Proceedings of the Edinburgh Mathematical Society (2023) **66**, 862–867
doi:10.1017/S0013091523000433

EVERY SALEM NUMBER IS A DIFFERENCE OF TWO PISOT NUMBERS

ARTŪRAS DUBICKAS

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania (arturas.dubickas@mif.vu.lt)

(Received 17 October 2022)

Abstract In this note, we prove that every Salem number is expressible as a difference of two Pisot numbers. More precisely, we show that for each Salem number α of degree d, there are infinitely many positive integers n for which $\alpha^{2n-1} - \alpha^n + \alpha$ and $\alpha^{2n-1} - \alpha^n$ are both Pisot numbers of degree d and that the smallest such n is at most $6^{d/2-1} + 1$. We also prove that every real positive algebraic number can be expressed as a quotient of two Pisot numbers. Earlier, Salem himself had proved that every Salem number can be written in this way.

Keywords: Salem number; Pisot number; Dirichlet's approximation theorem

2020 Mathematics subject classification: 11R04; 11R06; 11J72

1. Introduction

Recall that a *Salem number* is a real algebraic integer $\alpha > 1$ whose conjugates over $\mathbb{Q}$ except for α itself all lie in the disc $|z| \leq 1$ with at least one conjugate lying on the boundary $|z| = 1$. The Salem number α is reciprocal, so it has even degree $d \geq 4$ over $\mathbb{Q}$, the conjugate α^{-1} and $d-2$ unimodular conjugates of the form $\mathrm{e}^{\pm \mathrm{i}\phi_j}$, $j = 1, \ldots, d/2-1$, where $0 < \phi_1 < \cdots < \phi_{d/2-1} < \pi$. A *Pisot number* is a real algebraic integer greater than 1 whose other conjugates over $\mathbb{Q}$ (if any) all lie in the open disc $|z| < 1$.

Various properties of Salem numbers have been investigated in [6–8, 12, 13, 15, 17] (see also a survey [16]), while their relations with Pisot numbers have been explored in, for example, [1, 2, 5, 9, 10, 18, 19]. For example, an old result of Salem [12] asserts that every Pisot number is a limit point of the set of Salem numbers. In [14], Siegel showed that the smallest Pisot number is the root $\theta = 1.3247\ldots$ of $x^3 - x - 1 = 0$, while the smallest Salem number is not known, and it is not even known whether the set of Salem numbers is bounded away from 1.

In [5], the author investigated various sumsets and difference sets involving Salem and Pisot numbers. In this note, we will prove the following new result in this direction.

Theorem 1. *Every Salem number is expressible as a difference of two Pisot numbers.*

More explicitly, we will show the following:

Theorem 2. *For each Salem number α of degree $d \geq 4$, there exist infinitely many $n \in \mathbb{N}$ for which $\alpha^{2n-1} - \alpha^n + \alpha$ and $\alpha^{2n-1} - \alpha^n$ are both Pisot numbers of degree d. The smallest such n is at most $6^{d/2-1} + 1$.*

In [12, p. 69] (see also [13, p. 35]), Salem himself proved that every Salem number is expressible as a quotient of two Pisot numbers. On the other hand, the author showed that every positive algebraic number is a quotient of two Mahler measures [4, Theorem 1]. Recall that the *Mahler measure* $M(\alpha)$ of a non-zero algebraic number α is the modulus of the product of its conjugates lying outside the unit circle and the leading coefficient of its minimal polynomial in $\mathbb{Z}[x]$. Thus, for a real algebraic number $\alpha > 1$, we have $M(\alpha) \geq \alpha$ with equality if and only if α is a Salem or a Pisot number. Therefore, the following theorem generalizes both these results.

Theorem 3. *Every real positive algebraic number α of degree d is expressible as a quotient of two Pisot numbers of degree d from the field $\mathbb{Q}(\alpha)$.*

In the next section, we will recall a few simple results, which will be used in the proofs. Then, in § 3, we will prove Theorems 2 and 3. Evidently, Theorem 2 implies Theorem 1.

2. Auxiliary results

In the proof of Theorem 2, we will use the next version of Dirichlet's approximation theorem [4] (see, e.g., [11, p. 423]).

Lemma 4. *Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers. Then, for each $Q > 1$, there is a positive integer $q \leq Q$ such that*

$$\|\lambda_j q\| < Q^{-1/N}$$

for $j = 1, 2, \dots, N$.

Throughout, $\|y\|$ stands for the distance between $y \in \mathbb{R}$ and the nearest integer.

Let α be a Salem number of degree $d \geq 4$ with conjugates α^{-1} and $e^{\pm i\phi_j}$, $j = 1, \dots, N$, over $\mathbb{Q}$, where $0 < \phi_1 < \cdots < \phi_N < \pi$ and $d = 2N + 2$. In [13, p. 32], Salem showed that the numbers $\pi, \phi_1, \dots, \phi_N$ are linearly independent over $\mathbb{Q}$ (the argument is attributed to Pisot). In particular, Salem's result implies that

Lemma 5. *The numbers ϕ_j/π, $j = 1, \dots, N$, are all irrational.*

Note that in case $\phi_j/\pi \in \mathbb{Q}$, the conjugate $e^{i\phi_j}$ of a Salem number must be a root of unity, which is impossible, because all the conjugates of a root of unity over $\mathbb{Q}$ must be roots of unity themselves, but Salem number is not a root of unity. This also implies Lemma 5.

Next, we record the following observation:

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Lemma 6. *Let α be a real algebraic number of degree $d \geq 2$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ over $\mathbb{Q}$, and let f be a non-constant polynomial with rational coefficients such that $f(\alpha) > 0$ and $|f(\alpha_j)| < 1$ for $j = 2, \ldots, d$. If $f(\alpha) \in \mathbb{Q}(\alpha)$ is an algebraic integer, then it is a Pisot number of degree d.*

Proof. Note that

$$f(\alpha), f(\alpha_2), \ldots, f(\alpha_d)$$

is the list of conjugates of an algebraic integer $f(\alpha)$ over $\mathbb{Q}$, possibly repeated several times. In particular, this implies that $f(\alpha_j) \neq 0$ for $j = 2, \ldots, d$. Furthermore, $f(\alpha) \geq 1$, since otherwise $0 < f(\alpha) < 1$, and hence there is a non-zero algebraic integer $f(\alpha)$ with all conjugates in $|z| < 1$, including $f(\alpha)$. But then the modulus of the product of the conjugates of $f(\alpha)$ must be smaller than 1, which is impossible. Also, if $f(\alpha) = 1$, then its conjugates $f(\alpha_j)$, $j = 2, \ldots, d$, are all equal to 1, which is not the case. Consequently, $f(\alpha) > 1$. Since $f(\alpha)$ is the only conjugate of $f(\alpha)$ outside the unit circle, all $f(\alpha_j)$, $j = 2, \ldots, d$, lying in $|z| < 1$ must be distinct, whence the result. □

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let α be a Salem number of degree $d \geq 4$ with conjugates $\alpha_2 = \alpha^{-1}$ and $\{\alpha_3, \ldots, \alpha_d\} = \{e^{\pm i\phi_1}, \ldots, e^{\pm i\phi_N}\}$, where $N = d/2 - 1$. Applying Lemma 4 to the N irrational numbers $\lambda_1 = \phi_1/(2\pi), \ldots, \lambda_N = \phi_N/(2\pi)$ (see Lemma 5), we derive that for any $Q > 1$, there is an integer q in the range $1 \leq q \leq Q$ for which

$$0 < \|q\phi_j/(2\pi)\| < Q^{-1/N} = Q^{-2/(d-2)}. \tag{1}$$

Put $n = q + 1$ and consider the numbers

$$\beta = \alpha^{2n-1} - \alpha^n + \alpha \quad \text{and} \quad \gamma = \alpha^{2n-1} - \alpha^n. \tag{2}$$

We will show that β and γ are both Pisot numbers of degree d in the field $\mathbb{Q}(\alpha)$, provided that

$$Q^{-2/(d-2)} \leq \frac{1}{6}, \tag{3}$$

that is, $Q \geq 6^{d/2-1}$. Of course, by letting $Q \to \infty$ in Equation (1), we will produce infinitely many q satisfying Equation (1), and so infinitely many $n \in \mathbb{N}$ for which $\beta, \gamma \in \mathbb{Q}(\alpha)$ defined in Equation (2) are both Pisot numbers of degree d.

We begin with the number $\gamma = f(\alpha)$, where $f(x) = x^{2n-1} - x^n$ due to Equation (2). First, $\gamma = f(\alpha) > 0$ is an algebraic integer lying in the field $\mathbb{Q}(\alpha)$. In order to apply Lemma 6, we need to show that $|f(\alpha_j)| < 1$ for $j = 2, \ldots, d$.

Observe that, by Equation (2),

$$f(\alpha_2) = f(\alpha^{-1}) = \alpha^{-2n+1} - \alpha^{-n}.$$

It is clear that $-1 < \alpha^{-2n+1} - \alpha^{-n} < 0$ because $\alpha > 1$. So $f(\alpha_2)$ lies in $|z| < 1$. Next, fix a conjugate $\alpha' = e^{\pm i\phi_j}$ of α. It remains to check that for any choice of the sign $\pm$ the

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number

$$f(\alpha') = (\alpha')^{2n-1} - (\alpha')^n = \mathrm{e}^{\pm \mathrm{i}\phi_j n}(\mathrm{e}^{\pm \mathrm{i}\phi_j (n-1)} - 1) = \mathrm{e}^{\pm \mathrm{i}\phi_j (q+1)}(\mathrm{e}^{\pm \mathrm{i}\phi_j q} - 1)$$

lies in $|z| < 1$. In view of $|f(\alpha')| = 2|\sin(q\phi_j/2)|$, this is equivalent to $|\sin(q\phi_j/2)| < 1/2$. This happens if and only if

$$|q\phi_j/2 - \pi k| < \pi/6$$

for some $k \in \mathbb{Z}$ or, equivalently, $\|q\phi_j/(2\pi)\| < 1/6$, which is indeed the case by Equations (1) and (3). This completes our verification. Therefore, $\gamma = f(\alpha) \in \mathbb{Q}(\alpha)$ is a Pisot number of degree d by Lemma 6.

Now, let us consider the number $\beta = f(\alpha)$ defined in Equation (2), where $f(x) = x^{2n-1} - x^n + x$. It is clear that $f(\alpha) > \alpha > 1$ is an algebraic integer. This time, we find that

$$f(\alpha_2) = f(\alpha^{-1}) = \alpha^{-2n+1} - \alpha^{-n} + \alpha^{-1}.$$

In view of $\alpha > 1$ and $n \geq 2$, we obtain $0 < \alpha^{-2n+1} - \alpha^{-n} + \alpha^{-1} < 1$, so $f(\alpha_2)$ is in $|z| < 1$. Next, as above, fix a conjugate $\alpha' = \mathrm{e}^{\pm \mathrm{i}\phi_j}$ of α. This time, we need to show that for any choice of the sign $\pm$ the number

$$\begin{aligned} f(\alpha') &= (\alpha')^{2n-1} - (\alpha')^n + \alpha' = \mathrm{e}^{\pm \mathrm{i}\phi_j n}\left(\mathrm{e}^{\pm \mathrm{i}\phi_j (n-1)} - 1 + \mathrm{e}^{\mp \mathrm{i}\phi_j (n-1)}\right) \\ &= e^{\pm i\phi_j (q+1)}(2\cos(q\phi_j) - 1) \end{aligned}$$

lies in the open disc $|z| < 1$. This is true if and only if $0 < \cos(q\phi_j) < 1$. The latter inequalities hold whenever

$$0 < |q\phi_j - 2\pi k| < \pi/2$$

for some $k \in \mathbb{Z}$ or, equivalently, $0 < \|q\phi_j/(2\pi)\| < 1/4$. This is true by Equations (1), (3) and $1/6 < 1/4$. As before, by Lemma 6, we conclude that $\beta = f(\alpha) > 1$ is a Pisot number of degree d.

Finally, selecting $Q = 6^{d/2-1}$, by Equations (1) and (3), we see that the smallest $q \in \mathbb{N}$ for which Equation (1) is true satisfies $1 \leq q \leq 6^{d/2-1}$. This completes the proof of the last assertion of the theorem because the integer $n = q + 1$ is in the range $2 \leq n \leq 6^{d/2-1} + 1$. □

Proof of Theorem 3. Let α be a positive algebraic number of degree d over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. The claim is trivial for $d = 1$, since every integer $k \geq 2$ is a Pisot number and every positive rational number is a quotient of two such numbers. Assume that $d \geq 2$, and let m be a positive integer for which $m\alpha$ is an algebraic integer.

Fix a positive number $u < 1$ satisfying

$$mu \max(1, |\alpha_2|, \ldots, |\alpha_d|) < 1, \tag{4}$$

and a positive number $v > 1$ satisfying

$$mv\alpha > 1. \tag{5}$$

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Select a Pisot number $\beta \in \mathbb{Q}(\alpha)$ of degree d (see Theorem 2 in [13, p. 3]). A natural power of β is also a Pisot number of degree d, so by replacing β by its large power if necessary, we can assume that $\beta > v$ and that the other $d-1$ conjugates of β over $\mathbb{Q}$ are all in $|z| < u$.

Write this β in the form $\beta = f(\alpha)$, where f is a non-constant polynomial of degree at most $d-1$ with rational coefficients. Then, the numbers $\beta_j = f(\alpha_j)$, $j = 1, \dots, d$, are the conjugates of $\beta = \beta_1$ over $\mathbb{Q}$. Recall that, by the choice of β, we have

$$\beta = f(\alpha) > v \quad \text{and} \quad |\beta_j| = |f(\alpha_j)| < u \quad \text{for } j = 2, \dots, d.$$

We claim that under assumption on the constants $u \in (0,1)$ as in Equation (4) and $v > 1$ as in Equation (5), the numbers $m\alpha\beta \in \mathbb{Q}(\alpha)$ and $m\beta \in \mathbb{Q}(\alpha)$ are both Pisot numbers of degree d. This will complete our proof, since their quotient is α.

First, $m\beta$ is a Pisot number, since it is an algebraic integer greater than $m > 1$, whose other conjugates $m\beta_j$, $j = 2, \dots, d$, all lie in $|z| < 1$ by $|\beta_j| < u$ and Equation (4). Of course, $m\beta \in \mathbb{Q}(\alpha)$ is of degree d over $\mathbb{Q}$, since so is β.

Second, the number $m\alpha\beta = m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a positive algebraic integer, since so are $m\alpha$ and β. It is greater than 1 by $\beta > v$ and Equation (5). Its other conjugates are $m\alpha_j f(\alpha_j) = m\alpha_j\beta_j$, $j = 2, \dots, d$. They are all in $|z| < 1$ due to $|\beta_j| < u$ and Equation (4). Hence, $m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a Pisot number of degree d over $\mathbb{Q}$ by Lemma 6 applied to the polynomial $mxf(x) \in \mathbb{Q}[x]$.

Therefore, $m\alpha\beta \in \mathbb{Q}(\alpha)$ and $m\beta \in \mathbb{Q}(\alpha)$ indeed are both Pisot numbers of degree d, which finishes the proof. □

Competing Interests. The author declares none.

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