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# CHARACTERIZATIONS OF COMPLETELY HAUSDORFF-CLOSED SPACES VIA GRAPHS AND PROJECTIONS

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**1. Introduction.** L. L. Herrington recently characterized completely Hausdorff-closed topological spaces in terms of arbitrary filterbases and a type of convergence for filterbases called $f$-convergence [**2**]. Before this, characterizations of these spaces via filterbases was by open filterbases [**1**]. In this article, we employ Herrington's characterizations to obtain characterizations of completely Hausdorff-closed spaces in terms of projections and in terms of graphs of functions into the spaces; both of these—projections and graphs—are utilized in conjunction with a class $S$ of spaces containing as a subclass the Hausdorff completely normal fully normal spaces to effect the characterizations. See [**1**] for a survey of results on completely Hausdorff-closed spaces.

**2. Preliminaries.** The closure of a subset $K$ of a space will be denoted by $\text{cl}(K)$; ad $\Omega$ will represent the adherence of a filterbase $\Omega$ on the space. If $\psi, \lambda: X \to Y$ are functions, $E(\psi, \lambda, X, Y)$ will represent $\{x \in X: \psi(x) = \lambda(x)\}$ and $G(\psi)$ will represent the graph of $\psi$. The class of continuous real-valued functions on $X$ will be denoted by $C(X)$. If $f \in C(X)$, $x \in X$ and $H$ is an open set about $f(x)$, we will call $f$ and $H$ an *ordered pair for* $x$ and denote this by $(f, H)_x$ [**2**].

2.1. *Definition.* A point $x$ in a space is in the *$f$-closure of a subset* $K$ of the space ($x \in f\text{-cl}(K)$) if $K \cap f^{-1}(H) \neq \emptyset$ is satisfied by each pair $(f, H)_x$. $K$ is *$f$-closed* if $K$ contains its $f$-closure ($f\text{-cl}(K) \subset K$).

2.2. *Definition* [**2**]. A point $x$ in a space is in the *$f$-adherence of a filterbase* $\Omega$ on the space ($x \in f\text{-ad}\,\Omega$) if each $F \in \Omega$ and pair $(f, H)_x$ satisfy $F \cap f^{-1}(H) \neq \emptyset$.

2.3. *Definition* [**1**]. A space $X$ is *completely Hausdorff* if for each pair $x, y \in X$ with $x \neq y$, there is an $f \in C(X)$ satisfying $f(x) \neq f(y)$).

We state the following theorem without proof.

2.4. Theorem. *The following statements are equivalent for a space $X$:*
(a) *$X$ is completely Hausdorff.*
(b) $\{x\} = \bigcap_\Sigma f\text{-cl}(V)$ *for each $x \in X$ and open set base $\Sigma$ at $x$.*
(c) *Each point in $X$ is $f$-closed.*

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(d) *Each function* $\psi$ *into* $X$ *satisfies* $\psi^{-1}(x) = \bigcap_{C(x)}(g \circ \psi)^{-1}(g(x))$ *for each* $x \in X$;

(e) $\{x\} = \bigcap_{C(x)} g^{-1}(g(x))$ *for each* $x \in X$.

In stating our next theorem, which may be readily established, we recall that a point $x$ in a space is in the *θ-closure of a subset* $K$ of the space ($x \in \theta\text{-cl}(K)$) if each $V$ open about $x$ satisfies $K \cap \text{cl}(V) \neq \emptyset$; $K$ is *θ-closed* if $\theta\text{-cl}(K) \subset K$ [**6**]. We further recall from [**6**] that a point $x$ in a space is in the *θ-adherence of a filterbase* $\Omega$ on the space ($x \in \theta\text{-ad}\ \Omega$) if $x \in \theta\text{-cl}(F)$ for each $F \in \Omega$. In addition, we employ the following definition.

2.5. *Definition.* If $X$ and $Y$ are spaces, $(x, y) \in X \times Y$ is in the *first-coordinate f-closure* of $K \subset X \times Y((x, y) \in (1)f\text{-cl}(K))$ if $K \cap (f^{-1}(H) \times V) \neq \emptyset$ is satisfied for each pair $(f, H)_x$ and $V$ open about $y$. $K$ is *first-coordinate f-closed* (*(1)f-closed*) if $(1)f\text{-cl}(K) \subset K$.

2.6. Theorem. *The following statements hold for topological spaces* $X$ *and* $Y$, *subsets* $K$ *and* $M$ *of* $X$ *and subsets* $P$ *and* $Q$ *of* $X \times Y$:

(a) $X(X \times Y)$ *and* $\emptyset$ *are f-closed* (*(1)f-closed*).

(b) $K \subset \text{cl}(K) \subset \theta\text{-cl}(K) \subset f\text{-cl}(K)$; $P \subset \text{cl}(P) \subset (1)f\text{-cl}(P)$; *f-closed* (*(1)f-closed*) *subsets are θ-closed* (*closed*).

(c) $f\text{-cl}(K)$ ($(1)f\text{-cl}(P)$) *is f-closed* (*(1)f-closed*).

(d) *The intersection of any collection of f-closed* (*(1)f-closed*) *subsets of* $X(X \times Y)$ *is f-closed* (*(1)f-closed*).

(e) *If* $K \subset M(P \subset Q)$, *then* $f\text{-cl}(K) \subset f\text{-cl}(M)$ ($(1)f\text{-cl}(P) \subset (1)f\text{-cl}(Q)$).

(f) *For each filterbase* $\Omega$ *on* $X$, *we have* $f\text{-ad}\ \Omega = \bigcap_\Omega f\text{-cl}(F)$ *and* $\theta\text{-ad}\ \Omega \subset f\text{-ad}\ \Omega$.

It is known that a function $\psi: X \to Y$ is continuous if and only if $\psi(\text{ad}\ \Omega) \subset \text{ad}\ \psi(\Omega)$ for each filterbase $\Omega$ on $X$. A function $\psi: X \to Y$ is *weakly-continuous* if for each $x \in X$ and $U$ open about $\psi(x)$, there is a $V$ open about $x$ satisfying $\psi(V) \subset \text{cl}(U)$ [**5**]. We assert that the following theorem is valid.

2.7. Theorem. *A function* $\psi: X \to Y$ *is weakly-continuous if and only if* $\psi(\text{ad}\ \Omega) \subset \theta\text{-ad}\ \psi(\Omega)$ *for each filterbase* $\Omega$ *on* $X$.

These last observations motivate the following definition.

2.8. *Definition.* A function $\psi: X \to Y$ is *f-weakly-continuous* if $\psi(\text{ad}\ \Omega) \subset f\text{-ad}\ \psi(\Omega)$ for each filterbase $\Omega$ on $X$.

Theorem 2.9 gives various characterizations of $f$-weakly-continuous functions.

2.9. Theorem. *The following statements are equivalent for spaces* $X$, $Y$ *and function* $\psi: X \to Y$:

(a) $\psi$ *is f-weakly-continuous*;

(b) $\psi(\text{cl}(K)) \subset f\text{-cl}(\psi(K))$ *for each* $K \subset X$;

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(c) *$g \circ \psi$ is continuous for each $g \in C(Y)$.*

In [3], a function $\psi: X \to Y$ is said to have a *strongly-closed graph* if for each $(x, y) \in (X \times Y) - G(\psi)$, there is a $V$ open about $x$ and $U$ open about $y$ satisfying $(V \times \mathrm{cl}(U)) \cap G(\psi) = \emptyset$. We assert that the following theorem holds. The characterizations in this theorem motivate Definition 2.11.

2.10. THEOREM. *The following statements are equivalent for a space $X$, a $T_1(T_2)$ space $Y$ and function $\psi: X \to Y$;*

(a) *$\psi$ has a closed graph (strongly-closed graph);*

(b) $\mathrm{ad}\,\psi(\Omega) \subset \{\psi(x)\}$ ($\theta$-$\mathrm{ad}\,\psi(\Omega) \subset \{\psi(x)\}$) *for each $x \in X$ and filterbase $\Omega$ on $X$ with $\Omega \to x$;*

(c) $\mathrm{ad}\,\psi(\Omega) \subset \{\psi(x)\}$ ($\theta$-$\mathrm{ad}\,\psi(\Omega) \subset \{\psi(x)\}$) *for each $x \in X$ and filterbase $\Omega$ on $X - \{x\}$ with $\Omega \to x$.*

2.11. *Definition.* A function $\psi: X \to Y$ has an *$f$-strongly-subclosed graph* if $f$-$\mathrm{ad}\,\psi(\Omega) \subset \{\psi(x)\}$ for each $x \in X$ and filterbase $\Omega$ on $X - \{x\}$ with $\Omega \to x$.

It is clear from above that a function into a $T_2$ space with an $f$-strongly-subclosed graph has a strongly-closed graph.

Our next theorem gives several characterizations of functions with $f$-strongly-subclosed graphs.

2.12. THEOREM. *The following statements are equivalent for spaces $X$, $Y$ and function $\psi: X \to Y$;*

(a) *$\psi$ has an $f$-strongly-subclosed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\psi)$, there is a $V$ open about $x$ and a pair $(f, H)_y$ satisfying $((V - \{x\}) \times f^{-1}(H)) \cap G(\psi) = \emptyset$.*

(c) *For each $(x, y) \in (X \times Y) - G(\psi)$, there is a $V$ open about $x$ and a pair $(f, H)_y$ satisfying $(V \times (f^{-1}(H) - \{\psi(x)\})) \cap G(\psi) = \emptyset$.*

(d) *For each $(x, y) \in (X \times Y) - G(\psi)$, there is a $V$ open about $x$ and a pair $(f, H)_y$ satisfying $\psi(V - \{x\}) \cap f^{-1}(H) = \emptyset$.*

(e) *For each $(x, y) \in (X \times Y) - G(\psi)$, there is a $V$ open about $x$ and a pair $(f, H)_y$ satisfying $\psi(V) \cap (f^{-1}(H) - \{\psi(x)\}) = \emptyset$.*

(f) *For each $x \in X$ and each (some) open set base $\Sigma$ at $x$, we have* $\bigcap_{\Sigma} f\text{-}\mathrm{cl}(\psi(V - \{x\})) \subset \{\psi(x)\}$.

2.13. *Definition.* Let $X$ be a nonempty set, let $x_0 \in X$, and let $\Omega$ be a filterbase on $X$; $\{A \subset X: x_0 \in X - A$ or $F \cup \{x_0\} \subset A$ for some $F \in \Omega\}$ is a topology on $X$ which will be called *the topology on $X$ associated with $x_0$ and $\Omega$*. $X$ equipped with this topology will be called *the space $X$ associated with $x_0$ and $\Omega$*. We will denote this space by $X(x_0, \Omega)$.

The following easily established theorem is used frequently in the sequel.

2.14. THEOREM. *Let $X$ be a nonempty set, let $x_0 \in X$ be a filterbase on $X$ with empty intersection on $X - \{x_0\}$. Then $X(x_0, \Omega)$ is in class $S$.*

**3. Characterizations of completely Hausdorff-closed spaces via graphs.** In [3], it is proved that a Hausdorff space $Y$ is $H$-closed if and only if all functions with strongly-closed graphs from a space in class $S$ to $Y$ are weakly-continuous. In this section, we present—via graphs—some characterizations of spaces satisfying condition $CH$ (below), which has been observed by Herrington [**2**] to be equivalent to completely Hausdorff-closedness for completely Hausdorff spaces.

($CH$) *Each filterbase on the space has a nonempty f-adherence.*

We now use graphs to give several characterizations of $CH$ spaces which are not necessarily completely Hausdorff.

3.1. THEOREM. *A space $Y$ is CH if and only if for each space $X$ in class $S$, each bijection $\psi: X \to Y$ with an f-strongly-subclosed graph is f-weakly-continuous.*

*Proof.* (*Strong necessity*). Let $Y$ be a $CH$ space, $X$ be any space, and $\psi: X \to Y$ be any function with an $f$-strongly-subclosed graph; let $\Omega$ be a filterbase on $X$, choose $y \in \psi(\text{ad}\,\Omega)$ and $x \in \text{ad}\,\Omega$ with $\psi(x) = y$. If $\Omega_1 = \{(V \cap F) - \{x\}: V \text{ open about } x, F \in \Omega\}$ is not a filterbase on $X - \{x\}$,then $x \in F$ for each $F \in \Omega$, so $y = \psi(x) \in \psi(F)$ for each $F \in \Omega$. If $\Omega_1$ is a filterbase on $X - \{x\}$, then $\Omega_1 \to x$ and, since $Y$ is $CH$ and $G(\psi)$ is $f$-strongly-subclosed, we have $\emptyset \neq f\text{-ad}\,\psi(\Omega_1) \subset \{\psi(x)\}$. This gives $y \in f\text{-ad}\,\psi(\Omega_1) \subset f\text{-ad}\,\psi(\Omega)$.

(*Sufficiency*). Suppose $\Omega$ is a filterbase on $Y$ with $f\text{-ad}\,\Omega = \emptyset$. Choose $x_0 \in Y$ and let $\psi: Y(x_0, \Omega) \to Y$ be the identity function. Let $x \in Y$ and let $\Omega_1$ be a filterbase on $Y(x_0, \Omega) - \{x\}$ with $\Omega_1 \to x$. Then $x = x_0$ and $\Omega_1$ is stronger than $\Omega$; so we have $f\text{-ad}\,\psi(\Omega_1) = f\text{-ad}\,\Omega_1 \subset f\text{-ad}\,\Omega = \emptyset \subset \{\psi(x_0)\}$. This means that $G(\psi)$ is $f$-strongly-subclosed. However, $\psi$ is not $f$-weakly-continuous since $x_0 \in \psi(\text{ad}\,\Omega_1) - (f\text{-ad}\,\psi(\Omega_1))$. Thus $Y$ does not satisfy the condition of the theorem.

The proof is complete.

3.2. THEOREM. *A space $Y$ is CH if and only if for each space $X$ in class $S$ and bijections $\psi, \lambda: X \to Y$ with f-strongly-subclosed graphs, $E(\psi, \lambda, X, Y)$ is closed in $X$.*

*Proof.* (*Strong necessity*). Let $Y$ be $CH$, $X$ be any space, and $\psi, \lambda: X \to Y$ be any functions with $f$-strongly-subclosed graphs. Let $x \in \text{cl}(E(\psi, \lambda, X, Y)) - E(\psi, \lambda, X, Y)$. There is a filterbase $\Omega$ on $E(\psi, \lambda, X, Y)$ with $\Omega \to x$. Since $\lambda$ has an $f$-strongly-subclosed graph, and $\psi$ is $f$-weakly-coninuous from Theorem 3.1, we get $\{\psi(x)\} \subset \psi(\text{ad}\,\Omega) \subset f\text{-ad}\,\psi(\Omega) = f\text{-ad}\,\lambda(\Omega) \subset \{\lambda(x)\}$. This is a contradiction.

(*Sufficiency*). Suppose $\Omega$ is a filterbase on a space $Y$ with $f\text{-ad}\,\Omega = \emptyset$. Choose $x_0, y_0 \in Y$ with $x_0 \neq y_0$ and let $\Omega_1 = \{F - \{x_0, y_0\}: F \in \Omega\}$. Then $\Omega_1$ is a filterbase on $Y$. Let $\psi: Y(x_0, \Omega_1) \to Y$ be the identity function and define $\lambda: Y(x_0, \Omega_1) \to Y$ by $\lambda(x_0) = y_0$, $\lambda(y_0) = x_0$ and $\lambda(x) = x$ otherwise. Then $E(\psi, \lambda, Y(x_0, \Omega_1), Y) = Y - \{x_0, y_0\}$ which is not closed in $Y(x_0, \Omega_1)$. We see that $\psi$ and $\lambda$ are bijections and we show that both have $f$-strongly-subclosed graphs. This will establish that $Y$ does not satisfy the condition of the theorem.

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(a) $G(\psi)$ *is f-strongly-subclosed.* An argument similar to that for $\psi$ in the proof of the sufficiency of Theorem 3.1 will verify this.

(b) $G(\lambda)$ *is f-strongly-subclosed.* Let $x \in Y$ and let $\Omega_2$ be a filterbase on $Y(x_0, \Omega_1) - \{x\}$ with $\Omega_2 \to x$. Then $x = x_0$ and $\Omega_2$ is stronger than $\Omega_1$; so we have $f\text{-ad}\,\lambda(\Omega_2) = f\text{-ad}\,\Omega_2 \subset f\text{-ad}\,\Omega_1 \subset f\text{-ad}\,\Omega = \emptyset \subset \{\lambda(x)\}$. Thus $G(\lambda)$ is $f$-strongly-subclosed.

The proof is complete.

3.3. THEOREM. *A space $Y$ is CH if and only if for each space $X$ in class $S$ and functions (one a bijection) $\psi, \lambda: X \to Y$ with f-strongly-subclosed graphs, $E(\psi, \lambda, X, Y) = X$ whenever $E(\psi, \lambda, X, Y)$ is dense in $X$.*

*Proof.* (*Strong necessity*). Let $Y$ be a *CH* space, $X$ be any space and $\psi, \lambda: X \to Y$ be any functions with $f$-strongly-subclosed graphs. From Theorem 3.2, $E(\psi, \lambda, X, Y)$ is closed in $X$; so $E(\psi, \lambda, X, Y) = X$ if $E(\psi, \lambda, X, Y)$ is dense in $X$.

(*Sufficiency*). We follow the proof of the sufficiency of Theorem 3.2 to the point immediately preceding the definition of $\lambda$. We define $\lambda: Y(x_0, \Omega_1) \to Y$ by $\lambda(x_0) = y_0$, and $\lambda(x) = x$ otherwise. Then $E(\psi, \lambda, Y(x_0, \Omega_1), Y) = Y - \{x_0\}$ which is dense in $Y(x_0, \Omega_1)$. By arguments similar to those in the proof of the sufficiency of Theorem 3.2, we can show that $G(\psi)$ and $G(\lambda)$ are $f$-strongly-subclosed.

The proof is then complete.

**4. A characterization of completely Hausdorff-closed spaces via projections.** In [**4**], the author has proved that a Hausdorff space $X$ is $H$-closed if and only if the projection, $\pi_y: X \times Y \to Y$, takes $\theta$-closed subsets onto $\theta$-closed subsets for every space $Y$ in class $S$. In this section, we give a similar characterization of spaces satisfying condition *CH*.

4.1. THEOREM. *A space $X$ is CH if and only if $\pi_y: X \times Y \to Y$ maps (1) f-closed subsets of $X \times Y$ onto closed subsets of $Y$ for every space $Y$ in class $S$.*

*Proof.* (*Strong necessity*). Let $X$ be a *CH* space and let $Y$ be any space. Let $K \subset X \times Y$ be $(1)f$-closed and let $y \in \text{cl}(\pi_y(K))$. Then $\Omega = \{\pi_x((X \times V) \cap K): V \text{ open about } y\}$ is a filterbase on $X$. Let $x \in f\text{-ad}\,\Omega$. Then $(x, y) \in (1)f\text{-cl}(K) \subset K$, so $y \in \pi_y(K)$.

(*Sufficiency*). Let $\Omega$ be a filterbase on $X$ with $f\text{-ad}\,\Omega = \emptyset$, choose $y_0 \notin X$, let $Y = X \cup \{y_0\}$ and $K = \{(x, x): x \in X\}$. Then $(1)f\text{-cl}(K)$ is $(1)f$-closed in $X \times Y(y_0, \Omega)$ and $K \subset (1)f\text{-cl}(K)$. Thus $y_0 \in \pi_y((1)f\text{-cl}(K))$. Let $x \in X$ with $(x, y_0) \in (1)f\text{-cl}(K)$. For each $(f, H)_x$ and $F \in \Omega$, we have $(f^{-1}(H) \times (F \cup \{y_0\})) \cap K \neq \emptyset$. So $f^{-1}(\text{H}) \cap F \neq \emptyset$ and, consequently, $f\text{-ad}\,\Omega \neq \emptyset$.

The proof is complete.

**5. First countable completely Hausdorff spaces.** See [**1**] for definitions and results used but not given here. Observing that $X(x_0, \Omega)$ is metrizable when

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$\Omega = \{F_n\}$ is countable and $X(x_0, \Omega)$ is $T_1(X(x_0, \Omega)$ is regular and $\{V(n)\}$ defined by $V(n) = \{F_n \cup \{x_0\}\} \cup \{\{x\}: x \in X - (F_n \cup \{x_0\})\}$ is a $\sigma$-locally finite base), we may establish the following theorems by appropriate use of the first countability and arguments similar to those in the last section.

5.1. THEOREM. *The following statements are equivalent for a first countable space $Y$. $A$ may represent the class of first countable spaces, the class of first countable spaces in class $S$, or the class of metric spaces.*

(a) *Each countable filterbase on $Y$ has nonvoid $f$-adherence.*

(b) *Each sequence in $Y$ $f$-accumulates* [2] *to some point in $Y$.*

(c) *For each $X \in A$, each function (bijection) $\psi: X \to Y$ with an $f$-strongly-subclosed graph is $f$-weakly-continuous.*

(d) *For each $X \in A$ and any two functions (bijections) $\psi, \lambda: X \to Y$ with $f$-strongly-subclosed graphs, $E(\psi, \lambda, X, Y)$ is closed in $X$.*

(e) *For each $X \in A$ and any two functions (one a bijection) $\psi, \lambda: X \to Y$ with $f$-strongly-subclosed graphs, $E(\psi, \lambda, X, Y) = X$ when $E(\psi, \lambda, X, Y)$ is dense in $X$.*

We utilize the following notion in our next theorem.

5.2. *Definition.* A function $\psi: X \to Y$ has a *subclosed graph* if $\text{ad}\,\psi(\Omega) \subset \{\psi(x)\}$ for each $x \in X$ and filterbase $\Omega$ on $X - \{x\}$ with $\Omega \to x$.

5.3. THEOREM. *The following statements are equivalent for a first countable space $Y$. $X$ is restricted as in Theorem* 5.1.

(a) *Each countable filterbase on $Y$ with at most one $f$-adherent point is convergent.*

(b) *Each sequence in $Y$ with at most one $f$-accumulation point converges.*

(c) *For each $X \in A$, each function (bijection) $\psi: X \to Y$ with an $f$-strongly-subclosed graph is continuous.*

(d) *For each $X \in A$ and any two functions (bijections) $\psi, \lambda: X \to Y$ with $f$-strongly-subclosed graph and subclosed graph, respectively, $E(\psi, \lambda, X, Y)$ is closed in $X$.*

(e) *For each $X \in A$ and any two functions (one a bijection) $\psi, \lambda: X \to Y$ with $f$-strongly-subclosed graph and subclosed graph, respectively, $E(\psi, \lambda, X, Y) = X$ when $E(\psi, \lambda, X, Y)$ is dense in $X$.*

*Proof.* That (a) and (b) are equivalent is fairly immediate. We remark that (a), (c), (d), and (e) are established as equivalent statements by the appropriate use of the first countability and arguments similar to those in Section 3. We prove only the equivalence of (a) and (c). To establish a strong (c), suppose $Y$ satisfies (a), let $X$ be any first countable space and let $\psi: X \to Y$ have an $f$-strongly-subclosed graph. Let $x \in X$ and let $\Sigma$ be a countable open set base at $x$. If $\{x\} \in \Sigma$, then $\psi$ is continuous at $x$. If $\{x\} \notin \Sigma$, then $\Omega = \{V - \{x\}: V \in \Sigma\}$ is a filterbase on $X - \{x\}$ with $\Omega \to x$. So $\psi(\Omega)$ is a countable filterbase on $Y$ and, consequently, $\emptyset \neq f\text{-ad}\,\psi(\Omega) \subset \{\psi(x)\}$. This gives $\psi(\Omega) \to \psi(x)$. So for each $W$ open about $\psi(x)$, there is a $V \in \Sigma$ satisfying $\psi(V) \subset W$.

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Now, suppose a space $Y$ satisfies (c) but not (a). Then there is a $y_0 \in Y$, countable filterbase $\Omega$ on $Y$, and a $V_0$ open about $y_0$ with $f$-ad $\Omega \subset \{y_0\}$ and $\Omega_1 = \{F \cap (Y - V_0): F \in \Omega\}$ a filterbase on $Y$. Let $\psi: Y(y_0, \Omega_1) \rightarrow Y$ be the identity function. Let $y \in Y$ and let $\Omega_2$ be a filterbase on $Y(y_0, \Omega_1) - \{y\}$ with $\Omega_2 \rightarrow y$. Then $y = y_0$ and we have $f$-ad $\psi(\Omega_2) = f$-ad $\Omega_2 \subset f$-ad $\Omega_1 \subset \{y_0\}$; so $G(\psi)$ is $f$-strongly-subclosed. However, $\psi$ is a bijection and is not continuous since $\Omega_1 \rightarrow y_0$ in $Y(y_0, \Omega_1)$ but $\Omega_1 \not\rightarrow y_0$ in $Y$.

The proof that (a) and (c) are equivalent is complete.

5.4. COROLLARY. *A first countable completely Hausdorff space $Y$ is first countable and completely Hausdorff-closed (minimal completely Hausdorff) if $Y$ satisfies any of the equivalent statements of Theorem* 5.1 (5.3).

*Proof*. Follows from Theorem 4.2 (4.4) of [**2**].

**6. Some examples.** In this section, we give some examples to indicate some of the limitations on the weakening of hypotheses in the theorems in this paper. In the examples below, let $Y$ be a completely Hausdorff-closed space which is not $H$-closed [**1**]. Let $y_0 \in Y$ and let $\Omega$ be a filterbase on $Y$ satisfying $\theta$-ad $\Omega = \emptyset$ and $\Omega \rightarrow_f y_0$ [**2**]. Choose $x_0 \in Y - \{y_0\}$ and define $\alpha, \psi, \lambda: Y(y_0, \Omega) \rightarrow Y$ by $\psi(x) = x$ for all $x$, $\alpha(y_0) = x_0$ and $\alpha(x) = x$ otherwise, $\lambda(x_0) = y_0$, $\lambda(y_0) = x_0$ and $\lambda(x) = x$ otherwise.

6.1. *Example. No "f-strongly-subclosed" can be replaced by "strongly-closed" in either of Theorems* 3.1, 3.2 *or* 3.3. $\psi$ has an $f$-strongly-subclosed graph and $\alpha$ and $\lambda$ both have strongly-closed graphs which are not $f$-strongly-subclosed. However, $\lambda$ is not $f$-weakly-continuous. $E(\psi, \lambda, Y(y_0, \Omega), Y)$ is not closed in $Y$, and $E(\psi, \alpha, Y(y_0, \Omega), Y) = Y - \{y_0\}$ which is dense in $Y(y_0, \Omega)$.

6.2. *Example. The phrase "f-weakly-continuous cannot be replaced by "weakly-continuous" in Theorem* 3.1. It is clear that a weakly-continuous function is $f$-weakly-continuous. However, $\psi$ is not weakly-continuous.

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