RELATIONS BETWEEN (N, p_n) AND $(\overline{N}, p_n)$ SUMMABILITY

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Let $\{p_n\}$ be a positive sequence. The Nörlund transformation (N, p_n) maps the sequence $\{s_n\}$ into the sequence $\{t_n\}$ by means of the equation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k, \tag{1}$$

where $P_n = \sum_{k=0}^{n} p_k$.

The transformation $(\overline{N}, p_n)$ maps a sequence $\{s_n\}$ into the sequence $\{u_n\}$ by means of the equation

$$u_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k. \tag{2}$$

A matrix method is said to be regular if it is limit preserving for convergent sequences. Necessary and sufficient conditions for the regularity of (1) and (2) are, respectively, $p_n = o(P_n)$ and $P_n \to +\infty$.

Let A and B denote two regular matrix methods, and $A_n(x) = \Sigma_k a_{nk} x_k$, the nth transform of a sequence x, We say that B is stronger than A if

$$A_n(x) \to l \text{ implies } B_n(x) \to l, \ l \text{ finite}. \tag{3}$$

If (3) continues to hold for $l = \pm\infty$, we say that B is totally stronger than A (written B t.s. A).

The purposes of this paper are to extend the theorems of [**8**] to total comparison, and to establish additional properties between the two methods of summability.

For completeness we quote the theorems from [**8**].

Theorem I1. *Suppose that $\{p_n\}$ is positive non-increasing. Then in order that (N, p_n) should include $(\overline{N}, p_n)$, it is necessary and sufficient that* $\inf_n p_n > 0$.

[*Note*. Necessity is not stated by Ishiguro in his main theorem, but is given by his corollary.]

Theorem I2. *If $\{p_n\}$ is non-decreasing, and $p_n = o(P_n)$, then (N, p_n) includes $(\overline{N}, p_n)$.*

Let $\{p_n\}$, $\{q_n\}$ be positive sequences such that $(\overline{N}, p_n)$ and (N, q_n) are regular. We shall first establish conditions for (N, q_n) to be totally stronger than $(\overline{N}, p_n)$. Let $A = (a_{nk})$, $B = (b_{nk})$ be defined by $a_{nk} = p_k/P_n$, $k \leqq n$, $a_{nk} = 0$, $k > n$,

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$b_{nk} = q_{n-k}/Q_n$, $k \leqq n$, $b_{nk} = 0$, $k>n$, $v_n = \Sigma_k b_{nk} s_k$, and u_n as in (2). Then, from (2), $s_n = p_n^{-1}(P_n u_n - P_{n-1} u_{n-1})$, $P_{-1} = 0$, and

$$v_n = \sum_{k=0}^{n} \frac{q_{n-k}}{Q_n}\left(\frac{P_k u_k - P_{k-1} u_{k-1}}{p_k}\right)$$

$$= \frac{1}{Q_n}\sum_{k=0}^{n-1}\left(\frac{q_{n-k}}{p_k} - \frac{q_{n-k-1}}{p_{k+1}}\right) P_k u_k + \frac{q_0 P_n u_n}{p_n Q_n}. \tag{4}$$

Therefore $B = DA$, where

$$d_{nk} = \left(\frac{q_{n-k}}{p_k} - \frac{q_{n-k-1}}{p_{k+1}}\right)\frac{P_k}{Q_n}, \quad k<n;$$

$$d_{nn} = \frac{q_0 P_n}{p_n Q_n}; \quad d_{nk} = 0, \ k>n.$$

From Hurwitz [7], D will be totally regular if and only if there exists an integer k_0 such that $d_{nk} \geqq 0$ for all $k>k_0$; i.e.,

$$\frac{q_{n-k}}{p_k} - \frac{q_{n-k-1}}{p_{k+1}} \geqq 0 \text{ for all } n>k>k_0.$$

(Note that the regularity of (N, q_n) guarantees that $\lim_n d_{nk} = 0$ for each k.) Observing that $n-k$ may be any positive integer we can formalize these remarks as

Lemma 1. *Let $\{p_n\}$, $\{q_n\}$ be positive sequences satisfying* (i) $P_n \to +\infty$, (ii) $q_n = o(Q_n)$. *Then (N, q_n) t.s. $(\overline{N}, p_n)$ if and only if*

$$\frac{p_{k+1}}{p_k} \geqq a = \max_{m \geqq 1}\left(\frac{q_{m-1}}{q_m}\right), \quad k \geqq k_0. \tag{5}$$

Theorem 1. *Let $\{p_n\}$ be a real positive sequence satisfying $P_n \to +\infty$ and $p_n = o(P_n)$. Then (N, p_n) t.s. $(\overline{N}, p_n)$ if and only if $p_{n+1} \geqq p_n$ for all n.*

Proof. Consider Lemma 1 for $q_n = p_n$. If $p_{n+1} \geqq p_n$ for all n, then $a \leqq 1$, and (5) is satisfied.

To show the converse, suppose there exists an integer n for which $p_{n+1} < p_n$. Then $a>1$, and the condition $p_{k+1}/p_k \geqq a$ violates $p_n = o(P_n)$.

Theorem 1 includes Theorem I2 mentioned above, and corrects and strengthens the statement of Theorem 1 of [**14**].

An alternate proof of the sufficiency of Theorem 1 is the following.

Let $r_n = 1$ for all n. Then $(N, r_n) = (\overline{N}, r_n) = (C, 1)$. Using the proof of [**6**, Theorem 20, p. 67] or [**4**, Theorem 2, p. 136], (N, p_n) t.s. $(C, 1)$. Using a result of [**12**], $(C, 1)$ t.s. $(\overline{N}, p_n)$. Since t.s. is transitive, (N, p_n) t.s. $(\overline{N}, p_n)$.

Theorem 2. *Let $\{p_n\}$ be a non-increasing positive sequence. Then $(\overline{N}, p_n)$ t.s. (N, p_n) if and only if $P_n \to \infty$ as $n \to \infty$.*

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Proof. Suppose $(\bar{N}, p_n)$ t.s. (N, p_n). Since $\{p_n\}$ is non-increasing, $p_n = o(P_n)$ and (N, p_n) is regular. Also the sequence $\{p_n\}$ is positive. Therefore (N, p_n) is totally regular. It then follows (see, e.g. [15, Theorem 2.2, p. 398]) that $(\bar{N}, p_n)$ must be totally regular, hence regular. Thus, we must have $P_n \to \infty$ as $n \to \infty$.

To show that the condition is sufficient, we shall make use of the following lemma.

Lemma 2. *Let $\{p_n\}$ be a non-increasing positive sequence such that $P_n \to \infty$ as $n \to \infty$. Write*

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad q(z) = \frac{1}{p(z)} = \sum_{n=0}^{\infty} q_n z^n$$

($1/p(z)$ is clearly regular in some *neighbourhood of the origin and thus can be expanded in a power series),*

$$Q(z) = \frac{q(z)}{1-z} = \sum_{n=0}^{\infty} Q_n z^n, \quad where \quad Q_n = \sum_{k=0}^{n} q_k.$$

Then $Q_n = o(n)$.

The method of proof is suggested by examining a paper of Krishnaiah [9], particularly equation (4) on page 316. For $|z| < 1$,

$$(1-z)p(z) = \sum_{n=1}^{\infty} (p_{n-1} - p_n)(1 - z^n) + p_\infty,$$

where $p_\infty = \lim_n p_n$. If we write $z = re^{i\theta}$,

$$\begin{aligned} \mathscr{R}\{(1-z)p(z)\} &= \sum_{n=1}^{\infty} (p_{n-1} - p_n)(1 - r^n \cos n\theta) + p_\infty \\ &\geqq \sum_{n=1}^{\infty} (p_{n-1} - p_n)(1 - r^n) + p_\infty \\ &= (1-r)p(r). \end{aligned}$$

Thus $|(1-z)p(z)| \geqq (1-r)p(r)$. Since $P_n \to \infty$, $p(r) \to \infty$ as $r \to 1-$. Hence

$$|Q(z)| = o\left(\frac{1}{1-r}\right)$$

uniformly in arg z as $r = |z| \to 1-$. Since

$$Q_n = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{Q(z)}{z^{n+1}} dz,$$

where Γ_n denotes a circle of radius $1 - n^{-1}$, centre the origin, the conclusion follows.

Using the notation of Lemma 2, and (1), we may write $s_n = \sum_{k=0}^{n} q_{n-k} P_k t_k$.

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Substituting in (2), we have

$$u_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r \sum_{k=0}^{r} q_{r-k} P_k t_k$$
$$= \sum_{k=0}^{n} \beta_{nk} t_k,$$

where

$$\beta_{nk} = \frac{P_k}{P_n} \sum_{r=k}^{n} p_r q_{r-k}. \tag{6}$$

The theorem will be proved provided we can show that the matrix (β_{nk}) is totally regular. We first appeal to the following

Lemma K [10, p. 488]. *Let* $t_n = \sum_{k=0}^{n} \alpha_{nk} u_k$ *(the matrix* $C = (\alpha_{nk})$ *not necessarily regular), with* $\alpha_{nn} \neq 0$ *for all n. Denote the inverse transformation, which exists, by* $u_n = \sum_{k=0}^{n} \beta_{nk} t_k$. *If, for all n,* $\alpha_{nn} > 0$, $\alpha_{nk} \leqq 0$ $(0 \leqq k < n)$, *then* $\beta_{nk} \geqq 0$ *for all n, k.*

The matrix corresponding to (N, p_n) $(\bar{N}, p_n)^{-1}$, which is given by (4) with $q_n = p_n$, satisfies the conditions of Lemma *K*.

Moreover, $\sum_{k=0}^{n} |\beta_{nk}| = \sum_{k=0}^{n} \beta_{nk} = 1$. Therefore $B^{-1} = (\beta_{nk})$ has finite norm, and it only remains to show that

$$\lim_n \beta_{nk} = 0 \text{ for each } k; \text{ i.e., } \sum_{r=k}^{n} p_r q_{r-k} = o(P_n).$$

We may write

$$\sum_{r=k}^{n} p_r q_{r-k} = \sum_{r=k}^{n-1} Q_{r-k}(p_r - p_{r+1}) + Q_{n-k} p_n. \tag{7}$$

From Lemma 2 (k being fixed), for each $\varepsilon > 0$ there exists a natural number r_0 such that $|Q_{r-k}| \leqq \varepsilon(r-k)$ for $r > r_0$. The sum of those terms on the right of (7) for which $r \leqq r_0$ is fixed; since $P_n \to \infty$, this sum is $o(P_n)$. Thus, for all n sufficiently large,

$$\left| \sum_{r=k}^{n} p_r q_{r-k} \right| \leqq \varepsilon \left\{ \sum_{r=r_0+1}^{n-1} (r-k)(p_r - p_{r+1}) + (n-k)p_n \right\} + o(P_n)$$
$$= \varepsilon \{ P_n - P_{r_0+1} + (r_0 + 1 - k) p_{r_0+1} \} + o(P_n)$$
$$= \varepsilon P_n + o(P_n).$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{P_n} \left| \sum_{r=k}^{n} p_r q_{r-k} \right| < \varepsilon,$$

and since ε is arbitrary, the conclusion follows.

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Two regular matrix methods A and B are said to be equivalent if A is stronger than B and B is stronger than A. Combining Theorems 2 and I1 we have the following

Corollary 1. *If $\{p_n\}$ is non-increasing and $p_n \geqq \sigma > 0$, $n = 0, 1, 2, \ldots$, then (N, p_n) and $(\bar{N}, p_n)$ are equivalent.*

A method B is said to be strictly stronger than A if B is stronger than A, but there is a sequence x for which $\lim_n B_n(x)$ exists and $\lim_n A_n(x)$ does not. For triangles (that is, infinite matrices with all elements zero above the main diagonal, and all main diagonal elements non-zero) it is well known that the condition B is strictly stronger than A is equivalent to (i) BA^{-1} is regular, and (ii) AB^{-1} has infinite norm.

Corollary 2. *Let $\{p_n\}$ be a positive non-decreasing sequence with $p_n = o(P_n)$. Then, if $\sup_n p_n = +\infty$, (N, p_n) is strictly stronger than $(\bar{N}, p_n)$.*

Proof. Note that, from the hypotheses on $\{p_n\}$, $P_n \geqq (n+1)p_0$ and hence $P_n \to +\infty$.

With $A = (p_{n-k}/P_n)$, $B = (p_k/P_n)$, and $C = AB^{-1}$, then $c_{nn} = p_0/p_n$.

$$\| C^{-1} \| \geqq \sup_n | 1/c_{nn} | = \sup_n | p_n/p_0 | = +\infty,$$

and (N, p_n) is strictly stronger than $(\bar{N}, p_n)$.

Corollary 3. *Let $\{p_n\}$ be a positive non-decreasing sequence with $p_n \to c$, where $c < 2p_0$. Then (N, p_n) and $(\bar{N}, p_n)$ are equivalent.*

Proof. Note that the hypotheses on $\{p_n\}$ not only ensure that $P_n \to +\infty$, as in the proof of Corollary 2, but also that $p_n = o(P_n)$.

We shall need the following result from **[2]**, which also appears in **[13]**.

Theorem A. *Let C denote a regular triangle. If*

$$\liminf_n \{| c_{nn} | - \sum_{k<n} | c_{nk} |\} > \lambda > 0,$$

then C is equivalent to convergence.

If we let C be as defined in Corollary 2, then C is regular, because (N, p_n) is stronger than $(\bar{N}, p_n)$. From **[8**, p. 122],

$$| c_{nn} | - \sum_{k<n} | c_{nk} | = \frac{p_0}{p_n} - \left(1 - \frac{p_0}{p_n}\right) = \frac{2p_0}{p_n} - 1,$$

and the result follows since $c < 2p_0$.

The condition on c in Corollary 3 is the best possible. For, let $p_0 = 1$, $p_n = c > 1$ for $n > 0$. Then, with the notation used in the proof of Lemma 2, $p(z) = (1+(c-1)z)/(1-z)$, giving us $Q(z) = [(1-z)p(z)]^{-1} = 1/(1+(c-1)z)$, so that $Q_n = (-1)^n(c-1)^n$. For $k \geqq 1$, (6) becomes

$$\beta_{nk} = \frac{cP_k}{P_n} \sum_{r=k}^{n} q_{r-k} = \frac{cP_k}{P_n} Q_{n-k}$$

$$= \frac{(-1)^{n-k} c(ck+1)(c-1)^{n-k}}{(cn+1)}. \tag{8}$$

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Using (8) one can demonstrate that the transformation is not regular when $c \geqq 2$.

Two matrices A and B are said to be totally equivalent if and only if A t.s. B and B t.s. A. Lorch **[12]** has shown that $(\overline{N}, q_n)$ t.s. $(\overline{N}, p_n)$ if and only if $q_{n+1}/q_n \leqq p_{n+1}/p_n$ for almost all n. Therefore, if $(\overline{N}, p_n)$, $(\overline{N}, q_n)$ are totally equivalent, there exists an integer m such that

$$\frac{p_m}{q_m} = \frac{p_{m+1}}{q_{m+1}} = \frac{p_{m+2}}{q_{m+2}} = \dots;$$

i.e., $p_k = cq_k$ for all $n > m$, and $c = p_m/q_m$.

Formalizing these remarks we have

Theorem 3. *Let $\{p_n\}$, $\{q_n\}$ be positive sequences, with $P_n \to +\infty$, $Q_n \to +\infty$. Then $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ are totally equivalent if and only if $p_n = cq_n$ for almost all n, and some constant c.*

Theorem 4. *The methods (N, p_n) and $(\overline{N}, p_n)$ are identical if and only if $p_n = \text{const.}$ for all n.*

Proof. By hypothesis $p_{n-k}/P_n = p_k/P_n$, which implies $p_{n-k} = p_k$ for $0 \leqq k \leqq n$; i.e., $p_k = p_0$ for all $k > 0$. The converse is trivial.

Ullrich **[16]** showed that the only Nörlund matrices which are also Hausdorff matrices are the Cesàro matrices. Agnew re-proved this result in **[1]**. It is of interest to note which matrices of the form $(\overline{N}, p_n)$ are also Hausdorff matrices.

Theorem 5. *The only $(\overline{N}, p_n)$ matrices that are also Hausdorff matrices are those of the form $p_n = 0$ for $n > 0$ or $p_n = \Gamma(n+a)/\Gamma(n+1)\Gamma(a)$, $a > 0$.*

Proof I. Let A denote the matrix corresponding to a $(\overline{N}, p_n)$ method. Assume also that A is a Hausdorff matrix generated by a sequence μ. Then

$$a_{nn} = \frac{p_n}{P_n} = \mu_n,\ a_{n,n-1} = \frac{p_{n-1}}{P_n} = n\Delta\mu_{n-1} = n(\mu_{n-1} - \mu_n).$$

We may write p_{n-1}/P_n in the form $(p_{n-1}/P_{n-1})(P_{n-1}/P_n) = \mu_{n-1}(1-\mu_n)$. We then have $\mu_{n-1}(1-\mu_n) = n(\mu_{n-1} - \mu_n)$, or

$$(n-1)\mu_{n-1} = (n - \mu_{n-1})\mu_n. \tag{9}$$

Let $\mu_0 = c$. Then, from (9), $(1-\mu_0)\mu_1 = 0$, and either $\mu_1 = 0$ or $\mu_0 = 1$. If $\mu_1 = 0$, then $\mu_n = 0$ for all $n > 1$, and the sequence is $p_0 = \mu_0 = c$, $p_n = \mu_n = 0$, $n > 0$. If $\mu_1 \neq 0$, then we must have $\mu_0 = 1$. μ_1 can then be arbitrary. Let $\mu_1 = \alpha \neq 0$. For all $n > 1$, from (9)

$$\mu_n = \frac{(n-1)\mu_{n-1}}{n - \mu_{n-1}},$$

or

$$\mu_n = \frac{\alpha}{n-(n-1)\alpha} = \frac{\alpha}{(1-\alpha)n+\alpha} = \frac{a}{n+a}, \tag{10}$$

where $a = \alpha/(1-\alpha)$. We cannot have $\alpha = 1$, for then, from (10), we would have $\mu_n = 1$ for all n, giving rise to the identity matrix. But it is impossible to generate the identity matrix with a $(\bar{N}, p_n)$ method.

A straightforward calculation will verify that the sequence $\{p_n\}$ corresponding to (10) is $p_n = \Gamma(n+a)/\Gamma(n+1)\Gamma(a)$, hence the restriction that $a>0$.

Proof II. Associated with any triangular transformation

$$t_n = \sum_{k=0}^{n} \alpha_{nk}s_k \tag{11}$$

is the " reverse " transformation $u_n = \sum_{k=0}^{n} \alpha_{n,\,n-k}s_k$, formed by reversing the order of the elements on each row of the matrix corresponding to (11). Using the elementary properties of the forward difference operator Δ, defined by $\Delta u_n = u_n - u_{n+1}$, it is easy to show that the reverse of any Hausdorff method (H, μ) is a Hausdorff method (H, λ), where $\lambda_n = \Delta^n \mu_0$. Since the reverse of a $(\bar{N}, p_n)$ method is (N, p_n), it follows that a matrix is both $(\bar{N}, p_n)$ and Hausdorff if and only if the matrix of the reverse transformation is both Nörlund and Hausdorff. The result of Theorem 5 can then be deduced directly from the results of [**1**] and [**16**].

An analogous result relating Hausdorff matrices and generalized Norlund methods (N, p, q) (see [**3**] for the definition of (N, p, q)) appears in [**11**].

The following theorem appears in [**14**], where Γ_c' denotes the Hausdorff matrix generated by $\mu_n = c/(n+c)$.

Theorem R. *Let $\{p_n\}$ be a sequence of positive numbers such that*

$$(k+c)p_k > (k+1)p_{k+1}, \quad c>0,$$

for almost all k. Then $(\bar{N}, p_n)$ t.s. Γ_c', but not conversely.

In light of Theorem 5 we observe that Γ_c' is a $(\bar{N}, p_n)$ method with p_n as described in the discussion following equation (10). The theorem then follows immediately from the result of [**12**] quoted earlier.

For completeness we point out that Dikshit [**5**] has established a number of theorems comparing the relative strengths of the (N, p_n) and $(\bar{N}, q_n)$ methods for both ordinary and absolute summability. His principal result is the following. Let $q_n>0$, $p_n \geqq 0$, $p_0>0$, $Q_n \to +\infty$, $p_n = o(P_n)$. Then (N, p_n) includes $(\bar{N}, q_n)$ if and only if

$$\sum_{k=0}^{n} |\Delta_k(p_{n-k}Q_k/q_k)| = O(P_n),\ p_{-1} = 0.$$

However, he does not consider questions of total inclusion, and so there is no overlap in content with this paper.

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