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On Partitions into Powers of Primes and Their Difference Functions

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Abstract. In this paper, we extend the approach first outlined by Hardy and Ramanujan for calculating the asymptotic formulae for the number of partitions into r-th powers of primes, $p_{\mathbb{P}^{(r)}}(n)$, to include their difference functions. In doing so, we rectify an oversight of said authors, namely that the first difference function is perforce positive for all values of n, and include the magnitude of the error term.

1 Introduction

For a given subset $\Lambda \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we denote the number of partitions of n into elements from Λ by $p_\Lambda(n)$. That is, $p_\Lambda(n)$ is the number of solutions to the equation

$$a_1\lambda_1 + \cdots + a_m\lambda_m = n,$$

where each $\lambda_i \in \Lambda$, $\lambda_i > \lambda_{i+1}$, and each $a_i \in \mathbb{N}$. We set $p_\Lambda(0) = 1$, corresponding to the empty partition, and we assume that $p_\Lambda(n) = 0$, for $n < 0$.

The k-th difference function of $p_\Lambda(n)$ is defined inductively as follows:

$$\begin{aligned} p_\Lambda^{(0)}(n) &= p_\Lambda(n);\\ p_\Lambda^{(k)}(n) &= p_\Lambda^{(k-1)}(n) - p_\Lambda^{(k-1)}(n-1), \qquad \text{for } k \geq 1. \end{aligned}$$

It is easily established that the generating functions for $p_\Lambda^{(k)}(n)$ can be expressed in the following form:

$$\sum_{n=0}^{\infty} p_\Lambda^{(k)}(n)x^n = (1-x)^k \prod_{\lambda\in\Lambda}(1-x^\lambda)^{-1}.$$

Convergence is absolute when $|x| < 1$.

Products and sums with index p are taken over the set of primes, which we denote by $\mathbb{P}$. We write the set of r-th powers of primes as $\mathbb{P}^{(r)}$.

The purpose of this paper is to prove the following asymptotic formula:

$$\log p_{\mathbb{P}^{(r)}}^{(k-1)}(n) = (r+1)\Big[\Gamma\Big(\frac{1}{r}+2\Big)\zeta\Big(\frac{1}{r}+1\Big)\Big]^{r/(r+1)} n^{1/(r+1)}(\log n)^{-r/(r+1)} \times \Bigg(1 + O_\epsilon\Bigg(\sqrt{\frac{(\log\log n)^{1+\epsilon}}{\log n}}\Bigg)\Bigg), \text{ as } n\to\infty,$$

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for fixed $k, r \geq 1$. The asymptotic, for $k = 1$, without the error term was first given by Hardy and Ramanujan [5]. However, they did not provide a rigorous proof of this fact, and, as has been observed, they assumed that for a given r, $p^{(1)}_{\mathbb{P}^{(r)}}(n) \geq 0$ for all n. This is readily seen to be false for r as low as 2, $n = 5$. Bateman and Erdős [2] showed, however, that if Λ is a set such that the removal of any k elements leaves a set with greatest common divisor 1, then $\lim_{n\to\infty} p^{(k)}_{\Lambda}(n) = \infty$. Hence, for any $r \geq 1, k \geq 0$, $\lim_{n\to\infty} p^{(k)}_{\mathbb{P}^{(r)}}(n) = \infty$. We shall use this fact to rectify the dilemma. The theorem is of the Tauberian type: we shall first prove estimates for the generating functions, and then use them to yield information about the coefficients.

We shall use the following version of the prime number theorem:

$$\pi(x) = Li(x) + E(x),$$

where

$$E(x) = O_\delta\Big(\frac{x}{\log^\delta x}\Big), \text{ for all } \delta \geq 2.$$

2 Asymptotic Formula for the Generating Function

In the following argument, s is assumed to be a small positive quantity approaching 0. Define $\phi(s) = \sum_p e^{-sp^r}$.

Lemma 2.1 *As $s \to 0^+$,*

$$\phi(s) = \int_2^\infty \frac{e^{-su^r}}{\log u} du + O_\delta\Big(\frac{s^{-1/r}}{\log^\delta (1/s)}\Big),$$

for any $\delta \geq 2$.

Proof Using Riemann–Stieltjes integration, we have

$$\begin{aligned}
\phi(s) &= \int_{2^-}^\infty e^{-su^r} d\pi(u) \\
&= \int_{2^-}^\infty e^{-su^r} d(Li(u) + E(u)) \\
&= \int_2^\infty \frac{e^{-su^r}}{\log u} du + \int_{2^-}^\infty e^{-su^r} dE(u).
\end{aligned} \tag{2.1}$$

Let $C = C(s) = \log^{-\delta}(1/s)$. Note that as $s \to 0^+$, $s = o(C(s))$. Assume that

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$2^r s < C$. Integration by parts gives

$$\int_{2^-}^{\infty} e^{-su^r} dE(u) = rs \int_2^{\infty} u^{r-1} e^{-su^r} E(u) du + O(1) \tag{2.2}$$

$$\ll_\delta rs \int_2^{\infty} \frac{u^r e^{-su^r}}{\log^\delta u} du + O(1)$$

$$\ll_\delta r^\delta \int_{2^r s}^{\infty} \Big(\frac{t}{s}\Big)^{1/r} \frac{e^{-t}}{\log^\delta (t/s)} dt + O(1), \text{ via the substitution } t = su^r$$

$$= Cs^{-1/r} \int_{2^r s}^{\infty} \frac{t^{1/r} e^{-t}}{\big(\frac{\log t}{\log(1/s)} + 1\big)^\delta} dt + O(1)$$

$$= Cs^{-1/r} \Big[\int_{2^r s}^{C} \frac{t^{1/r} e^{-t}}{\big(\frac{\log t}{\log(1/s)} + 1\big)^\delta} dt + \int_C^{\infty} \frac{t^{1/r} e^{-t}}{\big(\frac{\log t}{\log(1/s)} + 1\big)^\delta} dt \Big] + O(1).$$

Now,

$$\int_{2^r s}^{C} \frac{t^{1/r} e^{-t}}{\big(\frac{\log t}{\log(1/s)} + 1\big)^\delta} dt \ll \int_{2^r s}^{C} \frac{t^{1/r} e^{-t}}{\big(\frac{\log(2^r s)}{\log(1/s)} + 1\big)^\delta} dt$$

$$= \int_{2^r s}^{C} \frac{t^{1/r} e^{-t}}{\big(\frac{r \log 2}{\log(1/s)}\big)^\delta} dt$$

$$\ll_\delta \log^\delta (1/s) \int_{2^r s}^{C} t^{1/r} e^{-t} dt$$

$$\ll_\delta C \log^\delta (1/s) = 1.$$

On the other hand for $C \le t < \infty$, we have that $\log t / \log(1/s) + 1$ is minimized when $t = C$, so that

$$\int_C^{\infty} \frac{t^{1/r} e^{-t}}{\big(\frac{\log t}{\log(1/s)} + 1\big)^\delta} dt \ll \int_C^{\infty} \frac{t^{1/r} e^{-t}}{\big(\frac{-\delta \log\log(1/s)}{\log(1/s)} + 1\big)^\delta} dt$$

$$\ll_\delta \int_0^{\infty} t^{1/r} e^{-t} dt \ll_\delta 1.$$

Hence by (2.2),

$$\int_{2^-}^{\infty} e^{-su^r} dE(u) \ll_\delta Cs^{-1/r},$$

which together with (2.1) completes the proof. ■

Lemma 2.2 *As $s \to 0^+$,*

$$\int_2^{\infty} \frac{e^{-su^r}}{\log u} du = r\Gamma\Big(\frac{1}{r} + 1\Big) s^{-1/r} (\log(1/s))^{-1} + O\Big(\frac{s^{-1/r} \log\log(1/s)}{\log^2 (1/s)}\Big).$$

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Proof Making the substitution $t = su^r$ into the integral gives

$$\int_2^\infty \frac{e^{-su^r}}{\log u}du = s^{-1/r}\int_{2^r s}^\infty \frac{t^{1/r-1}e^{-t}}{\log(1/s)}dt = s^{-1/r}(\log(1/s))^{-1}(I_1+I_2+I_3), \tag{2.3}$$

where

$$I_1 = \int_{2^r s}^{1/\log^{2r}(1/s)} \frac{t^{1/r-1}e^{-t}}{1+\frac{\log t}{\log(1/s)}}dt,$$

$$I_2 = \int_{1/\log^{2r}(1/s)}^{\log^2(1/s)} \frac{t^{1/r-1}e^{-t}}{1+\frac{\log t}{\log(1/s)}}dt,$$

$$I_3 = \int_{\log^2(1/s)}^{\infty} \frac{t^{1/r-1}e^{-t}}{1+\frac{\log t}{\log(1/s)}}dt.$$

We will consider each of these integrals individually.

For $t \in [2^r s, 1/\log^{2r}(1/s)]$, $\log t/\log(1/s)$ is closest to -1 when $t = 2^r s$. Hence

$$\begin{aligned} I_1 &\ll \int_{2^r s}^{1/\log^{2r}(1/s)} \frac{t^{1/r-1}e^{-t}}{1+\frac{\log 2^r s}{\log(1/s)}}dt, \\ &\ll \log(1/s)\int_{2^r s}^{1/\log^{2r}(1/s)} t^{1/r-1}e^{-t}dt \\ &\ll \log(1/s)\int_{0}^{1/\log^{2r}(1/s)} t^{1/r-1}dt \\ &\ll \frac{1}{\log(1/s)}. \end{aligned} \tag{2.4}$$

Now we consider I_2. For $t \in [1/\log^{2r}(1/s), \log^2(1/s)]$, we have

$$\frac{1}{1+\frac{\log t}{\log(1/s)}} = 1 + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big),$$

and so using integration by parts,

$$\begin{aligned} I_2 =& \int_{1/\log^{2r}(1/s)}^{\log^2(1/s)} t^{1/r-1}e^{-t}dt + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big) \\ =& r\Big[t^{1/r}e^{-t}\Big]_{1/\log^{2r}(1/s)}^{\log^2(1/s)} + r\int_0^\infty t^{1/r}e^{-t}dt + O\Big(\int_0^{1/\log^{2r}(1/s)} t^{1/r}e^{-t}dt\Big) \\ &+ O\Big(\int_{\log^2(1/s)}^\infty t^{1/r}e^{-t}dt\Big) + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big). \end{aligned}$$

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But

$$\int_0^\infty t^{1/r} e^{-t} dt = \Gamma\Big(\frac{1}{r} + 1\Big),$$

and all the remaining terms are $O(\log\log(1/s)/\log(1/s))$, so

$$I_2 = r\Gamma\Big(\frac{1}{r} + 1\Big) + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big). \tag{2.5}$$

Finally,

$$I_3 \ll \frac{1}{\log(1/s)} \int_{\log^2(1/s)}^\infty t^{1/r-1} e^{-t} dt \ll \frac{1}{\log(1/s)}. \tag{2.6}$$

The proof is completed by combining (2.3), (2.4), (2.5), and (2.6). ∎

The previous two lemmas yield the following.

Corollary 2.3 *As $s \to 0^+$,*

$$\phi(s) = r\Gamma\Big(\frac{1}{r} + 1\Big) s^{-1/r} \Big(\log(1/s)\Big)^{-1} + O\Big(\frac{s^{-1/r}\log\log(1/s)}{\log^2(1/s)}\Big).$$

Let $k \in \mathbb{N}$, and define

$$f(s) = \sum_{n=0}^\infty p^{(k)}_{\mathbb{P}^{(r)}}(n) e^{-ns} = (1 - e^{-s})^k \prod_p (1 - e^{-sp^r})^{-1}.$$

That is, $f(s)$ is the generating function in e^{-s} of the k-th difference function of $p_{\mathbb{P}^{(r)}}(n)$. Taking logarithms we have

$$\begin{aligned} \log f(s) &= k\log(1 - e^{-s}) - \sum_p \log(1 - e^{-sp^r}) \\ &= k\log(1 - e^{-s}) + \sum_p \sum_{j=1}^\infty \frac{e^{-jsp^r}}{j} \\ &= k\log(1 - e^{-s}) + \sum_{j=1}^\infty \frac{1}{j} \sum_p e^{-jsp^r} \\ &= k\log(1 - e^{-s}) + \sum_{j=1}^\infty \frac{\phi(js)}{j}. \end{aligned} \tag{2.7}$$

We wish to use our approximations for $\phi(s)$ to evaluate the sum $\sum_{j=1}^\infty \frac{\phi(js)}{j}$. To do this we break up the sum into two parts. Let $N = (1/s)/(\log(1/s))$. Then by

Corollary 2.3,

$$\sum_{j\leq N}\frac{\phi(js)}{j}=r\Gamma\Big(\frac{1}{r}+1\Big)s^{-1/r}\times\Big[\sum_{j\leq N}\frac{1}{j^{1+1/r}\log(1/js)}+O\Big(\sum_{j\leq N}\frac{\log\log(1/js)}{j^{1+1/r}\log^2(1/js)}\Big)\Big]. \tag{2.8}$$

Now,

$$\begin{aligned}\sum_{j\leq N}\frac{1}{j^{1+1/r}\log(1/js)}&=\frac{1}{\log(1/s)}\sum_{j\leq N}\frac{1}{j^{1+1/r}\Big(1-\frac{\log j}{\log(1/s)}\Big)}\\&=\frac{1}{\log(1/s)}\Big[\sum_{j\leq N}\frac{1}{j^{1+1/r}}+\frac{1}{\log(1/s)}\sum_{j\leq N}\frac{\log j}{j^{1+1/r}\Big(1-\frac{\log j}{\log(1/s)}\Big)}\Big]\\&=(\log(1/s))^{-1}\Big(\zeta\Big(\frac{1}{r}+1\Big)+O\Big(\frac{1}{N^{1/r}}\Big)\Big)\\&\quad+\frac{1}{\log^2(1/s)}O\Bigg(\sum_{j\leq N}\frac{1}{j^{1+1/2r}\Big(1-\frac{\log j}{\log(1/s)}\Big)}\Bigg).\end{aligned} \tag{2.9}$$

We have,

$$\frac{(\log(1/s))^{-1}}{N^{1/r}}=O(s^{1/r}), \tag{2.10}$$

and

$$\sum_{j\leq N}\frac{1}{j^{1+1/2r}\Big(1-\frac{\log j}{\log(1/s)}\Big)}=\Sigma_1+\Sigma_2,$$

where

$$\Sigma_1=\sum_{j\leq 1/\sqrt{s}}\frac{1}{j^{1+1/2r}\Big(1-\frac{\log j}{\log(1/s)}\Big)},\text{ and }\Sigma_2=\sum_{1/\sqrt{s}<j\leq N}\frac{1}{j^{1+1/2r}\Big(1-\frac{\log j}{\log(1/s)}\Big)}.$$

But $\Sigma_1\ll\sum_{j\leq 1/\sqrt{s}}\frac{1}{j^{1+1/2r}}\ll 1$, and

$$\begin{aligned}\Sigma_2&\ll\sum_{1/\sqrt{s}<j\leq N}\frac{1}{j^{1+1/2r}\Big(1-\frac{\log N}{\log(1/s)}\Big)}\\&\ll\sum_{1/\sqrt{s}<j\leq N}\frac{\log(1/s)}{j^{1+1/2r}\log\log(1/s)}\ll\frac{s^{1/4r}\log(1/s)}{\log\log(1/s)},\end{aligned}$$

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Hence

$$\sum_{j\le N}\frac{1}{j^{1+1/2r}\left(1-\frac{\log j}{\log(1/s)}\right)}\ll 1. \tag{2.11}$$

We use a similar technique to bound the error term in (2.8). Write

$$\sum_{j\le N}\frac{\log\log(1/js)}{j^{1+1/r}\log^2(1/js)}=\frac{1}{\log^2(1/s)}(\Sigma_1'+\Sigma_2'),$$

where

$$\Sigma_1'=\sum_{j\le 1/\sqrt{s}}\frac{\log\log(1/js)}{j^{1+1/r}\left(1-\frac{\log j}{\log(1/s)}\right)^2},$$

and

$$\Sigma_2'=\sum_{1/\sqrt{s}<j\le N}\frac{\log\log(1/js)}{j^{1+1/r}\left(1-\frac{\log j}{\log(1/s)}\right)^2}.$$

Then

$$\Sigma_1'\ll\sum_{j\le 1/\sqrt{s}}\frac{\log\log(1/s)}{j^{1+1/r}}\ll\log\log(1/s),$$

and

$$\begin{aligned}\Sigma_2'&\ll\sum_{1/\sqrt{s}<j\le N}\frac{\log\log(1/s)}{j^{1+1/r}\left(1-\frac{\log N}{\log(1/s)}\right)^2}\\&\ll\sum_{1/\sqrt{s}<j\le N}\frac{\log^2(1/s)}{j^{1+1/r}\log\log(1/s)}\ll\frac{s^{1/2r}\log^2(1/s)}{\log\log(1/s)}.\end{aligned}$$

Hence,

$$\sum_{j\le N}\frac{\log\log(1/js)}{j^{1+1/r}\log^2(1/js)}\ll\frac{\log\log(1/s)}{\log^2(1/s)}. \tag{2.12}$$

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Next we must consider the tail of the sum $\sum \phi(js)/j$:

$$
\begin{aligned}
(2.13)\qquad \sum_{j>N} \frac{\phi(js)}{j} &\ll \sum_{n=2}^{\infty}\sum_{j>N}\frac{e^{-jsn}}{j} \ll \frac{1}{N}\sum_{n=2}^{\infty}\sum_{j>N} e^{-jsn} \ll \frac{1}{N}\sum_{n=2}^{\infty}\frac{e^{-Nsn}}{1-e^{-sn}}\\
&= \frac{1}{N}\sum_{2\le n\le 1/s}\frac{e^{-Nsn}}{1-e^{-sn}} + \frac{1}{N}\sum_{n>1/s}\frac{e^{-Nsn}}{1-e^{-sn}}\\
&\ll \frac{1}{N}\sum_{2\le n\le 1/s}\frac{e^{-Nsn}}{sn} + \frac{1}{N}\sum_{n>1/s} e^{-Nsn}\\
&\ll \log(1/s)\sum_{n=2}^{\infty} e^{-Nsn} + \frac{1}{N}\frac{e^{-N}}{1-e^{-Ns}}\\
&\ll \log(1/s)\frac{e^{-2Ns}}{1-e^{-Ns}} + \frac{s\log(1/s)e^{-1/(s\log(1/s))}}{1-e^{-Ns}}\\
&\ll \log^2(1/s)e^{-2/\log(1/s)} + s\log^2(1/s)e^{-1/(s\log(1/s))}\\
&\ll \log^2(1/s).
\end{aligned}
$$

Combining (2.7) through (2.13), and the fact that

$$\log(1-e^{-s}) \ll \log(1/s) \ll \frac{s^{-1/r}\log\log(1/s)}{\log^2(1/s)},$$

as $s \to 0^+$, we have the following theorem.

Theorem 2.4 *As $s \to 0^+$,*

$$\log f(s) = r\Gamma\Big(\frac{1}{r}+1\Big)\zeta\Big(\frac{1}{r}+1\Big)s^{-1/r}(\log(1/s))^{-1} + O\Big(\frac{s^{-1/r}\log\log(1/s)}{\log^2(1/s)}\Big).$$

3 Bounding from Above

Now we are in a position to prove our main theorem, which we do in two parts, the first being the simplest. First let us introduce some new notation.

Let $k, r \ge 1$, $a_n = p^{(k)}_{\mathbb{P}^{(r)}}(n)$, $A_n = \sum_{i=0}^{n} a_i = p^{(k-1)}_{\mathbb{P}^{(r)}}(n)$, and denote the following constants:

$$
\begin{aligned}
A &= r\Gamma\Big(\frac{1}{r}+1\Big)\zeta\Big(\frac{1}{r}+1\Big),\\
B &= (r+1)\Big[\Gamma\Big(\frac{1}{r}+2\Big)\zeta\Big(\frac{1}{r}+1\Big)\Big]^{r/(r+1)}.
\end{aligned}
$$

Furthermore, choose $C_1 > 0$ such that if

$$\delta(s) = C_1\frac{\log\log(1/s)}{\log(1/s)},$$

then

$$|1-(1/A)s^{1/r}\log(1/s)\log f(s)| < C_1\frac{\log\log(1/s)}{\log(1/s)}.$$

We begin by bounding $\log A_n$ from above.

Lemma 3.1 *There exists a function $\beta \ll \log\log n/\log n$ such that for all n sufficiently large,*

$$\log A_n < \frac{Bn^{1/(r+1)}}{(\log n)^{r/(r+1)}}(1+\beta).$$

Proof We have that

$$(3.1)\quad (1-\delta(s))As^{-1/r}(\log(1/s))^{-1} < \log f(s) < (1+\delta(s))As^{-1/r}(\log(1/s))^{-1}.$$

Since $\lim_{n\to\infty} a_n = \infty$, there exists an $N \in \mathbb{N}$ depending on k and r such that $n > N$ implies that $a_n \geq 0$. We define a constant C_2 by $C_2 = \sum_{j=0}^{N}|a_j|$. Thus if $n > N$, then

$$\begin{aligned} A_n e^{-ns} &= \sum_{j=0}^{N} a_j e^{-ns} + \sum_{j=N+1}^{n} a_j e^{-ns} < \sum_{j=0}^{N} a_j e^{-ns} + \sum_{j=N+1}^{n} a_j e^{-js} \\ &= \sum_{j=0}^{N} a_j(e^{-ns}-e^{-js}) + \sum_{j=0}^{n} a_j e^{-js} < f(s) + C_2, \end{aligned}$$

and so

(3.2)

$$\begin{aligned} \log A_n &< ns + (1+\delta(s))As^{-1/r}(\log(1/s))^{-1} + \log(1 + C_2 e^{-(1+\delta(s))As^{-1/r}(\log(1/s))^{-1}}) \\ &< ns + (1+\delta(s))As^{-1/r}(\log(1/s))^{-1} + O(e^{-(1+\delta(s))As^{-1/r}(\log(1/s))^{-1}}). \end{aligned}$$

For a large value of n, we can, by continuity, choose a corresponding $s > 0$ such that

$$(3.3)\quad \frac{1-\delta(s)}{r}As^{-(r+1)/r}(\log(1/s))^{-1} < n < \frac{1+\delta(s)}{r}As^{-(r+1)/r}(\log(1/s))^{-1}.$$

For these values of s and n, we deduce from (3.3) that

$$(3.4)\quad \frac{1}{s} = \Big[\Big(\frac{rn\log(1/s)}{A}\Big)\Big(1+O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big)\Big)\Big]^{r/(r+1)},$$

and hence

$$(3.5)\quad \log(1/s) = \frac{r\log n}{r+1}\Big(1+O\Big(\frac{\log\log(1/s)}{\log n}\Big)\Big).$$

Note that this implies that $\log(1/s) \ll \log n \ll \log(1/s)$ as $s \to 0$, or equivalently, as $n \to \infty$, so we may use $\log n$, and $\log(1/s)$ interchangeably in various error terms. This fact, together with equations (3.4) and (3.5) implies that

$$s = \Big[\Big(\frac{A}{rn\log(1/s)}\Big)\Big(1 + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big)\Big)\Big]^{r/(r+1)} \tag{3.6}$$
$$= \Big[\Big(\frac{A(r+1)}{r^2 n\log n}\Big)\Big(1 + O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big)\Big)\Big]^{r/(r+1)}$$
$$= \frac{B}{(r+1)(n\log n)^{r/(r+1)}}\Big(1 + O\Big(\frac{\log\log n}{\log n}\Big)\Big).$$

From (3.5) and (3.6), we infer that

$$ns + As^{-1/r}(\log(1/s))^{-1} = \frac{Bn^{1/(r+1)}}{(r+1)(\log n)^{r/(r+1)}}\Big(1 + O\Big(\frac{\log\log n}{\log n}\Big)\Big)$$
$$+ \frac{A(r+1)^{(r+1)/r}n^{1/(r+1)}}{rB^{1/r}(\log n)^{r/(r+1)}}\Big(1 + O\Big(\frac{\log\log n}{\log n}\Big)\Big)$$
$$= \frac{Bn^{1/(r+1)}}{(\log n)^{r/(r+1)}}\Big(1 + O\Big(\frac{\log\log n}{\log n}\Big)\Big).$$

Therefore, by (3.2),

$$\log A_n < \frac{Bn^{1/(r+1)}}{(\log n)^{r/(r+1)}}\Big(1 + O\Big(\frac{\log\log n}{\log n}\Big)\Big). \tag{3.7}$$

This completes the proof of the lemma. ■

4 Bounding from Below

Lemma 3.1 is one half of what we require. We use it to prove the other half.

Lemma 4.1 *Let $\epsilon > 0$ be given. Then there is a function*

$$\beta \ll_\epsilon \sqrt{\frac{(\log\log n)^{1+\epsilon}}{\log n}}$$

such that for all n sufficiently large,

$$\log A_n > \frac{Bn^{1/(r+1)}}{(\log n)^{r/(r+1)}}(1-\beta).$$

First let us introduce a convenient bit of notation. At times throughout the following argument, we are guaranteed the existence of certain positive functions which are $O(\log\log(1/s)/\log(1/s))$ in magnitude, as $s \to 0^+$. Rather than rename each such function, we may simply write η. Thus the precise η may vary, depending on the context, even within the same equation, but will always be used to denote such a positive function whose existence is guaranteed.

Proof Let $\mathbf{A}(x) = A_n$, for $n \le x < n+1$. Hence by (3.7), there is a constant $C_3 > 0$, such that if $\eta_1(x) = C_3 \frac{\log\log x}{\log x}$,then

$$\log \mathbf{A}(x) < \frac{Bx^{1/(r+1)}}{(\log x)^{r/(r+1)}}\big(1 + \eta_1(x)\big). \tag{4.1}$$

Now

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-ns} = \sum_{n=0}^{\infty} A_n(e^{-ns} - e^{-(n+1)s}) \tag{4.2}$$
$$= s\sum_{n=0}^{\infty} A_n \int_n^{n+1} e^{-sx}dx = s\int_0^{\infty} \mathbf{A}(x)e^{-sx}dx.$$

The inequalities in (3.1) together with equation (4.2) imply that

$$\exp\big((1-\delta(s))As^{-1/r}(\log(1/s))^{-1}\big) < s\int_0^{\infty}\mathbf{A}(x)e^{-sx}dx \tag{4.3}$$
$$< \exp\big((1+\delta(s))As^{-1/r}(\log(1/s))^{-1}\big).$$

Given a small value of $s > 0$, we can, by continuity, choose a corresponding $m > 0$ such that

$$\frac{1}{s} = \frac{r+1}{B}(m\log m)^{r/(r+1)}. \tag{4.4}$$

Now, denote

$$f(s) = s\int_0^{\infty}\mathbf{A}(x)e^{-sx}dx$$
$$= s\Big(\int_0^{m/H} + \int_{m/H}^{(1-\zeta)m} + \int_{(1-\zeta)m}^{(1+\zeta)m} + \int_{(1+\zeta)m}^{Hm} + \int_{Hm}^{\infty}\Big)$$
$$= J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$\zeta = \sqrt{\frac{(\log\log m)^{1+\epsilon}}{\log m}},$$

and $H > 1$ is a constant yet to be determined. We will see that the dominant term here is J_3, but first we shall prove that the terms J_1, J_2, J_4, and J_5 are negligible in comparison to the exponentials on either side of (4.3).

We first dispatch J_1 and J_5. From Lemma 3.1, we have

$$J_1 < s\int_0^{m/H}\mathbf{A}(x)e^{-sx}dx \tag{4.5}$$
$$< \exp\big[(1+\eta_1(m/H))B(m/H)^{1/(r+1)}(\log(m/H))^{-r/(r+1)}\big].$$

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Taking logarithms in (4.4), we see that

$$\frac{r+1}{r}(1-\eta) < \frac{\log m}{\log(1/s)}.$$

We can in light of this fact, select a positive function $\eta_3(s) \ll \log\log(1/s)/\log(1/s)$ such that for m sufficiently large relative to H (*i.e.*, s sufficiently small),

$$\left(\frac{r+1}{r}\right)\frac{1+\eta_1(m/H)}{\left(1-\frac{\log H}{\log m}\right)^{r/(r+1)}} < \frac{(1+\eta_3(s))\log m}{\log(1/s)}.$$

This leads to the following string of inequalities:

$$\frac{(1+\eta_1(m/H))A(r+1)^{1+(r+1)/r}}{r^2\log m\left(1-\frac{\log H}{\log m}\right)^{r/(r+1)}} < \frac{(1+\eta_3(s))A(r+1)^{(r+1)/r}}{r\log(1/s)},$$

$$\frac{(1+\eta_1(m/H))B^{(r+1)/r}}{\log m\left(1-\frac{\log H}{\log m}\right)^{r/(r+1)}} < \frac{(1+\eta_3(s))A(r+1)^{(r+1)/r}}{r\log(1/s)},$$

$$\frac{(1+\eta_1(m/H))Bm^{1/(r+1)}}{(\log m)^{r/(r+1)}\left(1-\frac{\log H}{\log m}\right)^{r/(r+1)}} < \frac{(1+\eta_3(s))A(r+1)^{(r+1)/r}}{rB^{1/r}\log(1/s)} \times m^{1/(r+1)}(\log m)^{1/(r+1)}$$

$$\frac{(1+\eta_1(m/H))Bm^{1/(r+1)}}{(\log(m/H))^{r/(r+1)}H^{1/(r+1)}} < \frac{(1+\eta_3(s))A(r+1)}{rH^{1/(r+1)}s^{1/r}\log(1/s)}.$$

Comparing the final inequality with (4.5) yields

$$J_1 < \exp\left[(1+\eta_3(s))AH^{-1/(r+1)}s^{-1/r}(\log(1/s))^{-1}\right],$$

for s sufficiently small. Choose H large enough such that for all s in the range in question,

$$\frac{1+\eta_3(s)}{H^{1/(r+1)}} \le \frac{1+\delta(s)}{2}.$$

Then

$$J_1 < \exp\left[((1+\delta(s))/2)As^{-1/r}(\log(1/s))^{-1}\right].$$

We now consider J_5. Note that $\max\{\eta_1(x) : x > 1\} = C_3/e$. We may choose H sufficiently large such that

$$\frac{1}{r+1} > \frac{2(1+C_3/e)}{H^{r/(r+1)}}.$$

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Then

$$s = \frac{B}{(r+1)(m\log m)^{r/(r+1)}} > \frac{2(1+C_3/e)B}{(Hm\log(Hm))^{r/(r+1)}}$$
$$\geq \frac{2(1+\eta_1(x))B}{(x\log x)^{r/(r+1)}} \text{ for all } x \geq Hm,$$

and so

$$\frac{(1+\eta_1(x))Bx^{1/(r+1)}}{(\log x)^{r/(r+1)}} < \frac{sx}{2},$$

for all $x \geq Hm$. Thus

$$J_5 = s\int_{Hm}^{\infty} \mathbf{A}(x)e^{-sx}dx < s\int_{Hm}^{\infty} \exp\Big[\frac{Bx^{1/(r+1)}(1+\eta_1(x))}{(\log x)^{r/(r+1)}} - sx\Big]\,dx$$
$$< s\int_0^{\infty} e^{-sx/2}dx = 2,$$

where the first inequality follows from (4.1).

Now we take a look at the integrals J_2, and J_4, beginning with the latter. By (4.1),

$$J_4(s) = s\int_{1+\zeta}^{Hm} \mathbf{A}(x)e^{-sx}dx < s\int_{1+\zeta}^{Hm} e^{\psi(x)}dx,$$

where

$$\psi(x) = (1+\eta_1(x))Bx^{1/(r+1)}(\log x)^{-r/(r+1)} - sx. \tag{4.6}$$

If the maximum for $\psi(x)$ occurs at x_0, then, via a straightforward differentiation, it transpires that

$$\frac{1}{s} = \Big(1 + O\Big(\frac{\log\log x_0}{\log x_0}\Big)\Big)\,\frac{r+1}{B}x_0^{r/(r+1)}(\log x_0)^{r/(r+1)}. \tag{4.7}$$

Comparing this with (4.4), we conclude that $\log m \asymp \log x_0$, and that

$$x_0 = \Big(1 + O\Big(\frac{\log\log x_0}{\log x_0}\Big)\Big)\,m,$$

and therefore, for s sufficiently small, $(1-\zeta)m < x_0 < (1+\zeta)m$.

Writing $x = x_0 + \xi$, Taylor's formula gives us

$$\psi(x) = \psi(x_0) + \frac{B}{2}\xi^2\frac{d^2}{dx^2}\big[(1+\eta_1(x))x^{1/(r+1)}(\log x)^{-r/(r+1)}\big]\Big|_{x=x_1},$$

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where $x_0 < x_1 < x$, and hence $(1-\zeta)m < x_1 < Hm$. From this, it is easily seen that there exist positive constants C_4, C_5 such that

$$\begin{aligned}\frac{d^2}{dx_1^2}(1+\eta_1(x_1))\left[x_1^{1/(r+1)}(\log x_1)^{-r/(r+1)}\right] &< -C_4 x_1^{1/(r+1)-2}(\log x_1)^{-r/(r+1)}\\ &< -C_5 m^{1/(r+1)-2}(\log m)^{-r/(r+1)}.\end{aligned}$$

Equations (4.6), and (4.7) yield that

$$\psi(x_0) = As^{-1/r}(\log(1/s))^{-1}\Big(1+O\Big(\frac{\log\log(1/s)}{\log(1/s)}\Big)\Big).$$

Combining the information on $\psi(x)$, we see that there is a constant $C_6 > 0$ such that

$$\begin{aligned}J_4(s) <& s\exp\left[(1+\eta)As^{-1/r}(\log(1/s))^{-1}\right]\\ &\times\int_{(\zeta-\eta)m}^{\infty}\exp\left[-C_6\xi^2 m^{1/(r+1)-2}(\log m)^{-r/(r+1)}\right]d\xi.\end{aligned}$$

The integral on the right-hand side of this inequality is simplified by observing that it is of the form $\int_D^\infty e^{-Cx^2}dx$, for $C, D > 0$. Substituting $u^2 = Cx^2 - CD^2$, we have that

$$\int_D^\infty e^{-Cx^2}dx = \frac{1}{\sqrt{C}}\int_0^\infty \frac{ue^{-CD^2-u^2}}{\sqrt{u^2+CD^2}}du < \frac{e^{-CD^2}}{\sqrt{C}}\int_0^\infty e^{-u^2}du = \frac{e^{-CD^2}}{2}\sqrt{\frac{\pi}{C}}.$$

Hence with $D = (\zeta-\eta)m$, and $C = C_6 m^{1/(r+1)-2}(\log m)^{-r/(r+1)}$, there is a $C_7 > 0$ such that

$$J_4(s) \ll \frac{s\exp\left[(1+\eta)As^{-1/r}(\log(1/s))^{-1} - C_7\zeta^2 m^{1/(r+1)}(\log m)^{-r/(r+1)}\right]}{\sqrt{m^{1/(r+1)-2}(\log m)^{-r/(r+1)}}}.$$

Now, by the definition of m,

$$\begin{aligned}\frac{s}{\sqrt{m^{1/(r+1)-2}(\log m)^{-r/(r+1)}}} &= s\sqrt{m(m\log m)^{r/(r+1)}}\\ &\ll \sqrt{sm} \ll \frac{1}{\sqrt{s^{1/r}\log(1/s)}}.\end{aligned}$$

As we similarly have $s^{-1/r}(\log(1/s))^{-1} \asymp m^{1/(r+1)}(\log m)^{-r/(r+1)}$, there is a constant $C_8 > 0$ such that

$$\begin{aligned}J_4(s) &\ll \frac{\exp\left[(1+\eta-C_8\zeta^2)As^{-1/r}(\log(1/s))^{-1}\right]}{\sqrt{s^{1/r}\log(1/s)}}\\ &\ll \exp\left[(1-C_8\zeta^2/2)As^{-1/r}(\log(1/s))^{-1}\right].\end{aligned}$$

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Virtually the same analysis works for $J_2(s)$ giving a bound of a similar form. The results thus far have guaranteed us the existence of a constant $C_9 > 0$, such that

$$J_1, J_2, J_4, J_5 \ll \exp\left((1 - C_9\zeta^2)As^{-1/r}(\log(1/s))^{-1}\right).$$

Hence by (4.3), we may select a new function $\delta_1(s)$ of the form

$$\frac{C \log\log(1/s)}{\log(1/s)}$$

($2\delta(s)$ works), such that for s sufficiently small,

$$\exp\left[(1 - \delta_1(s))As^{-1/r}(\log(1/s))^{-1}\right] < s\int_{(1-\zeta)m}^{(1+\zeta)m} \mathbf{A}(x)e^{-sx}dx$$

Since $\mathbf{A}(x)$ increases, we have

$$\exp\left[(1 - \delta_1(s))As^{-1/r}(\log(1/s))^{-1}\right] < s\mathbf{A}((1+\zeta)m)\int_{(1-\zeta)m}^{(1+\zeta)m} e^{-sx}dx.$$

Evaluating the integral leads to

$$(4.8)\qquad (e^{\zeta sm} - e^{-\zeta sm})\mathbf{A}((1+\zeta)m) > \exp\left[(1 - \delta_1(s))As^{-1/r}(\log(1/s))^{-1} + ms\right].$$

Substituting s in terms of m into the right-hand side of (4.8), we obtain an expression of the form

$$\exp\left[Bm^{1/(r+1)}(\log m)^{-r/(r+1)}\left(1 + O\left(\frac{\log\log m}{\log m}\right)\right)\right].$$

Now, equation (4.4) yields

$$e^{\zeta sm} - e^{-\zeta sm} = e^{\frac{\zeta B}{r+1}m^{1/(r+1)}(\log m)^{-r/(r+1)}}\left(1 - e^{-\frac{2\zeta B}{r+1}m^{1/(r+1)}(\log m)^{-r/(r+1)}}\right),$$

and so by (4.8),

$$(4.9)\quad \mathbf{A}((1+\zeta)m) > \exp\left[Bm^{1/(r+1)}(\log m)^{-r/(r+1)}\left(1 - \frac{\zeta}{r+1} + O\left(\frac{\log\log m}{\log m}\right)\right)\right].$$

But $(1+\zeta)m$ is a continuous function of m, which is ultimately increasing. Thus for all n sufficiently large, we may choose a unique value of s, and hence of m such that $(1+\zeta)m = n$. Substituting $m = \frac{n}{1+\zeta}$ into (4.9) and observing that $\log m \asymp \log n$, we have the lemma. ∎

Together, Lemmas 3.1 and 4.1 yield our main theorem.

Theorem 4.2 *For a fixed $k \geq 1$,*

$$\log p_{\mathbb{P}^{(r)}}^{(k-1)}(n) = (r+1)\left[\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right]^{r/(r+1)} n^{1/(r+1)}(\log n)^{-r/(r+1)} \times\left(1 + O_\epsilon\left(\sqrt{\frac{(\log\log n)^{1+\epsilon}}{\log n}}\right)\right),\ \text{as } n\to\infty.$$

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