4.12), (2.3) and $a_2 \neq 0$ we see that

$$|a_2 q_1| \geqslant q = P^{1-2\alpha}. \tag{4.13}$$

But by (4.7), (4.8), $x \in E_2$

$$\left| \frac{a_2}{q_2} \right| q_1 q_2 \leqslant (|\lambda_2 x| + \varepsilon q_2^{-1} P^{-1+\alpha}) P^{2\alpha} \leqslant 2|\lambda_2| P^{3\alpha}. \tag{4.14}$$

In view of (4.13), (4.14) we have a contradiction since by (2.1), (2.8) $\alpha < A/3 < 1/5$.

LEMMA 9. *If at least two* $\mathfrak{p}_j$ *in* $\Psi(x)$ *are of degree* 1 *then for any positive constant* B *we have*

$$\int_{E_2} |\Psi(x)|\, K_\tau(x)\, dx \ll \tau^2 L^{-B} P^S.$$

PROOF. It is known (Theorem 36, Hardy and Wright (1960)) that for $j = 1, 2$ and each $x \in E_2$ there are integers a_j, q_j with $(a_j, q_j) = 1$ and $1 \leqslant q_j \leqslant P^{1-\alpha} \varepsilon^{-1}$ such that

$$|\lambda_j x - a_j/q_j| \leqslant \varepsilon q_j^{-1} P^{-1+\alpha} \quad (j = 1, 2).$$

By Lemma 8 either $q_1 > P^\alpha$ or $q_2 > P^\alpha$. Let

$$E_{21} = \{x \in E_2 | q_1 > p^\alpha\}; \quad E_{22} = \{x \in E_2 | q_2 > P^\alpha\}.$$

Then

$$\int_{E_2} |\Psi|\, K_\tau\, dx \leqslant \int_{E_{21}} |\Psi|\, K_\tau\, dx + \int_{E_{22}} |\Psi|\, K_\tau\, dx = \mathscr{J}_1 + \mathscr{J}_2, \quad \text{say}. \tag{4.15}$$

By Lemma 7, (2.1), (2.5) we have, for any positive constant $B + C$ and each $x \in E_{2j}$ $(j = 1, 2)$

$$g_j(\lambda_j x) \ll Q_j P^{-\alpha\mu_j} \ll \tau Q_j L^{-(B+C)} \tag{4.16}$$

since in (4.6) $V \geqslant \min(Q_j^{1/3}, \varepsilon P^\alpha) = \varepsilon P^\alpha$ and $\mu_j = ((k_j + 1) 2^{2(k_j+1)})^{-1} \geqslant 1/192$. We come now to estimate $\mathscr{J}_1$. As it is given that among $\mathfrak{p}_h$ $(h \neq 1)$ there is a polynomial of degree 1, for simplicity we let $k_2 = 1$. By (4.16), Hölder's inequality and Lemma 2 we have

$$\begin{aligned} \mathscr{J}_1 &\ll \tau Q_1 L^{-(B+C)} \left(\int_{E_2} |g_2|^2 K_\tau\, dx \right)^{\frac{1}{2}} \left(\int_{E_2} |g_3 g_4|^2 K_\tau\, dx \right)^{\frac{1}{2}} \\ &\ll \tau Q_1 L^{-(B+C)} (\tau L^C Q_2^{(2-1)})^{\frac{1}{2}} (\tau L^C Q_3^{2-(k_3/2)} Q_4^{2-(k_4/2)})^{\frac{1}{2}} \\ &\ll \tau^2 L^{-B} P^S, \end{aligned}$$

where S is defined in (4.3). Similarly,

$$\mathscr{J} \ll \tau^2 L^{-B} P^S.$$

By (4.15) the lemma follows.

LEMMA 10. *Let*

$$\Omega(x) = \sum e(x\omega(y_1, \ldots, y_n)),$$

where ω is any real valued function and the summation is over any finite set of values of $y_1, \ldots, y_n$. Then for any $X > 4/\tau$ we have

$$\int_{|x|>X} |\Omega(x)|^2 K_\tau(x)\,dx \leqslant (8/X\tau) \int_{-\infty}^{\infty} |\Omega(x)|^2 K_\tau(x)\,dx.$$

PROOF. This lemma is due to Davenport and Roth (1955), p. 82. See, for example, Lemma 13 in Vaughan (1974b), p. 394.

LEMMA 11. *For any positive constant B we have*

$$\int_{E_3} |\Psi(x)|\, K_\tau(x)\,dx \ll \tau^2 L^{-B} P^S.$$

PROOF. By Hölder's inequality, (4.2), Lemmas 10, 2, (2.4) and (4.3) we have

$$\begin{aligned}\int_{E_3} |\Psi|\, K_\tau\,dx &\ll (\tau P^\alpha)^{-1} \left(\int_{-\infty}^{\infty} |g_1 g_2|^2 K_\tau\,dx\right)^{\frac12} \left(\int_{-\infty}^{\infty} |g_3 g_4|^2 K_\tau\,dx\right)^{\frac12}\\ &\ll (\tau P^\alpha)^{-1} (\tau L^C Q_1^{2-(k_1/2)}\, Q_2^{2-(k_2/2)})^{\frac12} (\tau L^C Q_3^{2-(k_3/2)}\, Q_4^{2-(k_4/2)})^{\frac12}\\ &\ll L^C P^{-\alpha} P^S \ll \tau^2 L^{-B} P^S\end{aligned}$$

since by (2.1) $\alpha > 2\beta$.

5. Completion of the proof of Theorem 1

LEMMA 12. *For any positive constant B we have*

$$\int_{x\notin E_1} |\Psi^*(x)|\, K_\tau(x)\,dx \ll \tau^2 L^{-B} P^S.$$

PROOF. By (3.8), (2.4), if $|x| > P^{-1+\alpha}$ we have $I_j(x) \ll Q_j^{1-k_j} |x|^{-1}$. Then, by (2.6), (4.3),

$$\int_{x\notin E_1} |\Psi^*|\, K_\tau\,dx \ll \tau^2 P^{3(1-\alpha)} \prod_1^4 Q_j^{1-k_j} \ll \tau^2 L^{-B} P^S.$$

https://doi.org/10.1017/S1446788700013458 Published online by Cambridge University Press

LEMMA. 13. *We have*

$$\int_{-\infty}^{\infty} e(\eta x)\,\Psi^*(x)\,K_\tau(x)\,dx \gg \tau^2 L^{-4} P^S.$$

PROOF. Without loss of generality, let $\lambda_1\lambda_2<0$. Then define the set $\mathscr{B}^*$ by the following conditions (5.1), (5.2), (5.3):

(5.1) $\varepsilon P \leqslant z_j \leqslant 2\varepsilon P \quad (j=3,4), \quad \sqrt{(\varepsilon)}\,|\lambda_1/\lambda_2|\,P \leqslant z_2 \leqslant \sqrt{(\varepsilon)}\,|\lambda_1/\lambda_2|\,P;$

and $z_1>0$ and satisfies

$$(5.2)\qquad \lambda_1\,\mathfrak{p}_1(z_1^{1/k_1}) = y-\eta-\sum_{j=2}^{4}\lambda_j\,\mathfrak{p}_j(z_j^{1/k_j}),$$

for some real y with

$$(5.3)\qquad |y|\leqslant \tfrac{1}{2}\tau.$$

Note that such z_1 is uniquely defined if the right-hand side of (5.2) is large enough. We shall show that

$$\varepsilon P < z_1 < P.$$

Hence if $\mathscr{B}$ denotes the cartesian product of the intervals $\varepsilon^{k_j}P\leqslant z_j\leqslant P$ $(1\leqslant j\leqslant 4)$ then

$$(5.4)\qquad \mathscr{B}\supset\mathscr{B}^*.$$

We see that for large z_j

$$(5.5)\qquad \tfrac{1}{2}\alpha_j z_j < \mathfrak{p}_j(z_j^{1/k_j}) < 2\alpha_j z_j,$$

where α_j is the positive leading coefficient of $\mathfrak{p}_j$. It follows from (5.2), (5.3), (5.5), (5.1) that

$$\begin{aligned}
\mathfrak{p}_1(z_1^{1/k_1}) &\geqslant \tfrac{1}{2}|\lambda_2/\lambda_1|\,\alpha_2 z_2 - \tfrac{1}{2}\tau|\lambda_1|^{-1} - \Big(|\eta| + 2\sum_{j=3}^{4}|\lambda_j|\,\alpha_j z_j\Big)|\lambda_1|^{-1}\\
&\geqslant \sqrt{(\varepsilon)}\,P\left\{\tfrac{1}{2}\alpha_2 - \left((2P\sqrt{(\varepsilon)})^{-1} + |\eta|\,(\sqrt{(\varepsilon)}P)^{-1} + 4\sqrt{(\varepsilon)}\sum_{j=3}^{4}|\lambda_j|\,\alpha_j\right)|\lambda_1|^{-1}\right\}\\
&> \tfrac{1}{3}\sqrt{(\varepsilon)}\,P\alpha_2.
\end{aligned}$$

So by (5.5)

$$z_1 > \mathfrak{p}_1(z_1^{1/k_1})\,(2\alpha_1)^{-1} > \varepsilon P.$$

Similarly, we have $\mathfrak{p}_1(z_1^{1/k_1}) < 5\sqrt{(\varepsilon)}\,P\alpha_2$ and hence $z_1<P$. This proves (5.4).

By Lemma 1, (4.3), (5.4), we have

$$\int_{-\infty}^{\infty} e(x\eta)\,\Psi^* K_\tau\,dx = \int_{-\infty}^{\infty}\left(\prod_{j=1}^{4}\int_{(\varepsilon Q_j)^{k_j}}^{P} e(x\lambda_j\,\mathfrak{p}_j(z_j^{1/k_j}))\,z_j^{(1/k_j)-1}(\log z_j)^{-1}\,dz_j\right) \times e(\eta x)\,K_\tau\,dx$$

https://doi.org/10.1017/S1446788700013458 Published online by Cambridge University Press

$$\gg \prod_{j=1}^{4} P^{(1/k_j)-1} L^{-1} \int_{\mathscr{B}} \max\left(0, \tau - \left|\eta + \sum_{j=1}^{4} \lambda_j \mathfrak{p}_j(z_j^{1/k_j})\right|\right) dz_1 \ldots dz_4$$

$$\gg P^{S-3} L^{-4} \int_{\mathscr{B}^*} \tfrac{1}{2}\tau \, dy \, dz_2 \, dz_3 \, dz_4 \gg \tau^2 L^{-4} P^S.$$

This proves Lemma 13.

We come now to the proof of Theorem 1. By (4.1), Lemma 1, we have

$$\mathscr{I} = \int_{-\infty}^{\infty} e(x\eta)\Psi K_\tau \, dx = \sum_{\substack{\varepsilon Q_j \leqslant p_j \leqslant Q_j \\ 1 \leqslant j \leqslant 4}} \max\left(0, \tau - \left|\eta + \sum_1^4 \lambda_j \mathfrak{p}_j(p_j)\right|\right) \leqslant \tau N,$$

where N is the number of solutions (p_1, p_2, p_3, p_4) of the inequalities $\varepsilon Q_j \leqslant p_j \leqslant Q_j$ $(1 \leqslant j \leqslant 4)$ and $|\eta + \sum_1^4 \lambda_j \mathfrak{p}_j(p_j)| < \tau$. So it suffices to show that $\mathscr{I} \to \infty$ as $P \to \infty$.

By Lemmas 13, 12, 6, we have

$$(5.6) \qquad \int_{E_1} e(x\eta)\Psi K_\tau \, dx = \int_{-\infty}^{\infty} e(x\eta)\Psi^* K_\tau \, dx - \int_{x \notin E_1} e(x\eta)\Psi^* K_\tau \, dx - \int_{E_1} e(x\eta)(\Psi^* - \Psi) K_\tau \, dx$$

$$\gg \tau^2 P^S L^{-4}(1 - L^{-B+4} - L^4 \exp(-L^{1/5})) \gg \tau^2 L^{-4} P^S.$$

It follows from (5.6), Lemmas 9, 11 that

$$\mathscr{I} = \sum_{h=1}^{3} \int_{E_h} e(x\eta)\Psi K_\tau \, dx \gg \tau^2 L^{-4} P^S (1 - 2L^{-B+4}) \gg L^{-4} P^{S-2\beta}.$$

This completes the proof of Theorem 1.

6. Remark

K.W. Lau and the author are able to replace the 1/10 in (1.2) by any constant $< 1/9$ (to appear in *Bull. Austral. Math. Soc.*).

References

A. Baker (1967), 'On some diophantine inequalities involving primes', *J. Reine Angew. Math.* **228**, 166–181.

H. Davenport and H. Heilbronn (1946), 'On indefinite quadratic forms in five variables', *J. London Math. Soc.* **21**, 185–193.

H. Davenport and K. F. Roth (1955), 'The solubility of certain diophantine inequalities', *Mathematika* **2**, 81–96.

G. Hardy and E. M. Wright (1960), *An introduction to the theory of numbers* (4th ed., Clarendon Press, Oxford).

L. K. Hua (1965), *Additive theory of prime numbers*, Translations of Mathematical Monographs, Vol. 13 (Amer. Math. Soc., Providence, R.I.).

A. E. Ingham, 'On the estimation of $N(\sigma, T)$', *Quart. J. Math. Oxford* **11** (1940), 291–292.
M. C. Liu (1977), 'Diophantine approximation involving primes', *J. Reine Angew. Math.* **289**, 199–208.
M. C. Liu (1978), 'Approximation by a sum of polynomials involving primes', *J. Math. Soc. Japan*, **30**, 395–412.
K. Ramachandra (1973), 'On the sums $\sum_{j=1}^{K} \lambda_j f_j(p_j)$', *J. Reine Angew. Math.* **262/263**, 158–165.
W. Schwarz (1963), 'Über die Lösbarkeit gewisser Ungleichungen durch Primzahlen', *J. Reine Angew. Math.* **212**, 150–157.
R. C. Vaughan (1974a), 'Diophantine approximation by prime numbers, I', *Proc. London Math. Soc.* (3) **28**, 373–384.
R. C. Vaughan (1974b), 'Diophantine approximation by prime numbers, II', *Proc. London Math. Soc.* (3) **28**, 385–401.
I. M. Vinogradov (1938), 'A new estimation of a trigonometric sum containing primes' (in Russian with English summary), *Bull. Acad. Sci. USSR, Sér. Math.* **2**, 3–13.

Mathematics Department
University of Hong Kong
Hong Kong