SHORT PROOF OF A MAP-COLOUR THEOREM

G. A. DIRAC

Heawood **(3)** showed that for $h > 1$ the countries of any map on a surface of connectivity $h = 3 - \chi$ can be coloured using at most

$$n_h = [\tfrac{1}{2}(7 + \sqrt{(24h - 23)})]$$

colours. In a previous paper **(1)** I proved the following

THEOREM. *For $h = 3$ and for $h \geqslant 5$ a map on a surface of connectivity h with chromatic number n_h always contains n_h mutually adjacent countries.*

A new short proof of this will now be given. It is based on the following result **(2)**:

If a critical k-chromatic graph contains $N > k$ nodes and E edges then

$$(1) \qquad 2E \geqslant (k - 1)N + k - 3.$$

(A graph is called critical if deleting any arbitrary node or edge reduces the chromatic number. A k-chromatic graph always contains a critical k-chromatic subgraph, and a critical graph is finite and connected **(1**, p. 481**)**.)

To prove the above map-colour theorem it is sufficient to show that for $h = 3$ and $h \geqslant 5$, if a critical k-chromatic graph with more than k nodes is drawn without intersection of edges on a surface of connectivity h, then $k \leqslant n_h - 1$. Let such a graph have N nodes and E edges. By (1) and Euler's Theorem **(1**, 2.1**)**, we have

$$(2) \qquad (k - 1)N + k - 3 \leqslant 2E \leqslant 6N + 6(h - 3).$$

Further **(1**, 1.3**)**, $N \geqslant k + 2$.

For the case when $h = 3$, we observe that $n_3 = 7$ and, from (2), $k \leqslant 6$. For $h \geqslant 5$ it may be assumed that $k > 6$, so it follows from (2) that

$$(k - 7)(k + 2) \leqslant (k - 7)\,N \leqslant 6h - 15 - k,$$

whence

$$k \leqslant 2 + [\sqrt{(6h + 3)}].$$

Thus $k \leqslant n_h - 1$ when

$$[\sqrt{(6h + 3)}] \leqslant [\tfrac{1}{2}(1 + \sqrt{(24h - 23)})].$$

This is clearly true if

$$\sqrt{(6h + 3)} \leqslant \tfrac{1}{2}(1 + \sqrt{(24h - 23)}),$$

that is, if $h \geqslant 13$. It can be verified also for $h = 5, 6, \ldots, 12$. This completes the proof of the Theorem.

Received December 31, 1956.

REFERENCES

1. G. A. Dirac, *Map-colour theorems*, Can. J. Math., *4* (1952), 480–490.
2. ———, *A theorem of R. L. Brooks and a conjecture of H. Hadwiger*, Proc. London Math. Soc. (3), *7* (1957).
3. P. J. Heawood, *Map-colour theorem*, Quart. J. Math., *24* (1890), 332–338.

Vienna XIX (*Hardtgasse* 19), *Austria*

CORRECTION TO "A MINIMUM-MAXIMUM PROBLEM FOR DIFFERENTIAL EXPRESSIONS"

D. S. CARTER

The author takes this opportunity to correct some misprints and to add a note to his paper "*A minimum-maximum problem for differential expressions*," in this Journal, 9 (1957), 132-140.

Page 134, Equation (2.5): for this equation read

$$\| \mathfrak{L}x_0 \| = \inf \{ \| \mathfrak{L}x \| \mid x \epsilon X \}.$$

Page 137, line -5: for "e_0^i" read "e^i".

Page 138, line -7: for "$|\xi_0^i|$" read "$|\zeta_0|$".

Added note: Since the preparation of this manuscript it has come to the author's attention that the present problem bears a close relationship to the "Bang-Bang" control problem (**3**). Choosing $c = 0$, $\eta_b = 0$, and allowing the endpoint b to vary, it is easy to show that the value of $\|g_0\|$ at the solution is a continuous monotone function of b. The value of b for which $\|g_0\| = 1$ provides the solution to a "Bang-Bang" problem of a rather general type.

REFERENCE

3. R. Bellman, I. Glicksburg, and O. Gross, *On the "Bang-Bang" Control Problem*, Quart. Appl. Math. *14* (1956), 11-18.

Los Alamos Scientific Laborotory

https://doi.org/10.4153/CJM-1957-027-2 Published online by Cambridge University Press