THE EFFECT OF SMOOTHNESS ON VARIATION

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(Received 6 April 1972)

Communicated by J. Virsik

1. Introduction

Let R be the set of real numbers, and let S_1 denote the class of all real valued functions f on R which are smooth to the first order (i.e. the derivative $f^{(1)}$ exists and is continuous) and have compact support. The first order variation of f on an open set U is given by

$$I_1(f,U) = \int_U |f^{(1)}(x)|\,dx$$

and in the case where $U = R$ we have the total first order variation of f, usually denoted by $I_1(f)$.

$$I_1(f) = \int |f^{(1)}(x)|\,dx$$

We wish to establish an alternative expression for the total first order variation. Let P be the set on which $f^{(1)}$ is non zero. Since P is open we can find a countable collection of mutually disjoint open intervals (a_{1j}, b_{1j}) such that

$$P = \bigcup_{j=1}^{\infty} (a_{1j}, b_{1j})$$

Now we define $d_j = |f(b_{1j}) - f(a_{1j})| > 0$ and hence d_j is the first order variation of f on the interval (a_{1j}, b_{1j}). Thus it is easily seen that

$$I_1(f) = \sum_{j=1}^{\infty} d_j$$

We will show that the sequence $\{d_j\}$ can be rearranged to give a sequence $\{d_{j_n}\}$ such that $d_{j_n} \geqq d_{j_{n+1}}$ for all $n = 1, 2, \cdots$ and such that

$$d_{j_n} = \frac{c_n}{n} \quad all\ n = 1, 2, \cdots$$

where $c_n \to 0$ as $n \to \infty$. If, for each $\lambda > 0$ we define the open set $F_\lambda = \{x \mid x \in R$ and $0 < f(x) < \lambda\}$ then it is possible to show that

$$\lim_{\lambda \to 0} I_1(f, F_\lambda) = 0.$$

Let m be a natural number. We use S_{m+1} to denote the subclass of S_1 containing all functions f which are smooth to order $m+1$ (i.e. the $(m+1)$th derivative $f^{(m+1)}$ exists and is continuous). It seems intuitively reasonable that increased smoothness in f will be associated with decreased first order variation. We will show that it is now possible to write

$$d_{j_n} = \frac{k_n}{n^{m+1}} \quad all\ n = 1, 2, \cdots$$

where $k_n \to 0$ as $n \to \infty$ and hence to show that

$$\lim_{\lambda \to 0} \lambda^{1/(m+1)-1} \,.\, I_1(f, F_\lambda) = 0.$$

It is now possible to explain the origin of the problem and to suggest an extension and application of the above results. Let $f \in L$ (the class of all integrable functions). The first order variation of f on an open set U can be defined as

$$I_1(f, U) = \sup_{\psi} \int f(x)\psi^{(1)}(x)dx$$

where the supremum is taken over all infinitely differentiable functions ψ such that spt. $(\psi) \subset U$ and with $\|\psi\| \leqq 1$ (we are using the uniform norm). This definition can now be extended to cover more general sets. When $U = R$ we have the total first order variation $I_1(f)$. We use B_1 to denote the subclass of L consisting of those f with compact support for which $I_1(f)$ is finite. We can now use the co-area formula ([1]) to show that

$$\lim_{\lambda \to 0} I_1(f, F_\lambda) = 0$$

Following investigations by Michael ([5], [6], [7],) and Goffman ([2], [3]) which make implicit use of this result it is possible to state the following theorem

> "*Let $f \in B_1$ and choose $\varepsilon > 0$. We can find $g \in S_1$ such that the set $\{x \mid x \in R$ and $f(x) \neq g(x)\}$ has measure less than ε and such that $I_1(g) < I_1(f) + \varepsilon$.*"

Let m be a natural number. The $(m+1)$th order variation of f on an open set U can be defined as

https://doi.org/10.1017/S1446788700019182 Published online by Cambridge University Press

$$I_{m+1}(f,U) = \sup_{\psi} \int f(x)\psi^{(m+1)}(x)dx$$

where the supremum is taken over all infinitely differentiable functions ψ such that $\mathrm{spt}(\psi) \subset U$ and with $\|\psi\| \leqq 1$. This definition can now be extended to cover more general sets. When $U = R$ we obtain the total $(m+1)$th order variation. Let B_{m+1} denote the subclass of B_1 for which $I_{m+1}(f)$ is finite. For $m = 1$, the method described above for smooth functions can be modified ([4]) to show that

$$\lim_{\lambda \to 0} \lambda^{1/(m+1)-1} \; I_1(f,F_\lambda) = 0$$

This result is then used to establish the theorem stated below

> "*Let $f \in B_{m+1}$ and choose $\varepsilon > 0$. We can find $g \in S_{m+1}$ such that the set $\{x \mid x \in R$ and $f(x) \neq g(x)\}$ has measure less than ε and such that $I_{m+1}(g) < I_{m+1}(f) + \varepsilon$.*"

It is believed that this theorem can be proved for all values of m and that the work in this paper can be modified to provide a basis for the proof. It should be noted however that, as in [4], these result may only provide a partial answer to the corresponding theorem in R^2.

2. Variation in S_1

Let $f \in S_1$. Since $f^{(1)}$ is continuous and has compact support it follows from the integral definition that

$$I_1(f) < \infty$$

LEMMA 2.1. *The sequence $\{d_j\}$ can be rearranged to a sequence $\{d_{j_n}\}$ such that $d_{j_n} \geqq d_{j_{n+1}}$ for all $n = 1, 2, \cdots$ and in fact we can write*

$$d_{j_n} = \frac{c_n}{n} \quad \text{all } n = 1, 2, \cdots$$

where $\{c_n\}$ is bounded and has limit zero.

PROOF. Let $D = \{d_j \text{ for all } j = 1, 2, \cdots\}$

$$D_r = \left\{ d_j \,\middle|\, \frac{1}{r-1} > d_j \geqq \frac{1}{r} \right\} \text{ each } r = 1, 2, 3, \cdots$$

Since $d_j > 0$ all $j = 1, 2, \cdots$ it follows that $D = \bigcup_{r=1}^{\infty} D_r$.

Because $\sum_{j=1}^{\infty} d_j < \infty$ it follows that each set D_r is finite and hence the sequence $\{d_j\}$ can be rearranged to give a sequence $\{d_{j_n}\}$ where $d_{j_n} \geqq d_{j_{n+1}}$.

We define the sequence $\{c_n\}$ by letting $c_n = nd_{j_n}$. Now for each $p = 1, 2, \cdots$ we have

$$\sum_{n=1}^{p} d_{j_n} < I_1(f)$$

$$\therefore\ p.\, d_{j_p} < I_1(f)$$

$$\therefore\ c_p < I_1(f)$$

On the other hand if we take $\varepsilon > 0$ we can choose N such that for each $p = 1, 2, \cdots$ we have

$$\sum_{n=N+1}^{N+p} d_{j_n} < \varepsilon$$

$$\therefore\ p.\, d_{j_{N+p}} < \varepsilon$$

$$\therefore\ c_{N+p} < \varepsilon \left(1 + \frac{N}{p}\right)$$

$$\therefore\ \limsup_{p\to\infty} c_{N+p} \leqq \varepsilon$$

Since ε is arbitrary it follows that $\{c_n\}$ has limit zero.

LEMMA 2.2. *For each $\lambda > 0$ we have*

$$I_1(f, F_\lambda) \leqq N\lambda + \sum_{n=N+1}^{\infty} d_{j_n}$$

for all $N = 1, 2, 3, \cdots$

PROOF. Since F_λ is open

$$\begin{aligned} I_1(f, F_\lambda) &= \int_{F_\lambda} |f^{(1)}(x)|\, dx \\ &= \sum_{j=1}^{\infty} \int_{F_\lambda \cap (a_{1j}, b_{1j})} |f^{(1)}(x)|\, dx \\ &= \sum_{j=1}^{\infty} \left| \int_{F_\lambda \cap (a_{1j}, b_{1j})} f^{(1)}(x) dx \right| \end{aligned}$$

Now suppose $d_j \geqq \lambda$ Consider the case where

$$f(a_{1j}) \leqq 0 < \lambda \leqq f(b_{1j})$$

We can choose (ξ_j, ζ_j) such that $(\xi_j, \zeta_j) = F_\lambda \cap (a_{1j}, b_{1j})$ and also $f(\xi_j) = 0$, $f(\zeta_j) = \lambda$.

$$\therefore \int_{F_\lambda \cap (a_{1j}, b_{1j})} f^{(1)}(x)dx = \int_{(\xi_j, \zeta_j)} f^{(1)}(x)dx = \lambda.$$

By using similar reasoning it can be shown in all cases where $d_j \geqq \lambda$ that

$$\left| \int_{F_\lambda \cap (a_{1j}, b_{1j})} f^{(1)}(x)dx \right| \leqq \lambda.$$

Now if we choose N_λ such that

$$d_{j_{N_\lambda}} \geqq \lambda > d_{j_{N_\lambda + 1}}$$

then we have

$$I_1(f, F_\lambda) \leqq N_\lambda . \lambda + \sum_{n=N_\lambda+1}^{\infty} \left| \int_{F_\lambda \cap (a_{1j_n}, b_{1j_n})} f^{(1)}(x)dx \right|$$

$$\leqq N_\lambda . \lambda + \sum_{n=N_\lambda+1}^{\infty} \left| \int_{(a_{1j_n}, b_{1j_n})} f^{(1)}(x)dx \right|$$

$$= N_\lambda . \lambda + \sum_{n=N_\lambda+1}^{\infty} d_{j_n}$$

and it is easily seen that for each $N = 1, 2, \cdots$

$$N_\lambda . \lambda + \sum_{n=N_\lambda+1}^{\infty} d_{j_n} \leqq N . \lambda + \sum_{n=N+1}^{\infty} d_{j_n}$$

THEOREM 2.3 $\lim_{\lambda \to 0} I_1(f, F_\lambda) = 0.$

PROOF. Let $\varepsilon > 0$. Choose N such that $\Sigma_{n=N+1}^{\infty} d_{j_n} < \varepsilon/2$ and then choose λ sufficiently small such that $N\lambda < \varepsilon/2$. It follows that

$$I_1(f, F_\lambda) \leqq N\lambda + \sum_{n=N+1}^{\infty} d_{j_n} < \varepsilon$$

$$\therefore \lim_{\lambda \to 0} I_1(f, F_\lambda) = 0.$$

3. Variation in S_{m+1}

Let m be a natural number and suppose that $f \in S_{m+1}$. The $(m+1)$th order variation of f on an open set U can be defined as

$$I_{m+1}(f, U) = \int_U \left| f^{(m+1)}(x) \right| dx$$

https://doi.org/10.1017/S1446788700019182 Published online by Cambridge University Press

and in the case $U = R$ we have the total $(m+1)$th order variation usually denoted by $I_{m+1}(f)$. Since $f^{(m+1)}$ is continuous and has compact support it follows from the definition that

$$I_{m+1}(f) < \infty .$$

Consider the intervals (a_{1j}, b_{1j}). If we write $h_{1j} = b_{1j} - a_{1j}$ and if we choose L such that $\text{spt}(f) \subset [-L/2, L/2]$ then

$$\sum_{j=1}^{\infty} h_{1j} \leqq L$$

We define $a_{2j} \in (a_{1j}, b_{1j})$ such that $|f^{(1)}(a_{2j})| \geqq |f^{(1)}(x)|$ for all $x \in (a_{1j}, b_{1j})$. Since a_{2j} is a local extremum of $f^{(1)}(x)$ it follows that $f^{(2)}(a_{2j}) = 0$. Define $b_{2j} = \inf\{x \,|\, x \geqq b_{1j}$ and $f^{(2)}(x) = 0\}$. Obviously $f^{(2)}(b_{2j}) = 0$ and we also know that if $b_{1j} \leqq a_{1k}$ then from the definition of b_{2j} we have $b_{2j} \leqq a_{2k}$. Hence the intervals (a_{2j}, b_{2j}) are mutually disjoint. We also know that $b_{1j} \in (a_{2j}, b_{2j}]$. If we write $h_{2j} = b_{2j} - a_{2j}$ then we have

$$\sum_{j=1}^{\infty} h_{2j} \leqq L$$

We can deduce in addition that

$$(1) \qquad |f^{(1)}(a_{2j})| \geqq \frac{|f(b_{1j}) - f(a_{1j})|}{b_{1j} - a_{1j}} = \frac{d_j}{h_{1j}}$$

If $m = 1$ we need not continue. Otherwise suppose that for some natural number i with $2 \leqq i \leqq m$ we have mutually disjoint intervals (a_{ij}, b_{ij}) with $f^{(i)}(a_{ij}) = f^{(i)}(b_{ij}) = 0$ and with a point $b_{i-1j} \in (a_{ij}, b_{ij}]$ such that $f^{(i-1)}(b_{i-1j}) = 0$. Define $a_{i+1j} \in (a_{ij}, b_{ij})$ such that $|f^{( )}(a_{i+1j})| \geqq |f^{(i)}(x)|$ for all $x \in (a_{ij}, b_{ij})$. Since a_{i+1j} is a local extremum of $f^{(i)}(x)$ it follows that $f^{(i+1)}(a_{i+1j}) = 0$. We define $b_{i+1j} = \inf\{x \,|\, x \geqq b_{ij}$ and $f^{(i+1)}(x) = 0\}$. Obviously $f^{(i+1)}(b_{i+1j}) = 0$ and we also know that if $b_{ij} \leqq a_{ik}$ then from the definition of b_{i+1j} we have $b_{i+1j} \leqq a_{i+1\,k}$. Hence the intervals (a_{i+1j}, b_{i+1j}) are mutually disjoint. We know that $b_{ij} \in (a_{i+1j}, b_{i+1j}]$ and if we write $h_{i+1j} = b_{i+1j} - a_{i+1j}$ then it is true that

$$\sum_{j=1}^{\infty} h_{i+1j} \leqq L$$

We also have

$$|f^{(i)}(a_{i+1j})| \geqq \frac{|f^{(i-1)}(b_{i-1j}) - f^{(i-1)}(a_{ij})|}{b_{i-1j} - a_{ij}}$$

and since $f^{(i-1)}(b_{i-1j}) = 0$ and $h_{ij} = b_{ij} - a_{ij} \geqq b_{i-1j} - a_{ij}$ it follows that

$$|f^{(i)}(a_{i+1j})| \geqq \frac{|f^{(i-1)}(a_{ij})|}{h_{ij}} \tag{2}$$

By induction it follows that the intervals (a_{i+1j}, b_{i+1j}) can be defined for each $i = 2, 3, \cdots, m$. Now since the intervals (a_{mj}, a_{m+1j}) are disjoint it follows that

$$I_{m+1}(f) = I_1(f^{(m)}) \geqq \sum_{j=1}^{\infty} |f^{(m)}(a_{m+1j}) - f^{(m)}(a_{mj})|$$

$$\therefore I_{m+1}(f) \geqq \sum_{j=1}^{\infty} |f^{(m)}(a_{m+1j})|$$

and bý repeated use of (2) and finally using (1) this becomes

$$I_{m+1}(f) \geqq \sum_{j=1}^{\infty} \frac{d_j}{h_{mj}h_{m-1j} \cdot \cdots \cdot h_{2j}h_{1j}} \tag{3}$$

It is convenient now to quote a standard inequality

LEMMA 3.1. *Let $x_{ij} > 0$ each $i = 1, \cdots, m$ and each $j = 1, \cdots, p$. Suppose that $\sum_{j=1}^{p} x_{ij} \leqq L$ for each $i = 1, \cdots, m$. Then*

$$\sum_{j=1}^{p} \frac{1}{x_{mj}x_{m-1j} \cdot \cdots \cdot x_{2j}x_{1j}} \geqq \frac{p^{m+1}}{L^m}$$

LEMMA 3.2. *The sequence $\{d_j\}$ can be rearranged to a sequence $\{d_{j_n}\}$ such that $d_{j_n} \geqq d_{j_{n+1}}$ for all $n = 1, 2, \cdots$ and in fact we can write*

$$d_{j_n} = \frac{k_n}{n^{m+1}} \quad all\ n = 1, 2, \cdots$$

where $\{k_n\}$ is bounded and has limit zero.

PROOF. Define $k_n = n^{m+1} d_{j_n}$. Now for each $p = 1, 2, \cdots$ we have

$$\sum_{n=1}^{p} \frac{d_{j_n}}{h_{mj_n}h_{m-1j_n} \cdots h_{2j_n}h_{1j_n}} \leqq I_{m+1}(f)$$

$$\therefore d_{j_p} \left(\sum_{n=1}^{p} \frac{1}{h_{mj_n} \cdot \cdots \cdot h_{2j_n}h_{1j_n}} \right) \leqq I_{m+1}(f)$$

$$\therefore \frac{k_p}{p^{m+1}} \cdot \frac{p^{m+1}}{L^m} \leqq I_{m+1}(f)$$

$$\therefore k_p \leqq L^m I_{m+1}(f)$$

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On the other hand if we take $\varepsilon > 0$ we can choose N such that for each $p = 1, 2, \cdots$ we have

$$\sum_{n=N+1}^{N+p} \frac{d_{j_n}}{h_{mj_n} \cdot \cdots \cdot h_{2j_n} h_{1j_n}} < \frac{\varepsilon}{L^m}$$

Thus as above we deduce that

$$\frac{k_{N+p}}{(N+p)^{m+1}} \cdot \frac{p^{m+1}}{L^m} < \frac{\varepsilon}{L^m}$$

$$\therefore k_{N+p} < \varepsilon \left(1 + \frac{N}{p}\right)^{m+1}$$

$$\therefore \limsup_{p \to \infty} k_{N+p} \leqq \varepsilon$$

Since ε is arbitrary it follows that $\{k_n\}$ has limit zero.

THEOREM 3.3 *For each λ with $0 < \lambda < 1$ we have*

$$\lambda^{1/(m+1)-1} I_1(f, F_\lambda) \leqq 1 + \frac{(2L)^m I_{m+1}(f)}{m}$$

and in addition

$$\lim_{\lambda \to 0} \lambda^{1/(m+1)-1} I_1(f, F_\lambda) = 0$$

PROOF. Choose N such that $1/2N \leqq \lambda^{1/(m+1)} < 1/N$

$$I_1(f, F_\lambda) \leqq N\lambda + \sum_{n=N+1}^{\infty} d_{j_n}$$

$$\leqq \lambda^{m/(m+1)} + \sum_{n=N+1}^{\infty} \frac{L^m . I_{m+1}(f)}{n^{m+1}}$$

$$\leqq \lambda^{m/(m+1)} + L^m . I_{m+1}(f) \cdot \int_N \frac{dx}{x^{m+1}}$$

$$= \lambda^{m/(m+1)} + \frac{L^m . I_{m+1}(f)}{mN^m}$$

$$\leqq \{1 + \frac{(2L)^m . I_{m+1}(f)}{m}\} . \lambda^{m/(m+1)}$$

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Now choose $\varepsilon > 0$, and find N such that

$$k_n \leqq 2m \left(\frac{\varepsilon}{4}\right)^{m+1} \; all \; n \geqq N.$$

Take λ sufficiently small that $0 < \lambda^{1/(m+1)} < \varepsilon/2N$. Choose $N_1 \geqq N$ with

$$\frac{\varepsilon}{4N_1} \leqq \lambda^{1/(m+1)} < \frac{\varepsilon}{2N_1}.$$

Now it follows that

$$\begin{aligned} I_1(f, F_\lambda) &\leqq N_1 \lambda + \sum_{n=N_1+1}^{\infty} d_{j_n} \\ &\leqq \frac{\varepsilon}{2} . \lambda^{m/(m+1)} + \sum_{n=N_1+1}^{\infty} \frac{2m.(\varepsilon/4)^{m+1}}{n^{m+1}} \\ &\leqq \frac{\varepsilon}{2} . \lambda^{m/(m+1)} + 2m \left(\frac{\varepsilon}{4}\right)^{m+1} \int_{N_1}^{\infty} \frac{dx}{x^{m+1}} \\ &= \frac{\varepsilon}{2} . \lambda^{m/(m+1)} + 2m \left(\frac{\varepsilon}{4}\right)^{m+1} \frac{1}{mN_1{}^m} \\ &\leqq \varepsilon \lambda^{m/(m+1)} \end{aligned}$$

Since ε was arbitrarily chosen it follows that

$$\lim_{\lambda \to 0} \lambda^{1/(m+1)-1} I_1(f, F_\lambda) = 0$$

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