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GROWTH ESTIMATES FOR WARPING FUNCTIONS AND THEIR GEOMETRIC APPLICATIONS

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Abstract. By applying Wei, Li and Wu's notion (given in 'Generalizations of the uniformization theorem and Bochner's method in p-harmonic geometry', *Comm. Math. Anal. Conf.*, vol. 01, 2008, pp. 46–68) and method (given in 'Sharp estimates on $\mathcal{A}$-harmonic functions with applications in biharmonic maps, preprint) and by modifying the proof of a general inequality given by Chen in 'On isometric minimal immersion from warped products into space forms' (*Proc. Edinb. Math. Soc.*, vol. 45, 2002, pp. 579–587), we establish some simple relations between *geometric* estimates (the mean curvature of an isometric immersion of a warped product and sectional curvatures of an ambient m-manifold $\tilde{M}^m_c$ bounded from above by a non-positive number c) and *analytic* estimates (the growth of the warping function). We find a dichotomy between constancy and 'infinity' of the warping functions on complete non-compact Riemannian manifolds for an appropriate isometric immersion. Several applications of our growth estimates are also presented. In particular, we prove that if f is an L^q function on a complete non-compact Riemannian manifold N_1 for some $q > 1$, then for any Riemannian manifold N_2 the warped product $N_1 \times_f N_2$ does not admit a minimal immersion into any non-positively curved Riemannian manifold. We also show that both the geometric curvature estimates and the analytic function growth estimates in this paper are sharp.

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1. Introduction. In [**5**], B. Y. Chen established a new relationship between extrinsic quantities and intrinsic quantities of Riemannian manifolds via Nash's theorem. In particular, he obtained a necessary condition for an arbitrary isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c as follows.

THEOREM 1.1. (Chen [**5**]) *Let* $\phi : N_1 \times_f N_2 \to R^m(c)$ *be an isometric immersion of a warped product* $N_1 \times_f N_2$ *into a Riemannian m-manifold* $R^m(c)$ *of constant sectional* *curvature c. Then the warping function f satisfies*

$$-\frac{(n_1+n_2)^2}{4n_2}H^2 - n_1 c \leq \frac{\Delta f}{f}, \tag{1}$$

where $n_i = \dim N_i$, $i = 1, 2$; H^2 *is the squared mean curvature of* ϕ; *and* Δf *is the Laplacian of* f *on* N_1 (*defined as the divergence of the gradient vector field of* f). *The equality sign in* (1) *holds if and only if* ϕ *is a mixed totally geodesic immersion with trace* h_1 = *trace* h_2, *where* h_1 *and* h_2 *are the restriction of the second fundamental form* h *of* ϕ *restricted to* N_1 *and* N_2, *respectively.*

On the other hand, S. W. Wei, J. Li and L. Wu extended in [**9, 10**] the scope of L^q or q-integrable functions on complete non-compact Riemannian manifolds by generalising them to 'p-finite, p-mild, p-obtuse, p-moderate and p-small' functions that depend on q and introducing the concepts of their counterparts 'p-infinite, p-severe, p-acute, p-immoderate and p-large' growth.

The purposes of this paper are the following: First, by modifying the proof of Theorem 1.1 in [**5**], we prove that the same inequality (1) holds if the ambient space $R^m(c)$ is replaced by an arbitrary Riemannian m-manifold $\tilde{M}^m_c$ with sectional curvatures bounded from above by the constant c. Next, by applying this general inequality and the method of Wei, Li and Wu, we prove several simple relations between the growth of the warping function f on a complete non-compact Riemannian manifold N_1 and the squared mean curvature of the isometric immersion ϕ of $N_1 \times_f N_2$ into $\tilde{M}^m_c$. Finally, we provide several applications of our growth estimates. In particular, we prove that if f is an L^q function on a complete non-compact Riemannian manifold N_1 for some $q > 1$, then for any Riemannian manifold N_2 the warped product $N_1 \times_f N_2$ does not admit any minimal immersion into any non-positively curved Riemannian manifold. In the last section, we provide some examples to illustrate that both the geometric curvature estimates and the analytic function growth estimates in this paper are sharp.

2. Preliminaries. Let N be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that restricted to N, the vectors $e_1, \ldots, e_n$ are tangent to N and $e_{n+1}, \ldots, e_m$ are normal to N.

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of N is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j). \tag{2}$$

Let L be a subspace of T_xN of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The 'scalar curvature $\tau(L)$ of the r-plane section L', introduced in [**4**], is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{3}$$

For a submanifold N in $\tilde{M}^m$ we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of N and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are respectively given

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by (see, for instance, [**2**])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{5}$$

for vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection and A the shape operator of the submanifold. Let $\{h_{ij}^r\}$, $i, j = 1, \ldots, n$, $r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$.

Denote by R and $\tilde{R}$ the Riemann curvature tensor of N and $\tilde{M}^m$, respectively. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) &+ \langle h(X, W), h(Y, Z)\rangle \\ &- \langle h(X, Z), h(Y, W)\rangle \end{aligned} \tag{6}$$

for vectors X, Y, Z, W tangent to N.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \operatorname{trace} h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{7}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. The squared mean curvature is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product. A submanifold N is called minimal in $\tilde{M}^m$ if the mean curvature vector of N in $\tilde{M}^m$ vanishes identically.

Let M be a Riemannian k-manifold and $\{e_1, \ldots, e_k\}$ be orthonormal frame field on M. For a smooth function φ on M with Levi-Civita connection ∇^M, the Laplacian of φ is defined by the divergence of the gradient of φ or the trace of the Hessian of φ, i.e.

$$\Delta\varphi = \sum_{j=1}^{k} \{e_j e_j \varphi - (\nabla^M_{e_j} e_j)\varphi\}. \tag{8}$$

A C^2 function φ on M is said to be harmonic, subharmonic or superharmonic if we have $\Delta\varphi = 0$, $\Delta\varphi \geq 0$ or $\Delta\varphi \leq 0$ on M, respectively.

An isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}^m$ of a warped product $N_1 \times_f N_2$ into a Riemannian m-manifold $\tilde{M}^m$ is called *mixed totally geodesic* if the second fundamental form h of ϕ satisfies $h(X, Z) = 0$ for any X tangent N_1 and Z tangent to N_2.

3. A general inequality. By making a minor modification of the proof of Theorem 1.1 in [**5**] we have the following

THEOREM 3.1. *If $\tilde{M}^m_c$ is a Riemannian manifold with sectional curvatures bounded from above by a constant c, then for any isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}^m_c$ from a warped product $N_1 \times_f N_2$ into $\tilde{M}^m_c$ the warping function f satisfies*

$$-\frac{(n_1 + n_2)^2}{4n_2} H^2 - n_1 c \leq \frac{\Delta f}{f}, \tag{9}$$

where $n_1 = \dim N_1$ and $n_2 = \dim N_2$.

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Proof. Put $n = n_1 + n_2$ and $N = \nabla_1 \times_f N_2$. If we chose a local orthonormal frame $e_1, \ldots, e_n$ such that $e_1, \ldots, e_{n_1}$ are tangent to N_1 and $e_{n_1+1}, \ldots, e_n$ are tangent to N_2, then we have

$$\frac{-\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s) \tag{10}$$

for each $s \in \{n_1 + 1, \ldots, n\}$. It follows from the equation of Gauss that the scalar curvature τ of N and the squared mean curvature H^2 of N in $\tilde{M}^m_c$ satisfy (cf., e.g., **[2]**)

$$2\tau(x) = n^2 H^2(x) - \|h\|^2(x) + 2\tilde{\tau}(T_x(N)), \tag{11}$$

where $\|h\|^2$ is the squared norm of the second fundamental form h of N in $\tilde{M}^m_c$ and $\tilde{\tau}(T_x(N))$ is the scalar curvature of the subspace $T_x(N)$ in $\tilde{M}^m_c$ as defined in (3).

Let us put

$$\delta = 2\tau - 2\tilde{\tau}(T_x(N)) - \frac{n^2}{2} H^2. \tag{12}$$

Then (12) becomes

$$n^2 H^2 = 2\delta + 2\|h\|^2. \tag{13}$$

If we choose an orthonormal frame $e_{n+1}, \ldots, e_m$ of the normal bundle so that e_{n+1} is in the direction of the mean curvature vector, then we obtain

$$\left(\sum_{i=1}^{n} h^{n+1}_{ii}\right)^2 = 2\left[\delta + \sum_{i=1}^{n} \left(h^{n+1}_{ii}\right)^2 + \sum_{i\neq j} \left(h^{n+1}_{ij}\right)^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} \left(h^r_{ij}\right)^2\right], \tag{14}$$

where $h^r_{ij} = \langle h(e_i, e_j), e_r\rangle$. Equation (14) is equivalent to

$$\begin{aligned}(\bar{a}_1 + \bar{a}_2 + \bar{a}_3)^2 = 2\Bigg[&\delta + \bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2 + 2\sum_{1\le i<j\le n} \left(h^{n+1}_{ij}\right)^2 \\ &+ \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h^r_{ij})^2 - 2\sum_{2\le j<k\le n_1} h^{n+1}_{jj} h^{n+1}_{kk} - 2\sum_{n_1+1\le s<t\le n} h^{n+1}_{ss} h^{n+1}_{tt}\Bigg],\end{aligned} \tag{15}$$

where $\bar{a}_1 = h^{n+1}_{11}$, $\bar{a}_2 = h^{n+1}_{22} + \cdots + h^{n+1}_{n_1 n_1}$ and $\bar{a}_3 = h^{n+1}_{n_1+1 n_1+1} + \cdots + h^{n+1}_{nn}$.

Applying Lemma 3.1 of **[3]** to (15) yields

$$\begin{aligned}&\sum_{1\le j<k\le n_1} h^{n+1}_{jj} h^{n+1}_{kk} + \sum_{n_1+1\le s<t\le n} h^{n+1}_{ss} h^{n+1}_{tt} \\ &\qquad \ge \frac{\delta}{2} + \sum_{1\le \alpha<\beta\le n} \left(h^{n+1}_{\alpha\beta}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{m} \sum_{\alpha,\beta=1}^{n} \left(h^r_{\alpha\beta}\right)^2.\end{aligned} \tag{16}$$

From the equation of Gauss and (10), we have the following at point $x = (x_1, x_2) \in N$:

$$-\frac{n_2 \Delta f}{f} = \tau - \sum_{1 \le j < k \le n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \le s < t \le n} K(e_s \wedge e_t)$$
$$= \tau - \tilde{\tau}(T_{x_1}(N_1)) - \sum_{r=n+1}^{m} \sum_{1 \le j < k \le n_1} \left(h^r_{jj} h^r_{kk} - (h^r_{jk})^2\right)$$
$$- \tilde{\tau}(T_{x_2}(N_2)) - \sum_{r=n+1}^{m} \sum_{n_1+1 \le s < t < n} \left(h^r_{ss} h^r_{tt} - (h^r_{st})^2\right). \tag{17}$$

Therefore, by (12), (16) and (17), we obtain

$$-\frac{n_2 \Delta f}{f} \le \tau - \tilde{\tau}(T_x(N)) + n_1 n_2 \max \tilde{K} - \sum_{j \in I_1;\, t \in I_2} (h^{n+1}_{jt})^2$$
$$- \frac{1}{2} \sum_{r=n+2}^{m} \sum_{\alpha,\beta=1}^{n} (h^r_{\alpha\beta})^2 + \sum_{r=n+2}^{m} \sum_{1 \le j < k \le n_1} \left((h^r_{jk})^2 - h^r_{jj} h^r_{kk}\right)$$
$$+ \sum_{r=n+2}^{m} \sum_{n_1+1 \le s < t < n} \left((h^r_{st})^2 - h^r_{ss} h^r_{tt}\right) - \frac{\delta}{2}$$
$$= \tau - \tilde{\tau}(T_x(N)) + n_1 n_2 \max \tilde{K} - \sum_{r=n+1}^{m} \sum_{j \in I_1} \sum_{t \in I_2} (h^r_{jt})^2$$
$$- \frac{1}{2} \sum_{r=n+2}^{m} \Big(\sum_{j \in I_1} h^r_{jj}\Big)^2 - \frac{1}{2} \sum_{r=n+2}^{m} \Big(\sum_{t \in I_2} h^r_{tt}\Big)^2 - \frac{\delta}{2}$$
$$\le \tau - \tilde{\tau}(T_x(N)) + n_1 n_2 \max \tilde{K} - \frac{\delta}{2}$$
$$= \frac{n^2}{4} H^2 + n_1 n_2 \max \tilde{K}, \tag{18}$$

where $I_1 = \{1, \ldots, n_1\}$, $I_2 = \{n_1 + 1, \ldots, n\}$ and $\max \tilde{K}(x)$ denotes the maximum of the sectional curvatures of $\tilde{M}^m_c$ restricted to 2-plane sections of the tangent space $T_x(N_1 \times_f N_2)$ at $x \in N_1 \times N_2$. Since $\max \tilde{K} \le c$ by the assumption, we obtain inequality (9). □

REMARK 3.1. For the most recent survey on inequalities similar to (9), see **[6]**.

4. Growth of warping function and mean curvature. In the following, assume that N_1 is a complete non-compact Riemannian manifold and $B(x_0; r)$ is the geodesic ball of radius r centred at $x_0 \in N_1$.

We recall some notions from **[7, 9]**.

DEFINITION 4.1. A function f on N_1 is said to have *p-finite growth* (or, simply, *is p-finite*) if there exists $x_0 \in N_1$ such that

$$\liminf_{r \to \infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty; \tag{19}$$

it has *p-infinite growth* (or, simply, *is p-infinite*) otherwise (cf. (27) for $p = 2$).

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DEFINITION 4.2. A function f has *p-mild growth* (or, simply, *is p-mild*) if there exist $x_0 \in N_1$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=\ell_0}^{\infty}\left(\frac{(r_{j+1}-r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)}|f|^q dv}\right)^{\frac{1}{p-1}} = \infty\,; \tag{20}$$

it has *p-severe growth* (or, simply, *is p-severe*) otherwise (cf. (40) for $p = 2$).

DEFINITION 4.3. A function f has *p-obtuse growth* (or, simply, *is p-obtuse*) if there exists $x_0 \in N_1$ such that for every $a > 0$, we have

$$\int_a^{\infty}\left(\frac{1}{\int_{\partial B(x_0;r)}|f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty\,; \tag{21}$$

it has *p-acute growth* (or, simply, *is p-acute*) otherwise (cf. (44) for $p = 2$).

DEFINITION 4.4. A function f has *p-moderate growth* (or, simply, *is p-moderate*) if there exist $x_0 \in N_1$, and $F(r) \in \mathcal{F}$,such that

$$\limsup_{r\to\infty}\frac{1}{r^p F^{p-1}(r)}\int_{B(x_0;r)}|f|^q dv < \infty. \tag{22}$$

And it has *p-immoderate growth* (or, simply, *is p-immoderate*) otherwise, where

$$\mathcal{F} = \{F : [a,\infty) \longrightarrow (0,\infty) | \int_a^{\infty}\frac{dr}{rF(r)} = +\infty \ \text{ for some } \ a \geq 0\}\,. \tag{23}$$

(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

DEFINITION 4.5. A function f has *p-small growth* (or, simply, *is p-small*) if there exists $x_0 \in N_1$, such that for every $a > 0$,we have

$$\int_a^{\infty}\left(\frac{r}{\int_{B(x_0;r)}|f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty; \tag{24}$$

it has *p-large growth* (or, simply, *is p-large*) otherwise.

The above definitions of 'p-finite, p-mild, p-obtuse, p-moderate, p-small' and their counterparts 'p-infinite, p-severe, p-acute, p-immoderate and$p$-large' growth depend on q, and q will be specified in the context in which the definition is used.

From now on, we assume that N_1 is a complete non-compact Riemannian n_1-manifold and f is a C^2-function on N_1. Denote by $\tilde{M}_c^m$ a Riemannian m-manifold with sectional curvatures $\tilde{K} \leq c$ for some real number $c \leq 0$.

We have the following results.

THEOREM 4.1. *If f is non-constant and* 2-finite *for some $q > 1$, then for any Riemannian n_2-manifold N_2 and any isometric immersion ϕ of the warped product $N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}_c^m$ with $c \leq 0$, the mean curvature H of ϕ*

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satisfies

$$H^2 > \frac{4n_1n_2|c|}{(n_1+n_2)^2} \tag{25}$$

at some points.

COROLLARY 4.1. *Suppose the squared mean curvature of the isometric immersion* $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ *satisfies*

$$H^2 \le \frac{4n_1n_2|c|}{(n_1+n_2)^2} \tag{26}$$

everywhere on $N_1 \times_f N_2$*. Then the warping function* f *either is a constant or for every* $q > 1$ *has* 2-infinite *growth; i.e. for every* $x_0 \in N_1$*,*

$$\lim_{r\to\infty} \frac{1}{r^2} \int_{B(x_0;r)} |f|^q dv = \infty\,. \tag{27}$$

THEOREM 4.2. *If* f *is non-constant and* 2-mild *for some* $q > 1$*, then for any isometric immersion of* $N_1 \times_f N_2$ *into a Riemannian manifold* $\tilde{M}_c^m$ *with* $c \le 0$ *we have* (25) *at some points.*

COROLLARY 4.2. *Suppose that the squared mean curvature of the isometric immersion* $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ *satisfies* (26) *everywhere on* $N_1 \times_f N_2$*. Then the warping function* f *either is a constant or has* 2-severe *growth for every* $q > 1$.

THEOREM 4.3. *If* f *is non-constant and* 2-obtuse *for some* $q > 1$ *, then for any isometric immersion of* $N_1 \times_f N_2$ *into a Riemannian manifold* $\tilde{M}_c^m$ *with* $c \le 0$ *we have* (25) *at some points.*

COROLLARY 4.3. *Suppose the squared mean curvature of the isometric immersion* $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ *satisfies* (26) *everywhere on* $N_1 \times_f N_2$*. Then the warping function* f *either is a constant or has* 2-acute *growth for every* $q > 1$.

THEOREM 4.4. *If* f *is non-constant and* 2-moderate *for some* $q > 1$*, then for any isometric immersion of* $N_1 \times_f N_2$ *into a Riemannian manifold* $\tilde{M}_c^m$ *with* $c \le 0$ *we have* (25) *at some points.*

COROLLARY 4.4. *Suppose the squared mean curvature of the isometric immersion* $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ *satisfies* (26) *everywhere on* $N_1 \times_f N_2$*. Then the warping function* f *either is a constant or has* 2-immoderate *growth for every* $q > 1$.

THEOREM 4.5. *If* f *is non-constant and* 2-small *for some* $q > 1$*, then for any isometric immersion of* $N_1 \times_f N_2$ *into a Riemannian manifold* $\tilde{M}_c^m$ *with* $c \le 0$ *we have* (25) *at some points.*

REMARK 4.1. The assumption on Theorems 4.1–4.5 cannot be dropped. Otherwise, we have counter-examples that violate (25) (cf. Remark 7.1).

COROLLARY 4.5. *Suppose that the squared mean curvature of the isometric immersion* $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ *satisfies* (26) *everywhere on* $N_1 \times_f N_2$*. Then the warping function* f *either is a constant or has* 2-large *growth for every* $q > 1$.

REMARK 4.2. Corollaries 4.1–4.5 lead to a dichotomy between constancy and 'infinity' of the warping functions on complete non-compact Riemannian manifolds for isometric immersions of the warped products.

THEOREM 4.6. *Let f be a non-constant, L^q function on N_1 for some $q > 1$; then for any isometric immersion of $N_1 \times_f N_2$ into a Riemannian manifold $\tilde{M}^m_c$ with $c \leq 0$ we have* (25) *at some points.*

5. Proofs of Theorems 4.1–4.6 and Corollaries 4.1–4.5. Let N_1 be a complete non-compact Riemannian n_1-manifold. It is well known that for any $x_0 \in N_1$ and any pair of positive numbers s, t with $s < t$ there exists a rotationally symmetric Lipschitz continuous function $\psi(x) = \psi(x; s, t)$ and a constant $C_1 > 0$ (independent of x_0, s, t) with the following properties (cf. [**1**]):

(i) $\psi \equiv 1$ on $B(x_0; s)$ and $\psi \equiv 0$ off $B(x_0; t)$;

(ii) $|\nabla\psi| \leq \frac{C_1}{t-s}$, a.e. on N_1;

(iii) $0 \leq \psi \leq 1$.

Now, we assume that N_2 is a Riemannian n_2-manifold and that $\phi : N_1 \times_f N_2 \to \tilde{M}^m_c$ is an isometric immersion from $N_1 \times_f N_2$ into a Riemannian m-manifold $\tilde{M}^m_c$ with with sectional curvature $\tilde{K} \leq c \leq 0$.

Proof of Theorem 4.1. Assume on the contrary that the squared mean curvature of ϕ: $N_1 \times_f N_2 \to \tilde{M}^m_c$ satisfies (26). Then, Theorem 3.1 implies that the non-constant warping function $f > 0$ satisfies $\Delta f \geq 0$. Let $\psi \geq 0$ be a standard cut-off as above and $m > 1$. Then we have

$$0 \leq \psi^2 f^{m-1}\Delta f = \psi^2 f^{m-1}\mathrm{div}(\nabla f) = \mathrm{div}(\psi^2 f^{m-1}\nabla f) - \left\langle \nabla(\psi^2 f^{m-1}), \nabla f \right\rangle.$$

Then by integrating this expression over N_1 and applying Stoke's theorem we find

$$\begin{aligned} 0 &\geq \int_{B(t)} \langle \nabla(\psi^2 f^{m-1}), \nabla f\rangle dv \\ &\geq \int_{B(t)\setminus B(s)} 2\psi f^{m-1}\langle \nabla\psi, \nabla f\rangle dv + \int_{B(t)} (m-1)\psi^2 f^{m-2}|\nabla f|^2 dv, \end{aligned} \tag{28}$$

where we have used the fact $\psi \equiv 1$ on $B(s)$. Hence, by Cauchy–Schwarz inequality and also condition (ii) on ψ, we have

$$\begin{aligned} \int_{B(t)} \psi^2 f^{m-2}|\nabla f|^2 dv &\leq \frac{2}{m-1}\int_{B(t)\setminus B(s)} \psi f^{m-1}|\nabla\psi|\,|\nabla f| dv \\ &\leq \frac{2C_1}{(m-1)(t-s)}\int_{B(t)\setminus B(s)} \psi f^{m-1}|\nabla f| dv. \end{aligned} \tag{29}$$

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Thus, from (29) and by Cauchy–Schwarz inequality, we obtain

$$\int_{B(t)} \psi^2 f^{m-2}|\nabla f|^2 dv$$
$$\leq \frac{2C_1}{(m-1)(t-s)}\left(\int_{B(t)\setminus B(s)} f^m dv\right)^{\frac{1}{2}}\left(\int_{B(t)\setminus B(s)} \psi^2 f^{m-2}|\nabla f|^2 dv\right)^{\frac{1}{2}}. \tag{30}$$

Let $\{r_j\}$ be a sequence of strictly increasing positive numbers and $\varphi_j(x) := \psi(x; r_j, r_{j+1})$ be the cut-off function, and put $\widetilde{C} = (\frac{2C_1}{m-1})^2$. Let us define

$$A_j = \frac{1}{r_j^2}\int_{B(r_j)} f^m dv, \quad Q_{j+1} = \int_{B(r_{j+1})} \varphi_j^2 f^{m-2}|\nabla f|^2 dv. \tag{31}$$

Then we have

$$\int_{B(r_{j+1})\setminus B(r_j)} f^m dv = r_{j+1}^2 A_{j+1} - r_j^2 A_j \tag{32}$$

and

$$\int_{B(r_{j+1})\setminus B(r_j)} \varphi_j^2 f^{m-2}|\nabla f|^2 dv$$
$$\leq \int_{B(r_{j+1})} \varphi_j^2 f^{m-2}|\nabla f|^2 dv - \int_{B(r_j)} \varphi_{j-1}^2 f^{m-2}|\nabla f|^2 dv$$
$$= Q_{j+1} - Q_j, \tag{33}$$

since $\varphi_{j-1}^2 \leq 1 = \varphi_j^2$ on $B(r_j)$. Combining (30), (32) and (33) gives

$$Q_{j+1}^2 \leq \widetilde{C}\left(\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1} - r_j)^2}\right)[Q_{j+1} - Q_j]. \tag{34}$$

Since the warping function f is non-constant, there exists a real number $a > 0$ such that $f \not\equiv$ const. on $B(a)$. As a consequence of this and (34), we observe that for any strictly increasing positive sequence $\{r_j\}$, there exists an integer $\ell_0 > 0$ such that

$$\begin{aligned} &r_{\ell_0} \geq a\,, Q_j > \mathcal{Q}_a := \int_{B(a)} \varphi_j^2 f^{m-2}|\nabla f|^2 dv > 0, \\ &r_{j+1}^2 A_{j+1} - r_j^2 A_j > 0 \quad \text{and} \quad Q_{j+1} - Q_j > 0 \quad \text{whenever } j > \ell_0. \end{aligned} \tag{35}$$

If we choose a positive sequence $\{r_j\}$ satisfying $r_{j+1} \geq 2r_j$, then we have

$$\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1} - r_j)^2} \leq \frac{r_{j+1}^2 A_{j+1}}{(r_{j+1} - \frac{1}{2}r_{j+1})^2} \leq 4A_{j+1}. \tag{36}$$

It follows from (34) and (36) that

$$Q_{j+1}^2 \leq 4\widetilde{C}A_{j+1}[Q_{j+1} - Q_j] \quad \text{and} \quad Q_{j+1} \leq 4\widetilde{C}A_{j+1}. \tag{37}$$

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On the other hand, it follows from assumption '$\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} f^m dv < \infty$' that there exist a constant $K > 0$ and a sequence $\{r_j\}$ with $r_{j+1} \geq 2r_j$ such that $A_{j+1} \leq K$. From (37) we know that the corresponding sequence $\{Q_j\}$ is bounded from above by $4\widetilde{C} \cdot K$. So, after summing up (37) over j, we obtain

$$\sum_{j=1}^{N} Q_{j+1}^2 \leq 4\widetilde{C}K\big[Q_{N+1} - Q_1\big] \leq 4\widetilde{C}KQ_{N+1} \leq 16\widetilde{C}^2K^2 \tag{38}$$

for each integer $N > 1$. Therefore, we get $Q_j \to 0$ as $j \to \infty$. Consequently, f must be a constant, which is a contradiction. □

Proof of Theorem 4.2. Assume that (26) holds everywhere on $N_1 \times_f N_2$. Let a and $\mathcal{Q}_a := \mathcal{Q}_a(0) > 0$ be as in (35). It then follows from (34) and summing over j from ℓ_0 to ℓ that for every strictly increasing sequence $\{r_j\}_1^\infty$ going to infinity and every $r_{\ell_0} > a$, we have

$$\begin{aligned}\sum_{j=\ell_0}^{\ell} \frac{(r_{j+1}-r_j)^2}{r_{j+1}^2A_{j+1} - r_j^2A_j} &\leq \widetilde{C}\sum_{j=\ell_0}^{\ell} \frac{[Q_{j+1}-Q_j]}{Q_{j+1}^2}\\ &< \widetilde{C}\int_{\mathcal{Q}_a}^{\infty} \frac{1}{r^2}dr\\ &= \frac{\widetilde{C}}{\mathcal{Q}_a} < \infty.\end{aligned} \tag{39}$$

Letting $\ell \to \infty$, we get

$$\sum_{j=\ell_0}^{\infty} \frac{(r_{j+1}-r_j)^2}{\displaystyle\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv} < \infty \tag{40}$$

in which $q = m > 1$. Thus, f is 2-severe, which is a contradiction. □

Proof of Theorem 4.3. Assume that (26) holds everywhere on $N_1 \times_f N_2$. Consider the general term in the finite series (39); we have

$$\frac{r_{j+1}-r_j}{r_{j+1}^2A_{j+1} - r_j^2A_j} \leq \widetilde{C}\left(\frac{Q_{j+1}-Q_j}{r_{j+1}-r_j}\right) \cdot \frac{1}{Q_{j+1}^2}. \tag{41}$$

Since $\{r_j\}$ is arbitrary in (41), we can set a variable $r = r_j$, let $r_{j+1} \to r = r_j$ and obtain

$$\frac{1}{\frac{d}{dr}(r^2A_r)} \leq \widetilde{C}\frac{\frac{d}{dr}Q_r}{Q_r^2} \tag{42}$$

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for $r > a$ a.e., where integrals $Q_r := Q_j$ and $A_r := A_j$, with $B(r)$ as their domain of integrations. Integrating the above inequality over the interval $[a, t]$ yields

$$\int_a^t \left(\frac{1}{\frac{d}{dr}(r^2 A_r)} \right) dr \le \frac{\tilde{C}}{Q_a} < \infty. \tag{43}$$

Let $t \to \infty$; we get

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q dv} \right) dr < \infty \tag{44}$$

in which $q = m > 1$ by the Coarea formula. This contradicts the assumption that f is 2-obtuse. □

Proof of Theorem 4.4. Assume that (26) holds everywhere on $N_1 \times_f N_2$. We observe that for any sequence $\{r_j = 2^j \tilde{r}_0, \tilde{r}_0 > 0\}$ and any $F(r) > 0$,

$$\begin{aligned} \sum_{j=\ell_0}^{\ell} \frac{(r_{j+1} - r_j)^2}{r_{j+1}^2 A_{j+1} - r_j^2 A_j} &\ge \frac{1}{4} \sum_{j=\ell_0}^{\ell} \frac{r_{j+1}^2}{r_{j+1}^2 A_{j+1}} \\ &\ge \frac{1}{8} \sum_{j=\ell_0}^{\ell} \int_{r_{j+1}}^{r_{j+2}} \left(\frac{r}{r^2 A_r} \right) dr \\ &= \frac{1}{8} \int_{r_{\ell_0+1}}^{r_{\ell+2}} \frac{1}{rF(r)\left(\frac{r^2 A_r}{r^2 F(r)}\right)} dr. \end{aligned} \tag{45}$$

The second step follows from the mean-value theorem for integrals. Combining (40) and (45) together and letting $\ell \to \infty$, one gets

$$\int_{r_{\ell_0+1}}^\infty \frac{1}{rF(r)\left(\frac{r^2 A_r}{r^2 F(r)}\right)} dr < \infty.$$

Since f is assumed to have 2-moderate growth for $q = m > 1$, there exist constants $C > 0$, $a < r$ and $F \in \mathcal{F}$ such that

$$\infty = \frac{1}{C} \int_a^\infty \frac{1}{rF(r)} dr < \int_a^\infty \frac{1}{rF(r)\left(\frac{r^2 A_r}{r^2 F(r)}\right)} dr < \infty,$$

which is a contradiction. □

Proof of Theorem 4.5. Assume that (26) holds everywhere on $N_1 \times_f N_2$. By assumption f has 2-small growth for $q = m > 1$ and some $a > 0$. Thus, if we define $F_0(r) = \left(\frac{r^2 A_r}{r^2}\right)$, i.e. $\frac{r^2 A_r}{r^2 F_0(r)} = 1$, we would have $\int_a^\infty \frac{1}{rF_0(r)} dr = \infty$, *i.e.*, $F_0(r) \in$

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$\mathcal{F}$. In view of Theorem 4.4, this would lead to $1 = \limsup_{r\to\infty} \frac{1}{r^2 F_0(r)} r^2 A_r = \infty$, a contradiction. □

Proof of Corollaries 4.1–4.5. Follows at once from Theorem 9 and respectively from Theorems 4.1–4.5. □

Proof of Theorem 4.6. This follows from the fact that every L^q-function has 2-*finite*, 2-*mild*, 2-*obtuse*, 2-*moderate*, 2-*small* growth for the same q (cf. [**9**]) and any one of Theorems 4.1–4.5. Remark 4.2. For the most recent survey on the growth related to (19)–(24), see [**11**]. □

6. Some applications and remarks.

THEOREM 6.1. *Suppose $q > 1$ and the warping function f is one of the following:* 2-finite*,* 2-mild*,* 2-obtuse*,* 2-moderate *and* 2-small*. If N_2 is compact, then there does not exist an isometric minimal immersion from $N_1 \times_f N_2$ into any Euclidean space.*

Proof. Suppose the contrary; then inequality (26) would be true. Hence by Corollaries 4.1–4.5, f would be a constant. It follows from Theorem 3.1 that ϕ would be mixed totally geodesic and hence a product of minimal immersion (see [**5**]),

$$\phi = (\phi_1, \phi_2) : N_1 \times N_2 \to \mathbb{E}^{m_1} \times \mathbb{E}^{m_2} = \mathbb{E}^m.$$

This would contradict the fact that there is no compact minimal submanifold N_2 in the Euclidean space $\mathbb{E}^{m_2}$. □

Finally, from Theorem 4.6 we have the following.

THEOREM 6.2. *If f is an L^q function on N_1 for some $q > 1$, then for any Riemannian manifold N_2 the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any Riemannian manifold with non-positive sectional curvature.*

Proof. This follows immediately from Theorem 4.6. □

7. Remarks.

REMARK 7.1. In views of our results, it is interesting to point out that there do exist isometric minimal immersions from a warped product $N_1 \times_f N_2$ into $\tilde{M}^m_c$ with $c \le 0$ such that the warping function f is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any $q > 1$.

A simple example of this is the warped product $\mathbf{R} \times_{e^x} \mathbb{E}^{n-1}$ (or $\mathbf{R} \times_{e^{-x}} \mathbb{E}^{n-1}$) of constant sectional curvature -1 which can be isometrically immersed in $H^{n+1}(-1)$ as a totally geodesic (hence minimal) submanifold.

REMARK 7.2. Inequality (25) (resp. inequality (26)) on H^2 as the assumption for Theorems 4.1–4.5 (resp. assumption of Corollaries 4.1–4.5) is sharp. This can be seen from the following two examples (cf. [**2**]).

First, let us regard the Euclidean $2k$-space $\mathbb{E}^{2k}$ as the warped product $\mathbb{E}^k \times_f \mathbb{E}^k$ with a constant warping function f. Then $\mathbb{E}^k \times_f \mathbb{E}^k$ can be isometrically immersed in $H^{2k+1}(-1)$ as a totally umbilical hypersurface with $H^2 = 1$. Since $n_1 = n_2 = k$, the

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right-hand side of (26) is also equal to 1. Thus, this example satisfies the equality case of (26). For the case $k \leq 2$, this example also shows that the non-constant assumption on Theorems 4.1–4.5 cannot be dropped.

The second example is the warped product $\mathbf{R} \times_{\cosh bx} \mathbf{R}$ of constant negative curvature $-b^2$. This warped product admits an isometric immersion in $H^3(-1)$ as totally umbilical surface, for $0 < b < 1$. The squared mean curvature of the immersion satisfies

$$H^2 = 1 - b^2 < \frac{4n_1n_2|c|}{(n_1+n_2)^2} = 1 \quad \text{and} \quad H^2 \to 1 \quad \text{as} \quad b \to 0.$$

The warping function $\cosh bx$ with $0 < b < 1$ is a non-constant and non-harmonic function which is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any $q > 1$.

REMARK 7.3. Let φ be the function on $\mathbf{R}$ defined by $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = x^2$ for $|x| > 1$. Denote by f the smooth-out function of φ at ± 1. Then f is a subharmonic function on $\mathbf{R}$ which is 2-finite, 2-mild, 2-obtuse, 2-moderate and 2-small for any $q \leq 1$; but it is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for $q > 1$.

The sectional curvature K of the warped product $N = \mathbf{R} \times_f \mathbb{E}^{n-1}$ with this subharmonic warping function f satisfies $K \leq 0$. Let $\tilde{M}_0^{n+1} = \mathbf{R} \times N$ denote the Riemannian product of the real line and N. Clearly, the sectional curvatures of $\tilde{M}_0^{n+1}$ is bounded above by 0 and the warped product N can be trivially isometrically imbedded in $\tilde{M}_0^{n+1}$ as a totally geodesic hypersurface. This isometric imbedding of N in $\tilde{M}_0^{n+1}$ satisfies $H^2 = c = 0$, which shows that the condition '$q > 1$' given in Theorems 4.1–4.5 and Corollaries 4.1–4.5 is sharp as well.

REMARK 7.4. The assumption on the warping function f given in Theorem 6.1 cannot be dropped, since there do exist minimal hypersurfaces in $\mathbb{E}^{n+1}$ which are isometric to some warped products $N_1 \times_f N_2$ with compact N_2. A simple such example is the hypercaternoid in $\mathbb{E}^{n+1}$ (cf. [**5**]). The hypercaternoid is isometric to a warped product $\mathbf{R} \times_f S^{n-1}$ with compact $N_2 = S^{n-1}$ whose warping function is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any $q > 1$ according to Theorem 6.1.

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REFERENCES

1. A. Andreotti and E. Vesentini, Carleman estimates for the Laplace–Beltrami equation on complex manifolds, *Inst. Hautes Tudes Sci. Publ. Math.* **25** (1965), 81–130.

2. B. Y. Chen, *Geometry of submanifolds* (M. Dekker, New York, 1973).

3. B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* **60** (1993), 568–578.

4. B.Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Jpn J. Math.* **26** (2000), 105–127.

5. B. Y. Chen, On isometric minimal immersion from warped products into space forms, *Proc. Edinb. Math. Soc.* **45** (2002), 579–587.

6. B. Y. Chen, δ-invariants, inequalities of submanifolds and their applications, in *Topics in Differential Geometry* (Mihai A, Mihai I and Miron R, Editors), (Editura Academiei Române, Bucharest, Romania, 2008), 29–155.

7. B. Y. Chen and S. W. Wei, Submanifolds of warped product manifolds $I \times_f S^{m-1}(k)$ from a p-harmonic viewpoint, *Bull. Transilv. Univ. Braşov Ser. III*, **1**(50) (2008), 59–78.

8. L. Karp, Subharmonic functions on real and complex manifolds, *Math. Z.* **179** (1982), 535–554.

9. S. W. Wei, J. Li and L. Wu, Generalizations of the uniformization theorem and Bochner's method in p-harmonic geometry, *Comm. Math. Anal. Conf.* **01** (2008), 46–68.

10. S. W. Wei, J. F. Li and L. Wu, Sharp estimates on $\mathcal{A}$-harmonic functions with applications in biharmonic maps, preprint.

11. S. W. Wei, p-harmonic geometry and related topics, *Bull. Transilv. Univ. Brasov Ser. III* **1**(50) (2008), 415–453.