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FINITE GROUPS OF CONJUGATE RANK 2

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Introduction

In 1953 N. Itô defined the conjugate rank of a finite group as the number of distinct sizes, not equal to 1, of the conjugacy classes of the group [7].

Let G be a finite group and let $\{n_1, \cdots, n_r\}$ be a set of integers

$$n_i > 1 \forall i , \qquad n_i = n_j$$

if and only if $i = j$; and such that every conjugacy class in G has n_i elements for some i and for every i, n_i is the size of some conjugacy class of G, then r is the conjugate rank of G.

N. Itô showed that any group of conjugate rank 2 is soluble [8]. It is the purpose of this paper to strengthen this result.

A group G is called a group of type F, or of isolated type, if for every pair

$$x, y \in G ; \; x, y \notin Z(G), \; C_G(x) \cap C_G(y) = Z(G) \quad \text{or} \quad C_G(x) = C_G(y) .$$

R. Schmidt and J. Rebmann have completely classified such groups [9, 10]. The main theorem is the following:

THEOREM 2. *Let G be a finite group of conjugate rank* 2 *which is not isolated. Then G is a direct product of an Abelian group and a group whose order involves no more than* 2 *primes.*

Combining this with the work of Rebmann and the theorem of Burnside on groups whose orders are divisible by just 2 primes, Itô's theorem follows from Theorem 2. The proof of this depends on the main theorem of [3] and on an extension of this which is proved in section 3. Unhappily in the earlier paper the situation described by (*) is not sufficiently general for the application intended. The following is

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the required situation which is also needed as a basis for the proof of Theorem 2.

(**) Let G be a finite group containing a proper subgroup A satisfying the following conditions:

(i) $A = C_G(x)$ for some $x \in G$,

(ii) $\exists$ an integer r such that if $y \in G$ and $C_G(y) < A$ then $[A : C_G(y)] = r$.

(iii) There exists no element $z \in G$ satisfying

$$A < C_G(z) < G .$$

Let $\pi(A) = \{p \mid p$ is a prime such that $\exists$ a p-power element whose centralizer is $A\}$.

Then A satisfies the condition that every non-central element of A which has order prime to some element of $\pi(A)$ has index r in A. The index of an element is defined to be the number of elements in the conjugacy class containing it. In [3] the main theorem classified groups satisfying the above condition when $|\pi(A)| > 1$. In this paper a similar result is obtained for the situation when $|\pi(A)| = 1$. The result is stated as a corollary, as it is obtained as a consequence of a slightly more general theorem.

COROLLARY. *Let G be a finite group, p a prime number and n an integer. If every p'-element of G has 1 or n conjugates and $|G/Z(G)|$ is divisible by at least two primes different from p, then G is soluble with p-length $\leqslant 2$, q-length $= 1$, q a prime different from p and n is a power of p.*

It is convenient at this point to define for any group G a G-eccentric prime p to be a prime such that $G/Z(G)$ has a non-trivial p-element. If q divides $|G|$ and q is not G-eccentric then the Sylow q-subgroup of G is an Abelian direct factor of G. We can now state the theorem.

THEOREM 1. *Let G be a finite group such that the set of G-eccentric primes divides into two disjoint subsets, π and σ, and let n be an integer Further let G satisfy the following conditions:*

(i) *G has a non-trivial nilpotent Hall σ-subgroup*

(ii) *every non-central π-element of G has n conjugates, and*

(iii) $|\pi| > 1$.

Then G is soluble, n is a σ-number, and $G = O_\pi(G) \times Z_0$ where Z_0 is

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a central $\{\pi, \sigma\}$*-subgroup of* G. *Further either*:

(1) G *has a normal Abelian Hall* π*-subgroup* L_0 *such that* $G/C_G(L_0)$ *acts regularly on* $L_0/L_0 \cap Z(G)$, *or*

(2) $O_\pi(G) \leqslant Z(G)$ *and* $L/O_\pi(G)(Z(G) \cap L)$ *is cyclic and* $O_{\sigma\pi\sigma}(G)/L$ *acts regularly on* $L/O_\pi(G)(Z(G) \cap L)$, *where* $L = O_{\sigma\pi}(G)$.

The corollary follows by applying Theorem 1 with $\sigma = \{p\}$ and $\pi = \{G$-eccentric primes excluding $p\}$. The term regularly will be used in the sense of D. Gorenstein [5] to mean that each element acts fixed point freely. This avoids the possible confusion of using fixed point freely in two different senses.

In the second section a number of trivial but useful lemmas are proved and a very useful proposition which may well be of independent interest.

PROPOSITION 1. *Let* G *be a finite group with a subgroup* A_0 *such that* A_0 *is a characteristic subgroup of* A, *a subgroup of* G, *such that every element of* A_0 *has centralizer* A *or* G. *Let* π *be the set of primes dividing* $|A_0/A_0 \cap Z(G)|$ *and assume* $|\pi| > 1$. *Then either*

(i) $N_G(A)/A$ *is a* π'*-group or,*

(ii) $|N_G(A)/A| = p$ *for some* $p \in \pi$.

It is clear that this proposition could be used to give alternative proofs of the theorems concerning a group of type F [9], [10].

NOTATION. Most of the notation is standard, see for example [5] or [6]. Let G be a finite group. p is said to be a G-eccentric prime if $p \mid |G/Z(G)|$. If $x \in G$, $\mathrm{Ind}_G(x) = [G:C_G(x)] =$ the order of the conjugacy class containing x. If $A < G$ and $x \in A$ then to say that x is non-central will usually mean that $x \notin Z(G)$. For a definition of conjugate rank see [7].

2. Preliminary Lemmas and a Useful Proposition

We begin with a series of simple lemmas which contain results which are frequently used in the analysis of this type of problem.

LEMMA 1. *Let* q *be a prime dividing the order of* G.

(i) *If* x *is in* G *and* $\mathrm{Ind}_G(x) = \mathrm{Ind}_G(y)$ *for some non-central* q*-element* y, *then* $C_G(x)$ *contains non-central* q*-elements.*

(ii) *If* q *divides* $\mathrm{Ind}_G(w)$ *for all non-central* q*-elements* w *then*

$Z(Q) \leqslant Z(G)$ where Q is a Sylow q-subgroup of G.

Proof. (i) Let q^a be the highest power of q which divides $\mathrm{Ind}_G(x)$. Let Q_0 be a Sylow q-subgroup of $C_G(x)$. If (i) is false $Q_0 \leqslant Z(G)$. Let Q be a Sylow q-subgroup of G such that $Q \cap C_G(y)$ is a Sylow q-subgroup of $C_G(y)$. Now $[Q: Q_0] = q^a$ and $C_Q(y) \geqslant \langle y, Q_0 \rangle$. Thus $[Q: C_Q(y)] < q^a$ which contradicts the assumption that $\mathrm{Ind}_G(x) = \mathrm{Ind}_G(y)$.

(ii) If $x \in Z(Q)$, $\mathrm{Ind}_G(x)$ is prime to q. Thus $x \in Z(G)$.

LEMMA 2. *Let A be a proper subgroup of G which is the centralizer of an element in G, and let π be a set of at least two G-eccentric primes each of which divides the order of A. If the centralizer of each π-element of A has order $|G|$ or $|A|$ then A possesses an Abelian Hall π-subgroup. Further if there is a π-element whose centralizer is A then the Hall π-subgroup is central in A.*

Proof. Let p and q be G-eccentric primes in π and let x and y be non-central p- and q-elements of A respectively. Then xy is a non-central π-element lying in A. Thus $|C_G(xy)| = |A|$. But $C_G(xy) = C_G(x) \cap C_G(y)$ and $|C_G(x)| = |C_G(y)| = |A|$. So we conclude that $C_G(x) = C_G(y) = C_G(xy)$. The two results now follow.

LEMMA 3. *Let G be a group with a normal nilpotent subgroup H which has a nilpotent complement K of coprime order such that for all $x, y \in K \backslash C_K(H)$, $C_H(x) = C_H(y)$. Then $K/C_K(H)$ is cyclic or a direct product of a cyclic group with a generalised quaternion group.*

Proof. It is only necessary to show that there is some group on which K acts regularly. Consider $(C_H(x))^H$ for any $x \in K \backslash C_K(H)$. Since H is nilpotent, $(C_H(x))^H < H$ and clearly $K/C_K(H)$ acts regularly on $H/(C_H(x))^H$.

LEMMA 4. *Let P be an Abelian p-group, for some prime p and let K be a group of automorphisms of P whose order is divisible by p. If for all pairs $x, y \in K \backslash \{1\}$, $C_P(x) = C_P(y)$ then $O_{p'}(K) = 1$.*

Proof. Put $H = O_{p'}(K)$ and assume $H \neq 1$. Since K is not a p-group, $C_P(H) \neq 1$. Hence by a simple extension of the proof of Maschkes Theorem, $P = C_P(H) \times L$ where L is a K-invariant subgroup of P. However K would have to act regularly on L and this is false since K is not a p'-group.

The proof of Proposition 1 will be deduced from a sequence of Lemmas which will prove some stronger results than Proposition 1. It is convenient to assume a slightly weaker hypothesis to begin with.

(B) Let $A_0 \leqslant A \leqslant G$ be a sequence of finite groups where A_0 is characteristic in A. Let π be the set of G-eccentric primes dividing $|A_0|$. Finally assume the following three conditions:

(a) If $x \in A_0$ then $C_G(x) = A$ or G

(b) $|\pi| > 1$.

(c) $N_G(A) \neq A$.

LEMMA 5. *Let $A_0 \leqslant A \leqslant G$ satisfy (B). If $X \leqslant A_0$ and X is not central, $C_G(X) = A$ and $N_G(X) \leqslant N_G(A)$.*

Proof. Let $x \in X \backslash Z(G)$. Then $C_G(x) = C_G(X) = A$. $C_G(X) \lhd N_G(X)$, and so $A \lhd N_G(X)$.

LEMMA 6. *Let $A_0 \leqslant A \leqslant G$ satisfy (B). Let $W = N_G(A)/A$. Then*

(i) *if $U \leqslant W$ and U is a p'-group for some $p \in \pi$ then U acts regularly on some section of A_0;*

(ii) *every Sylow subgroup of W is cyclic or generalized quaternion;*

(iii) *any Sylow q-subgroup of W, for $q \in \pi$, has order q.*

Proof. (i) Let P_0 be a Sylow p-subgroup of A_0. Since A_0 is characteristic in A, P_0 is characteristic in A_0 and so U acts on P_0. Since $(|U|, |P_0|) = 1$ and $C_{P_0}(u) = P_0 \cap Z(G)$ for all $u \neq 1$, $u \in U$, U acts regularly on $P_0 | P_0 \cap Z(G)$.

(ii) This follows immediately from (i) and $|\pi|$ being greater than one.

(iii) Let V be a Sylow q-subgroup of W and Q_0 be a Sylow q-subgroup of P_0. V acts faithfully on Q_0 and $C_{Q_0}(V) = Q_0 \cap Z(G)$. Let T be a subgroup of Q_0 such that $|T/Q_0 \cap Z(G)| = q$ and $[T, V] \leqslant Q_0 \cap Z$. Such a subgroup exists because Q_0 and V are q-groups. From Lemma 5 it follows that V acts faithfully on T but $V/C_V(T)$ is elementary Abelian. From (ii) we know that if V has exponent q it is cyclic and so (iii) is proven.

LEMMA 7. *Let $A_0 \leqslant A \leqslant G$ satisfy (B). Then $N_G(A)/A$ is a π or π'-group. If $N_G(A)/A$ is a π-group then $|N_G(A)/A| = p$ for some prime $p \in \pi$.*

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Proof. Let $W = N_G(A)/A$. If W is soluble there exist subgroups of order rs for any pairs of primes r, s dividing $|W|$, from Lemma 6 (ii). If W is not soluble then the Sylow 2-subgroup of W is quaternion, again from Lemma 6 (ii), and so by [1] the involution in W is central. Thus we have subgroups of order $2r$ for any prime r dividing $|W|$.

Let U be a subgroup of rs for two primes dividing $|W|$ where at least one of the pair is in π. If one is not in π, U will be Abelian by Lemma 6 (i) and [6; V. 8.12]. Thus we can assume that U has a normal r-complement for $r \in \pi$. Let R_0 be the Sylow r-subgroup of A_0, which is clearly normalized by U. Hence by Lemma 4, U has no normal r'-subgroup which is false. Hence U does not exist and so either W is a π'-group or is a p-group for some prime $p \in \pi$. Then by Lemma 6 (iii), $|W| = p$.

This Lemma completes the proof of Proposition 1. However for the applications it is useful to have a slightly stronger hypothesis.

(C) Let $A_0 \leqslant A \leqslant G$ satisfy (B) and assume that A_0 is a Hall subgroup of A.

LEMMA 8. *Let $A_0 \leqslant A \leqslant G$ satisfy (C). Then A is the centralizer of a Sylow q-subgroup of G for any $q \in \pi$, $q \mid |N_G(A)/A|$. Further* (i) *if $|N_G(A)/A|$ is a π'-group, then A_0 is a Hall subgroup of G, or*

(ii) *If $|N_G(A)/A| = p$ every Sylow q-subgroup of A_0, $q \in \pi$, $q \neq p$ is a Sylow q-subgroup of G.*

Proof. The conclusion stated first follows from (i) and (ii). Let R_0 be a Sylow r-subgroup of A_0, $r \in \pi$. Then by Lemma 5 $N_G(R_0) \leqslant N_G(A)$ and so if $N_G(A)/A_0$ is an r'-group R_0 is a Sylow r-subgroup of G.

The situation described in Lemma 8 (ii) is the more exceptional and so it is useful to investigate it more thoroughly. (D) Let $A_0 \leqslant A \leqslant G$ satisfy (C) and let $|N_G(A)/A| = p$.

LEMMA 9. *If $A_0 \leqslant A \leqslant G$ satisfies (D) and p is odd and the Sylow p-subgroup of $A_0/A_0 \cap Z(G)$ has order greater than p then G has a normal π-complement.*

Proof. Let P_0 be a Sylow p-subgroup of A_0, and let P be a Sylow p-subgroup of $N_G(A)$. Then $P_0 \lhd P$ and $|P/P_0| = p$. P_0 is thus a normal Abelian maximal subgroup of P. If $P_0 \neq ZJ(P)$, P contains another normal Abelian maximal subgroup, say P_1. Then $P_1 \cap P_0 = Z(P)$. How-

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ever $|P_0/Z(P)| \neq p$, by assumption since $Z(P) = P \cap Z(G)$. Thus $P_0 = ZJ(P)$. Then P is a Sylow p-subgroup of G for otherwise $N_G(P_0) = N_G(ZJ(P))$ would be greater than $N_G(A)$ which is false by Lemma 5. Clearly $N_G(ZJ(P)) = N_G(A)$ has a normal p-complement and so by the Thompson-Glauberman Theorem [5; 8.3.1] so does G. Let this complement be K. Now $K \cap A_0$ is an Abelian Hall $(\pi - \{p\})$-subgroup of G which is contained in the centralizer of its normalizer and so by Burnside [6; IV, 2, 6] the Lemma follows.

3. Proof of Theorem 1.

We begin by showing that Proposition 1 can be applied to the centralizer of a π-element. Let A be the centralizer of a non-central π-element. So $[G:A] = n$. Further, there exists $x \in A$ such that x is a p-element for some $p \in \pi$ and $C_G(x) = A$. Also, since $|\pi| \geqslant 2$, there is a non-central q-element y say in A, with $q \neq p$, $q \in \pi$, by Lemma 1. Now $C_G(xy) = C_G(x) \cap C_G(y)$ and $[G: C_G(xy)] = n$ and so $C_G(x) = C_G(y) = A$. Clearly if we pick any π-element of prime power order in A it has order coprime to either x or y. Thus its centralizer is either A or G. Let A_0 be the Abelian characteristic Hall π-subgroup of A. We can now apply Proposition 1 to the centralizer A with A_0 as the appropriate subgroup.

If G is divisible by a prime s which is not G-eccentric then the Sylow s-subgroup S of G is central. So S is an Abelian direct factor of G. We will assume for the remainder of the proof that $|G|$ is not divisible by any primes which are not G-eccentric and so, in particular, that G is a $\{\pi, \sigma\}$-group.

Let x and y be the non-central q-elements of G. Then it is clear from Lemma 8 that $C_G(x)$ is conjugate to $C_G(y)$ unless $|\pi| = 2$ and $|N_G(C_G(x))/C_G(x)| = p$, $|N_G(C_G(y))/C_G(y)| = q$ for $\pi = \{p, q\}$. But from the first it could be deduced that the Sylow q-subgroup of G is Abelian which would contradict the second statement. Let A be a centralizer of some non-central q-element and let A_0 be the Hall π-subgroup of A.

Let ω be an element of a Sylow p-subgroup P of G $p \in \pi$ such that $C_P(\omega) \lhd P$ and such that $\omega \notin Z(G)$. From Lemma 5 it follows that $P \leqslant N_G(C_G(\omega))$. Thus $N_G(A)$ contains a Sylow p-subgroup for each $p \in \pi$. Hence $N_G(A)$ has π'-index. Let B_0 be a Hall σ-subgroup of A_0, so that $A = A_0 \times B_0$. Then $\exists$ a Hall σ-subgroup H of G such that $B_0 \leqslant H$.

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Clearly $H \cdot N_G(A) = G$. Then $B_0^G = B_0^{N_G(A)H} = B_0^H$ which is a σ-group. However H is nilpotent and so $B_0 \lhd\lhd G$. Thus $B_0 \leqslant O_{\pi'}(G) = O_\sigma(G)$. So $B_0 = O_{\pi'}(G) \cap A$. It is clear now that $G/O_{\pi'}(G)$ satisfies the same conditions as G with n replaced by $n|B_0|/|O_{\pi'}(G)|$.

It will now be proved that G is soluble. There are three cases.

(i) $N_G(A)/A$ is a π'-group.

Then by Lemma 8, A_0 is a Hall π-subgroup of G and so since G has a Hall σ-subgroup say D, $G = A_0 \cdot D$ D is nilpotent, A_0 is Abelian and so G is soluble by Kegel-Wielandt [6; VI. 4].

(ii) $|N_G(A)/A| = p > 2$; $p \in \pi$.

If the Sylow p-subgroup of $A_0/A_0 \cap Z(G)$ has order $>p$, there is nothing to prove since by Lemma 9 G has a normal π-complement. Let P be a Sylow p-subgroup of $N_G(A)$. Then $|P/P \cap Z(G)| \neq p^2$. Let P_1 be a Sylow p-subgroup of G containing P. Let $\omega \in Z_2(P_1)\backslash Z(P_1)$. Then $C_G(\omega)$ is conjugate to A and the Sylow p-subgroup of $N_G(A)$ is isomorphic to P. However $N(C_{P_1}(\omega)) \geqslant P_1$ and so $P_1 \leqslant N_G(C_G(\omega))$ by Lemma 5. Hence $P_1 = P$. Let $X_1, \cdots, X_{p+1}$ be the distinct maximal subgroup of P containing $Z(P)$. Now if $C_G(X_i) = C_G(X_j)$, $i \neq j$ then X_i and X_j commute and so P would be Abelian which is false. Hence $C_G(X_i)$ are $p+1$ distinct conjugate subgroups of G. Furthermore the X_i are all conjugate since they are the Sylow p subgroups of $C_G(X_1)$. Thus, as $X_i \lhd P$, $N_G(P)/C_G(P)$ is divisible by $p+1$. In particular there exists a 2-element u such that $u \in N_G(P)\backslash C_G(P)$ and $u^2 \in C_G(P)$. Hence u normalizes some X_i. However $N_G(X_i) = N_G(C_G(X_i))$ which has a normal p-complement and a normal Sylow 2 subgroup and hence $[u, P] = 1$, contradicting the choice of u. This proves that this situation cannot occur.

(iii) $|N_G(A)/A| = 2$.

Since $O_{\pi'}(G)$ is soluble we can assume $O_{\pi'}(G) = 1$. Let Q_0 be a Sylow 2-subgroup of A_0 and let Q be a Sylow 2-subgroup of $N_G(A)$. Then $|Q/Q_0| = 2$. Further if $y \in Q\backslash Q_0$, $y^2 \in X(G)$. Let $T > Z(G) \cap Q_0$ be such that $[T, y] \leqslant Z(G)$. Then $T \leqslant N_G(C_G(y))$ by Lemma 5. But since $|N_G(C_G(y))/C_G(y)| = 2$, and $C_T(y) \leqslant Z(G)$, $|T/Z(G) \cap Q_0| = 2$. Hence by [6; III. 11. 9] $Q/Z(G) \cap Q$ is a dihedral group. So the group $G/Z(G)$ has a dihedral Sylow 2-subgroup and the centralizer of an involution has an Abelian 2-complement. Thus the structure of $G/Z(G)$ is known by Gorenstein-Walter [4].

Assume that $K/Z(G)$ is soluble. Then there exists a normal non-central p-subgroup N, with $p \in \pi$. $N \leqslant N_G(A)$ since $[G : N_G(A)]$ is a π'-number. Let $q \in \pi$, $p \neq q$. Then N centralizes the Sylow q-subgroup of A_0 and so $N \leqslant A_0$. By Lemma 5 $A \lhd G$ and so G would be soluble.

Thus we may assume $G/Z(G)$ has no normal soluble subgroups. Then if $K/Z(G)$ is a minimal normal subgroup $K/Z(G) \cong PSL(2, r^e)$ of A_7 [4], where r^e is a power of some odd prime r. From the hypothesis it is clear that $\sigma = \{r\}$ or $\{7\}$ respectively. Assume $K/Z(G) \cong PSL(2, r^e)$. Since $r^e > 3$ we can pick x,y neither being involutions such that $|x| \mid r^e - 1$ and $|y| \mid r^e + 1$ and so that the orders of the centralizers are not conjugate as they should be since $K \leqslant G$. Similarly for A_7 by looking at elements of order 3 and 5.

Thus in all cases we have shown that G is soluble.

Assume that $O_\pi(G)$ is not central. Then there exists a non-central normal p-subgroup M of G, where $p \in \pi$. Then $M \leqslant N_G(A)$ and so $M \leqslant A$. Thus $A \lhd G$, from Lemma 5. If x is a non-central π-element of G, $C_G(x)$ is conjugate to A and so $C_G(x) = A$. Thus G/A is a π'-group and clearly acts regularly on $A_0/A_0 \cap Z(G)$. This is the situation described in (i) of Theorem 1, note $n = [G : A]$ is a π'-number.

Now assume that $O_\pi(G) \leqslant Z(G)$. However $G/O_{\pi'}(G)$ satisfies the hypothesis for the Theorem and $O_\pi(G/O_{\pi'}(G))$ is certainly not central so we can conclude from the previous paragraph that $AO_{\pi'}(G) \lhd G$, and that $AO_{\pi'}(G) = O_{\pi',\pi}(G)$. Hence again $N_G(A)/A$ is a π'-group and $[G : A]$ is a π'-number. Finally if $AO_{\pi'}(G) = L$ (2) of Theorem 1 holds by applying Lemma 3.

4. Finite Groups of Conjugate rank 2.

This section is devoted to a proof of Theorem 2. We assume that G is a minimal counter-example to this theorem. It follows that G satisfies the following three properties:—

(i) $|G|$ is divisible by at least 3 G-eccentric primes;

(ii) $|G|$ is divisible only by G-eccentric primes;

(iii) $\exists B < A < G$ with $[G : A] = n$, $[G : B] = m$ both A and B are centralizers of elements in G and every element of G has index 1, m or n.

Let $D \leqslant G$ and define $\pi(D) = \{p \mid p$ is a prime such that there exists a p-element whose centralizer is $D\}$. $\pi(D)$ could be empty. If x is a non-

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central element of G it can be classified according to the following types:

(I) $C_G(x)$ is isolated; and $C_G(x)$ is not isolated in the remaining four cases

(II) $[G: C_G(x)] = n$, $|\pi(C_G(x))| > 1$;

(III) $[G: C_G(x)] = n$, $|\pi(C_G(x))| = 1$;

(IV) $[G: C_G(x)] = m$, $|\pi(C_G(x))| > 1$;

(V) $[G: C_G(x)] = m$, $|\pi(C_G(x))| = 1$.

Since every element of G has index dividing m, every prime dividing $|G|$ also divides m by (ii).

(1) $m/n = p^a$ for some prime p and integer $a \geqslant 1$.

From (iii) there exists an $x \in G$ such that $C_G(x)$ is of type (II) or (III). If $C_G(x)$ is of type (II) then $C_G(x)$ is a direct product of a non-Abelian p-group and an Abelian p'-group and $m/n = p^a$ [3]. If $C_G(x)$ is of type (III), $m/n = p^a$ by the Corollary to Theorem 1, where $\{p\} = \pi(C_G(x))$.

(2) There is no centralizer of type II.

Let A be such a centralizer. Let A_0 be the normal Abelian Hall p'-subgroup of A. Clearly we can apply Lemma 8 to deduce that $N_G(A)$ contains a Sylow r subgroup of G, for some prime $r \neq p$, $r + [N_G(A): A]$. Since $r|m$ and $m = np^a$ and $n = [G: A]$ this leads to a contradiction.

We can now complete the proof of the theorem. Let X be a centralizer of type III. Let ω be a p'-element in X. $C_G(\omega)$ if of type III, IV or V. Note that $C_X(\omega)$ is the centralizer of an element of G, so that if $C_G(\omega)$ is of type IV or V, $C_{G(\omega)} = C_X(\omega)$. If $C_G(\omega)$ is of type III we would have that $m|n$ is of order prime to p which is false. Thus the centralizer of any non-central p'-element in $C_G(\omega)$ is precisely $C_G(\omega)$. From (1) and Lemma 1 we can apply Proposition 1 to $C_G(\omega)$ with the appropriate subgroup being the Hall p'subgroup of $C_G(\omega)$, the $C_G(\omega)$ being Abelian since it is a minimal centralizer. Finally by Lemma 8 $C_G(\omega)$ would contain some Sylow subgroup of G which is again false.

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