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A NEW CONSTRUCTION FOR REGULAR SEMIGROUPS WITH QUASI-IDEAL ORTHODOX TRANSVERSALS

XIANGJUN KONG ✉ and XIANZHONG ZHAO

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Abstract

In any regular semigroup with an orthodox transversal, we define two sets R and L using Green's relations and give necessary and sufficient conditions for them to be subsemigroups. By using R and L, some equivalent conditions for an orthodox transversal to be a quasi-ideal are obtained. Finally, we give a structure theorem for regular semigroups with quasi-ideal orthodox transversals by two orthodox semigroups R and L.

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1. Introduction and preliminaries

The concept of inverse transversal of a regular semigroup was first introduced by Blyth and McFadden in 1982 [3]. Since then, this class of regular semigroups has attracted several authors' attention and a series of important results have been obtained [1–3, 8–11]. If S is a regular semigroup, then an inverse transversal of S is an inverse subsemigroup S^o such that S^o meets $V(a)$ precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S \mid axa = a \text{ and } xax = x\}$ denotes the set of inverses of a. The intersection of $V(a)$ and S^o is denoted by $V_{S^o}(a)$ and the unique element of $V_{S^o}(a)$ is denoted by a^o. It is well known that the sets $I = \{e \in S \mid ee^o = e\}$ and $\Lambda = \{f \in S \mid f^o f = f\}$ are left regular and right regular bands, respectively, and play an important role in the study of regular semigroups with inverse transversals. Other interesting subsets of S are $R = \{x \in S \mid x^o x = x^o x^{oo}\}$ and $L = \{x \in S \mid xx^o = x^{oo} x^o\}$. Both R and L are subsemigroups with R left inverse (or $\mathcal{R}$-unipotent) and L right inverse (or $\mathcal{L}$-unipotent). Moreover, $R \cap L = S^o$ and $E(R) = I$, $E(L) = \Lambda$, where $E(S)$ denotes the idempotents of S. By using R and L, Saito [9, 10] gave some structure theorems of regular semigroups with inverse transversals, while Blyth and Almeida Santos [1, 2] classified the inverse transversals and gave some equivalent conditions for the inverse transversal S^o to be a quasi-ideal (defined below). Orthodox transversals were introduced by Chen [4] as a generalization of inverse transversals, and an excellent structure theorem for regular semigroups with quasi-ideal orthodox transversals was also given. Afterwards, Chen and Guo [5] considered the general case of orthodox transversals and investigated some properties concerning the sets I and Λ. Similarly two sets R and L (defined below) are shown to play an important role in the study of orthodox transversals. In this paper, we investigate some properties concerning R and L, and obtain some results that are parallel to the corresponding results on regular semigroups with inverse transversals. The main objective of this paper is to give a structure theorem for the class of regular semigroups with quasi-ideal orthodox transversals.

In a previous publication [7] we constructed regular semigroups with quasi-ideal orthodox transversals by a formal set (B, R), where R is a regular semigroup with a right ideal orthodox transversal S^o and B a band with a left ideal orthodox (in fact, band) transversal E^o. Evidently, there are different conditions on the structural 'brick' B and R. The present paper corrects this asymmetry by giving a new construction of regular semigroups with quasi-ideal orthodox transversals by way of two regular semigroups R and L. The semigroups R and L share a common orthodox transversal S^o, which is a right ideal of R and a left ideal of L. Many of the conditions on R and L are symmetric and one is weaker than that in [7] (that is, if $x \in S^o$ or $a \in S^o$ then $a * x = ax$ in this paper; instead of if $x \in E^o$ or $e \in E^o$, then $e * x = ex$ in [7]).

Let S be a semigroup and S^o a subsemigroup of S. Then S^o is said to be an orthodox transversal of S if the following conditions are satisfied.

(1.1) For all $a \in S$, $V_{S^o}(a) \neq \emptyset$.
(1.2) If $a, b \in S$ and $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Note that, if S^o is an orthodox transversal of S, then S is a regular semigroup by (1.1) and S^o is an orthodox subsemigroup of S by (1.2).

A subsemigroup S^o of S is said to be a quasi-ideal of S if $S^o S S^o \subseteq S^o$.

The following theorem will be frequently used without further mention.

(1.3) Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$.
(1.4) Let a, b be elements of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.

Finally, we list two basic results that are used in this paper.

LEMMA 1.1. [5] *Let S^o be a subsemigroup of S and $V_{S^o}(a) \neq \emptyset$ for each $a \in S$. Then the following conditions are equivalent:*

(1) *S^o is an orthodox transversal of S;*
(2) $IE(S^o) \subseteq I$, $E(S^o)\Lambda \subseteq \Lambda$, $E(S^o)I \subseteq E(S)$, $\Lambda E(S^o) \subseteq E(S)$.

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LEMMA 1.2. [5] *Let S^o be an orthodox transversal of S. Then the following conditions are equivalent:*

(1) *I is a band;*
(2) *$E(S^o)I \subseteq I$;*
(3) *$(\forall\, f \in I)\ (\exists\, f^* \in E(S^o),\ f^*\mathcal{L}f)\ f^*E(S^o)f \subseteq E(S^o)$;*
(4) *$(\forall\, f \in I)\ (\forall\, f^* \in E(S^o),\ f^*\mathcal{L}f)\ f^*E(S^o)f \subseteq E(S^o)$.*

We adopt the terminology and notation of [4, 6, 8].

2. Some properties

We begin this section by investigating some elementary properties of the sets R and L. For any result concerning R there is a dual result for L, which we list but omit its proof.

THEOREM 2.1. *Let S be a regular semigroup with an orthodox transversal S^o. Let*

$$R = \{x \in S \mid (\forall\, x^o \in V_{S^o}(x))\ (\exists\, x^{oo} \in V_{S^o}(x^o))\ x^o x = x^o x^{oo}\},$$
$$L = \{a \in S \mid (\forall\, a^o \in V_{S^o}(a))\ (\exists\, a^{oo} \in V_{S^o}(a^o))\ a a^o = a^{oo} a^o\}.$$

Then

$$\begin{aligned} R &= \{x \in S \mid (\exists\, y^o \in V_{S^o}(x), \exists\, y^{oo} \in V_{S^o}(y^o))\ y^o x = y^o y^{oo}\} \\ &= \{x \in S \mid (\exists\, e^o \in E^o)\ x\mathcal{L}e^o\}, \\ L &= \{a \in S \mid (\exists\, b^o \in V_{S^o}(a), \exists\, b^{oo} \in V_{S^o}(b^o))\ a b^o = b^{oo} b^o\} \\ &= \{a \in S \mid (\exists\, f^o \in E^o)\ a\mathcal{R}f^o\}. \end{aligned}$$

PROOF. It is evident that

$$R = \{x \in S \mid (\forall\, x^o \in V_{S^o}(x))\ (\exists\, x^{oo} \in V_{S^o}(x^o))\ x = x x^o x^{oo}\}.$$

For the first equation, we only need to show that, for $x \in S$, if there exist $y^o \in V_{S^o}(x)$, $y^{oo} \in V_{S^o}(y^o)$ such that $y^o x = y^o y^{oo}$, then $x \in R$. We notice that $x\mathcal{L}y^o x = y^o y^{oo}$ since $y^o \in V_{S^o}(x)$. For $x^o, y^o \in S^o$, $x \in V(x^o) \cap V(y^o) \neq \emptyset$, by [5, Lemma 2.2] we have $V_{S^o}(x^o) = V_{S^o}(y^o)$, so $y^{oo} \in V_{S^o}(x^o)$. So $x\mathcal{L}y^o y^{oo}\mathcal{L}x^o y^{oo}$ and hence $x = x x^o y^{oo}$. That is, $x \in R$.

For the second equation, if $x \in R$, then $x\mathcal{L}x^o x = x^o x^{oo} \in E(S^o)$. Conversely, if there exists $e^o \in E^o$ such that $x\mathcal{L}e^o$, then for any $x^o \in V_{S^o}(x)$, $x^o\mathcal{R}x^o x\mathcal{L}x\mathcal{L}e^o$, thus $x^o x \in E^o$ by [5, Theorem 2.4]. So $x^o x\mathcal{R}^{S^o}x^o$ and thus there exists $x^{oo} \in V_{S^o}(x^o)$ such that $x^o x = x^o x^{oo}$ since every idempotent in R_{x^o} is of the form $x^o x^{o\prime}$ for some $x^{o\prime} \in V_{S^o}(x^o)$. Therefore $x \in R$, and the theorem is proved. □

Notice that

$$I = \{e \in E(S) \mid (\exists\, e^* \in E^o)\ e\mathcal{L}e^*\}, \qquad \Lambda = \{f \in E(S) \mid (\exists\, f^+ \in E^o)\ f\mathcal{R}f^+\},$$

and by Theorem 2.1, we have the following result.

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COROLLARY. *Let R and L be as in Theorem 2.1. Then $R \cap L = S^o$ and $E(R) = I$, $E(L) = \Lambda$.*

As we know, I and Λ are subbands of S if S^o is an inverse transversal of S (see [11]). But in general, the corresponding result fails to be true if S^o is an orthodox transversal of S (see [5]). In [5], Chen and Guo proved, in general, that if S^o is an orthodox transversal of S, then the semibands $\overline{I}$ and $\overline{\Lambda}$ generated by I and Λ respectively are bands, and they also gave some equivalent conditions for I, Λ to be bands. By R and L, we obtain an equivalent condition for I and Λ to be bands, which is parallel to the result on regular semigroups with inverse transversals.

THEOREM 2.2. *Let S be a regular semigroup with an orthodox transversal S^o. Then R (L) is a subsemigroup of S if and only if I (Λ) is a subsemigroup of S.*

PROOF. Suppose that R is a subsemigroup of S. Let $e, f \in I$. Then $e, f \in R$ and so $ef \in R$ since R is a subsemigroup. Also we have $ef \in E(S)$ by [9, Theorem 2.6], whence $ef \in E(S) \cap R = I$.

Conversely, suppose that I is a subsemigroup of S and let $x, y \in R$. Then

$$\begin{aligned} xy &= xx^ox^{oo}yy^oy^{oo} \\ &= x \cdot x^ox^{oo}yy^o \cdot x^ox^{oo}yy^o \cdot y^{oo} \\ &= xy \cdot y^ox^o \cdot x^{oo}y. \end{aligned}$$

By the definition of an orthodox transversal, we have $y^ox^o \in V_{S^o}(x^{oo}y)$, and so

$$\begin{aligned} y^ox^o \cdot x^{oo}y &= y^o \cdot y^{oo}y^ox^ox^{oo}yy^o \cdot y^{oo} \\ &\in y^o \cdot y^{oo}y^oE^o \cdot yy^o \cdot y^{oo} \\ &\subseteq y^o \cdot E^o \cdot y^{oo} \quad (\text{since } yy^o \in I,\ yy^o\mathcal{L}y^{oo}y^o \in E^o) \\ &\subseteq E^o. \end{aligned}$$

So we have $xy = xy \cdot y^ox^o \cdot x^{oo}y\mathcal{L}y^ox^o \cdot x^{oo}y \in E^o$; by Theorem 2.1, $xy \in R$. □

LEMMA 2.3. *Let S be a regular semigroup with an orthodox transversal S^o. If $x \in R$ or $y \in L$, then $V_{S^o}(y)V_{S^o}(x) \subseteq V_{S^o}(xy)$.*

PROOF. If $x \in R$, then for any $x^o \in V_{S^o}(x)$ there exists $x^{oo} \in V_{S^o}(x^o)$ such that $x^ox = x^ox^{oo}$. For any $y^o \in V_{S^o}(y)$,

$$x^oxyy^o = x^ox^{oo}yy^o \in E(S^o)\Lambda \subseteq E(S)$$

and

$$yy^ox^ox = yy^ox^ox^{oo} \in IE(S^o) \subseteq E(S).$$

Thus

$$xy \cdot y^ox^o \cdot xy = x \cdot x^oxyy^o \cdot x^oxyy^o \cdot y = x \cdot x^oxyy^o \cdot y = xy$$

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and

$$y^o x^o \cdot xy \cdot y^o x^o = y^o \cdot yy^o x^o x \cdot yy^o x^o x \cdot x^o = y^o \cdot yy^o x^o x \cdot x^o = y^o x^o.$$

For the choice of x^o and y^o, we have $V_{S^o}(y)V_{S^o}(x) \subseteq V_{S^o}(xy)$. □

THEOREM 2.4. *Let S be a regular semigroup with an orthodox transversal S^o. Then the following statements are equivalent:*

(1) *S^o is a quasi-ideal;*
(2) *$E(S^o)I \subseteq E(S^o)$, $\Lambda E(S^o) \subseteq E(S^o)$;*
(3) *$\Lambda I \subseteq S^o$;*
(4) *$SS^o \subseteq R$, $S^o S \subseteq L$;*
(5) *R is a left ideal and L is a right ideal of S.*

PROOF. Obviously, (1), (2) and (3) are equivalent.

(1) implies (4). If (1) holds, then $yx^o \mathcal{L} (yx^o)^o yx^o \in S^o \cap E(S) = E(S^o)$, whence $SS^o \subseteq R$; and dually $S^o S \subseteq L$.

(4) implies (5). If (4) holds, then for any $x \in S$ and $y \in R$, we have $xy = xyy^o y^{oo} \in SS^o \subseteq R$, whence $SR \subseteq R$; and dually $LS \subseteq L$.

(5) implies (3). If (5) holds, then for $l \in \Lambda$ and $i \in I$, there exist $i^o, l^o \in E(S^o)$, such that $i = ii^o$, $l = l^o l$. Thus

$$li = lii^o \in SS^o \subseteq SR \subseteq R \quad \text{and} \quad li = l^o li \in S^o S \subseteq LS \subseteq L,$$

whence $li \in R \cap L = S^o$ and we have (3). □

THEOREM 2.5. *Suppose that $a, a' \in L$ and $a \mathcal{L} a'$, $y, y' \in R$ and $y \mathcal{R} y'$. Then*

$$y^o y' V_{S^o}(a'y') a' a^o \subseteq V_{S^o}(ay),$$

where $y^o \in V_{S^o}(y) \cap V_{S^o}(y')$, $a^o \in V_{S^o}(a) \cap V_{S^o}(a')$.

PROOF. Take $s \in V_{S^o}(a'y')$. Then

$$ay(y^o y' s a' a^o)ay = ay' s a' y = aa^o a' y' s a' y' y^o y = aa^o a' y' y^o y = ay$$

and

$$(y^o y' s a' a^o) ay (y^o y' s a' a^o) = y^o y' s a' y' s a' a^o = y^o y' s a' a^o.$$ □

3. The main theorem

The main objective in this section is to give a structure theorem for regular semigroups with quasi-ideal orthodox transversals. In what follows R denotes a regular semigroup with a right ideal orthodox transversal S^o. Then by [7, Lemma 1], $E(R) = I$ is a band, consequently R is an orthodox semigroup and we will denote the minimum inverse semigroup congruence on R by γ. For $a \in R$, the $\mathcal{R}$-class of R

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containing a will be denoted by R_a and the γ-class containing a will be denoted by $T(a)$. Then $T(a) \cap S^o = V_{S^o}(a)$ and by [5, Theorem 2.6] and since R is orthodox,

$$V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \iff V_{S^o}(a) = V_{S^o}(b) \iff T(a) = T(b) \quad \text{for all } a, b \in R.$$

We define $K(a) = K(b)$ if $R_a = R_b$ and $T(a) = T(b)$ for $a, b \in R$ and we define a relation $\mathcal{K}$ on R by $(a, b) \in \mathcal{K}$ if $K(a) = K(b)$. Then $\mathcal{K}$ is an equivalence relation on R.

THEOREM 3.1. *Let R and L be regular semigroups with a common orthodox transversal S^o. Suppose that S^o is a right ideal of R and a left ideal of L. Let $L \times R \longrightarrow S^o$ described by $(a, x) \longrightarrow a * x$ be a mapping such that for any $x, y \in R$ and for any $a, b \in L$:*

(1) $(a * x)y = a * xy$ *and* $b(a * x) = ba * x$;
(2) *if* $x \in S^o$ *or* $a \in S^o$, *then* $a * x = ax$; *and*
(3) *if* $a, a' \in L$ *and* $a\mathcal{L}a'$, $y, y' \in R$ *and* $y\mathcal{R}y'$, *then*

$$y^o y' V_{S^o}(a' * y') a' a^o \subseteq V_{S^o}(a * y),$$

where $y^o \in V_{S^o}(y) \cap V_{S^o}(y')$, $a^o \in V_{S^o}(a) \cap V_{S^o}(a')$.

Define a multiplication on the set

$$\Gamma = R/\mathcal{K} \,|\times|\, L/\mathcal{L} = \{(K_x,\ L_a) \in R/\mathcal{K} \times L/\mathcal{L} \mid V_{S^o}(x) \cap V_{S^o}(a) \neq \emptyset\}$$

by

$$(K_x, L_a)\,(K_y, L_b) = (K_{xx^o(a*y)}, L_{(a*y)y^o b}).$$

Then Γ is a regular semigroup with a quasi-ideal orthodox transversal that is isomorphic to S^o.

Conversely, every regular semigroup with a quasi-ideal orthodox transversal can be constructed in this way.

To prove this theorem, we give a sequence of lemmas as follows.

LEMMA 3.2. *The multiplication in Γ is well defined.*

PROOF. First it is easy to see that $(K_{xx^o(a*y)}, L_{(a*y)y^o b}) \in \Gamma$, since

$$(a * y)^o x^{oo} x^o \in V_{S^o}(xx^o(a * y)) \cap V_{S^o}((a * y)y^o b) \neq \emptyset.$$

Let $x^o, x_1^o \in V_{S^o}(x) \cap V_{S^o}(a)$, then

$$R_{xx^o(a*y)} = R_{xx_1^o(a*y)} \quad \text{and} \quad T(xx^o(a * y)) = T(xx_1^o(a * y)),$$

and hence the multiplication in Γ is not dependent on the choice of x^o. There is a dual result for y^o.

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Finally we prove that the multiplication in Γ is not dependent on the choice of x, a, y, b. Let

$$(K_x, L_a) = (K_{x'}, L_{a'}), \quad (K_y, L_b) = (K_{y'}, L_{b'}).$$

We then have

$$(K_x, L_a)\,(K_y, L_b) = (K_{xx^o(a*y)}, L_{(a*y)y^ob}),$$

and

$$(K_{x'}, L_{a'})\,(K_{y'}, L_{b'}) = (K_{x'x^o(a'*y')}, L_{(a'*y')y^ob'}),$$

where $x^o \in V_{S^o}(x) \cap V_{S^o}(x')$ and $y^o \in V_{S^o}(y) \cap V_{S^o}(y')$.

Next we prove that $T(xx^o(a*y)) = T(x'x^o(a'*y'))$. Take $s \in V_{S^o}(a'*y')$, then $y^oy'sa'a^o \in V_{S^o}(a*y)$ by (3). Since S^o is orthodox,

$$y^oy'sx^{oo}x^o \in V_{S^o}(x'x^o(a'*y')),$$
$$y^oy'sa'a^ox^{oo}x^o = y^oy'sx^{oo}x^o \in V_{S^o}(xx^o(a*y)),$$

where $x^{oo} \in V_{S^o}(x^o)$. So

$$V_{S^o}(xx^o(a*y)) \cap V_{S^o}(x'x^o(a'*y')) \neq \emptyset$$

and hence $V_{S^o}(xx^o(a*y)) = V_{S^o}(x'x^o(a'*y'))$, that is

$$T(xx^o(a*y)) = T(x'x^o(a'*y'))$$

as required.

To show that $R_{xx^o(a*y)} = R_{x'x^o(a'*y')}$, notice that $xx^o = x'x^o$ since $x\mathcal{R}x'$ and $x^o \in V_{S^o}(x) = V_{S^o}(x')$, and $x^oa = x^oa'$ since $a\mathcal{L}a'$ and $x^o \in V_{S^o}(a) = V_{S^o}(a')$. Take $s \in V_{S^o}(a'*y')$, then $(a*y)^o = y^oy'sa'a^o \in V_{S^o}(a*y)$ by (3). So

$$xx^o(a*y)\mathcal{R}xx^o(a*y)\,(a*y)^ox^{oo}x^o = e,$$
$$x'x^o(a'*y')\mathcal{R}x'x^o(a'*y')sx^{oo}x^o = f.$$

Thus

$$\begin{aligned} e &= xx^o(a*y)y^oy'sa'a^ox^{oo}x^o \\ &= xx^o(a*yy^oy')sx^{oo}x^o \quad (a'a^ox^{oo} = x^{oo} \text{ since } a' \in L) \\ &= x'x^o(a*y')sx^{oo}x^o \quad (xx^o = x'x^o \text{ and } y'\mathcal{R}y\mathcal{R}yy^o) \\ &= x'x^o(a'*y')sx^{oo}x^o \quad (x^o \in S^o \text{ and } x^oa = x^oa') \\ &= f. \end{aligned}$$

Therefore $R_{xx^o(a*y)} = R_{x'x^o(a'*y')}$. Dually we have $L_{(a*y)y^ob} = L_{(a'*y')y^ob'}$. □

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LEMMA 3.3. *The set Γ is a semigroup.*

PROOF. Let $e, f, g \in \Gamma$, where $e = (K_x, L_a)$, $f = (K_{x_1}, L_{a_1})$, $g = (K_{x_2}, L_{a_2})$. Then

$$\begin{aligned}(ef)g &= (K_{xx^o(a*x_1)}, L_{(a*x_1)x_1^o a_1})\,(K_{x_2}, L_{a_2})\\ &= (K_{xx^o(a*x_1)\,(a*x_1)^o x^{oo} x^o(((a*x_1)x_1^o a_1)*x_2)}, L_{(((a*x_1)x_1^o a_1)*x_2)x_2^o a_2})\\ &= (K_{xx^o(a*x_1)x_1^o(a_1*x_2)}, L_{(a*x_1)x_1^o(a_1*x_2)x_2^o a_2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}e(fg) &= (K_x, L_a)\,(K_{x_1x_1^o(a_1*x_2)}, L_{(a_1*x_2)x_2^o a_2})\\ &= (K_{xx^o(a*x_1)x_1^o(a_1*x_2)}, L_{(a*x_1)x_1^o(a_1*x_2)x_2^o a_2}).\end{aligned}$$

Therefore $(ef)g = e(fg)$. □

LEMMA 3.4. *Let $W = \{(K_x, L_x) \mid x \in S^o\}$. Then W is an orthodox subsemigroup of Γ isomorphic to S^o.*

PROOF. We only need to notice that, for $x, y \in S^o$, $(K_x, L_x) = (K_y, L_y)$ if and only if $x = y$. □

LEMMA 3.5. *Let $e = (K_x, L_a)$. Put*

$$M(e) = \{(K_{x^o}, L_{x^o}) \in W \mid x^o \in V_{S^o}(x)\}.$$

Then $V_W(e) = M(e)$.

PROOF. Take $f = (K_{x^o}, L_{x^o}) \in W$, where $x^o \in V_{S^o}(x)$. Then

$$\begin{aligned}(K_x, L_a)\,(K_{x^o}, L_{x^o})\,(K_x, L_a) &= (K_{xx^o(a*x_o)x^{oo}(x^o*x)}, L_{(a*x^o)x^{oo}(x^o*x)x^o a})\\ &= (K_{xx^o ax^o x^{oo} x^o x}, L_{ax^o x^{oo} x^o xx^o a})\\ &= (K_x, L_a).\end{aligned}$$

Also

$$\begin{aligned}(K_{x^o}, L_{x^o})\,(K_x, L_a)\,(K_{x^o}, L_{x^o}) &= (K_{x^o xx^o x^{oo} x^o ax^o}, L_{x^o xx^o ax^o x^{oo} x^o})\\ &= (K_{x^o}, L_{x^o}).\end{aligned}$$

Thus $f \in V_W(e)$.

Conversely, let $f = (K_{y^o}, L_{y^o}) \in V_W(e)$, then $efe = e$, $fef = f$. So

$$(K_x, L_a)\,(K_{y^o}, L_{y^o})\,(K_x, L_a) = (K_{xx^o ay^o x}, L_{ay^o xx^o a}) = (K_x, L_a),$$
$$(K_{y^o}, L_{y^o})\,(K_x, L_a)\,(K_{y^o}, L_{y^o}) = (K_{y^o xx^o ay^o}, L_{y^o xx^o ay^o}) = (K_{y^o}, L_{y^o}).$$

Therefore $x = xx^o ay^o x$ since x and $xx^o ay^o x$ have a common inverse by $T(xx^o ay^o x) = T(x)$. Similarly $y^o = y^o xx^o ay^o$. Then x has an inverse

$$x^\# = x^o y^{oo} x^o x^{oo} x^o = x^o y^{oo} x^o.$$

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On the other hand, $x^o y^{oo} x^o \in V_{S^o}(xy^o x)$; thus x and $xy^o x$ have a common inverse and so $x = xy^o x$. Similarly $y^o = y^o x y^o$. Hence $y^o \in V_{S^o}(x)$ and therefore $f \in M(e)$. Now the proof of the lemma is completed. □

LEMMA 3.6. *The set W is a quasi-ideal orthodox transversal of Γ.*

PROOF. Take $e = (K_x, L_a) \in \Gamma$, and $x^o \in V_{S^o}(x) \cap V_{S^o}(a)$. It follows from Lemma 3.5 that $V_W(e) \neq \emptyset$, and hence condition (1.1) holds. To check condition (1.2), take $f = (K_y, L_y) \in W$, where $y \in S^o$. Then $ef = (K_{xx^o ay}, L_{ay})$ since $xx^o(a * y) = xx^o ay$ and $(a * y)y^o y = ayy^o y = ay$ by the assumption $y \in S^o$. Now let

$$e' = (K_{x^o}, L_{x^o}) \in V_W(e), \qquad f' = (K_{y^o}, L_{y^o}) \in V_W(f).$$

Then $f'e' = (K_{y^o x^o}, L_{y^o x^o})$. Obviously $xx^o ay$ has an inverse

$$(xx^o ay)^{\#} = y^o x^o x^{oo} x^o = y^o x^o.$$

That is to say, $(K_{y^o x^o}, L_{y^o x^o}) \in M(ef)$ and thus $f'e' \in V_W(ef)$. Similarly we have $e'f' \in V_W(fe)$. Hence condition (1.2) holds and W is an orthodox transversal of Γ.

Take $w_1, w_2 \in W$ and $s \in \Gamma$. It is a routine matter to show that $w_1 s w_2 \in W$, so W is a quasi-ideal of Γ. □

Now we turn to prove the converse part of Theorem 3.1. Let S be a regular semigroup and S^o a quasi-ideal orthodox transversal of S. Let R and L be described as in Theorem 2.1. Then R and L are orthodox semigroups with an orthodox transversal S^o which is a right ideal of R and a left ideal of L. For every $(a, x) \in L \times R$, put $a * x = ax$. Then $a * x = ax = a^{oo} a^o a x x^o x^{oo} \in S^o$ since S^o is a quasi-ideal of S. Clearly the map satisfies (1) and (2). By Theorem 2.5 the condition (3) holds. Therefore we get a regular semigroup Γ in the same way as in the first part of Theorem 3.1. Finally we shall prove that Γ is isomorphic to S.

Let $(K_x, L_a) \in \Gamma$. Define $\theta : \Gamma \longrightarrow S$ by $(K_x, L_a)\theta = xx^o a$, where $x^o \in V_{S^o}(x)$. It is evident that, for every $y^o \in V_{S^o}(x)$, $xx^o a = xy^o a$ since $xx^o a \mathcal{H} xy^o a$ and

$$y^o x x^o \in V(xx^o a) \cap V(xy^o a).$$

We first have to show that θ is well defined. If $(K_x, L_a) = (K_y, L_b)$ then $R_x = R_y$, $V_{S^o}(x) = V_{S^o}(y)$, $L_a = L_b$ and so

$$xx^o a \mathcal{R} xx^o \mathcal{R} x \mathcal{R} y \mathcal{R} yy^o \mathcal{R} yy^o b,$$
$$xx^o a \mathcal{L} x^o a \mathcal{L} a \mathcal{L} b \mathcal{L} y^o b \mathcal{L} yy^o b.$$

Thus $xx^o a \mathcal{H} yy^o b$ and we also have

$$y^o x x^o \in V(xx^o a) \cap V(yy^o b).$$

Therefore $xx^o a = yy^o b$ since no $\mathcal{H}$-class contains more than one inverse of some element.

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Take $(K_x, L_a), (K_y, L_b) \in \Gamma$. Then

$$\begin{aligned}((K_x, L_a)\,(K_y, L_b))\theta &= (K_{xx^oay}, L_{ayy^ob})\theta \\ &= xx^oay(ay)^oa^{oo}a^o(ay)y^ob \\ &= xx^oayy^ob \\ &= (K_x, L_a)\theta(K_y, L_b)\theta,\end{aligned}$$

and so θ is a homomorphism.

For every $x \in S$,

$$xx^ox^{oo} \in R, \quad x^{oo}x^ox \in L \quad \text{and} \quad x^o \in V_{S^o}(xx^ox^{oo}) \cap V_{S^o}(x^{oo}x^ox),$$
$$(K_{xx^ox^{oo}}, L_{x^{oo}x^ox})\theta = xx^ox^{oo} \cdot x^o \cdot x^{oo}x^ox = x.$$

Therefore θ is surjective.

Now let $(K_x, L_a), (K_y, L_b) \in \Gamma$ such that $(K_x, L_a)\theta = (K_y, L_b)\theta$, that is $xx^oa = yy^ob$. So

$$x\mathcal{R}xx^o\mathcal{R}xx^oa = yy^ob\mathcal{R}yy^o\mathcal{R}y$$

and

$$a\mathcal{L}x^oa\mathcal{L}xx^oa = yy^ob\mathcal{L}y^ob\mathcal{L}b.$$

That is $R_x = R_y$ and $L_a = L_b$. It is easy to see that $x^o \in V_{S^o}(xx^oa)$ and $y^o \in V_{S^o}(yy^ob)$, so

$$V_{S^o}(x) = V_{S^o}(xx^oa) = V_{S^o}(yy^ob) = V_{S^o}(y).$$

Hence θ is injective.

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XIANGJUN KONG, School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China
and
Current address: Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China
e-mail: xiangjunkong97@163.com

XIANZHONG ZHAO, Department of Mathematics, Northwest University, Xi'an, Shaanxi 710069, People's Republic of China