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STRONG RADICAL CLASSES AND IDEMPOTENTS

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1. **Introduction.** All rings are associative but do not necessarily have identities. Definitions and basic results about radical classes can be found in [2]. A radical class $\mathscr{R}$ is *strong* [3] if for every ring A, $\mathscr{R}(A)$ contains all left and right $\mathscr{R}$-ideals of A.

THEOREM (Anderson [1]). *If $\mathscr{R}$ is a strong radical class, then eAe is $\mathscr{R}$-semisimple whenever A is an $\mathscr{R}$-semisimple ring and $e=e^2 \in A$.*

In this note we prove that for radical classes $\mathscr{R}$ which are *one-sided hereditary* (left and right ideals of $\mathscr{R}$-rings are also $\mathscr{R}$-rings) the converse of Anderson's Theorem is true. We also give an example to show that the converse of Anderson's Theorem is not in general true.

2. To see that the converse of Anderson's Theorem is not in general true we consider the Brown–McCoy radical class $\mathscr{G}$. This radical class is not strong [3, Example 3].

PROPOSITION. *If A is a $\mathscr{G}$-semisimple ring and $e=e^2 \in A$, then eAe is $\mathscr{G}$-semisimple.*

Proof. Let A be a $\mathscr{G}$-semisimple ring and $0 \neq e=e^2 \in A$. Since A is $\mathscr{G}$-semisimple there is a set $\{P_\lambda : \lambda \in \Lambda\}$ of ideals of A such that $\bigcap\{P_\lambda : \lambda \in \Lambda\}=(0)$ and A/P_λ is a simple ring with identity for each $\lambda \in \Lambda$. Also, for each $\lambda \in \Lambda$, $(eAe+P_\lambda)/P_\lambda = (e+P_\lambda)[A/P_\lambda](e+P_\lambda)$ and since A/P_λ is a simple ring, $(e+P_\lambda)A/P_\lambda(e+P_\lambda)$ is either a simple ring with identity $e+P_\lambda$ or, when $e \in P_\lambda$, the zero ring. Thus eAe is isomorphic to a subdirect product of simple rings with identity and is consequently $\mathscr{G}$-semisimple.

THEOREM. *Let $\mathscr{R}$ be a one-sided hereditary radical class. If eAe is $\mathscr{R}$-semisimple whenever A is $\mathscr{R}$-semisimple and $e=e^2 \in A$, then $\mathscr{R}$ is strong.*

Proof. Let $\mathscr{R}$ be a one-sided hereditary radical class and L a left ideal of a ring A. Suppose that $\mathscr{R}(L)=L$.

Define $B=\begin{bmatrix} A & L \\ A^* & L^* \end{bmatrix}$ where A^* (respectively L^*) is obtained by adjoining an

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identity to A (respectively L) in the usual way. Setting $e=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ we see that $\begin{bmatrix}0 & 0\\ 0 & L^*\end{bmatrix}=eBe$, and so

$$\begin{bmatrix}0 & 0\\ 0 & L\end{bmatrix} \subseteq \mathscr{R}\left(\begin{bmatrix}0 & 0\\ 0 & L^*\end{bmatrix}\right) = \mathscr{R}(eBe) \subseteq \mathscr{R}(B).$$

The first inclusion is true because $\begin{bmatrix}0 & 0\\ 0 & L\end{bmatrix}\simeq L \in \mathscr{R}$, and the second inclusion is true because we are assuming the conclusion of Anderson's Theorem. It follows that

$$M = \begin{bmatrix}0 & L\\ 0 & L\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & L\end{bmatrix}\begin{bmatrix}0 & 0\\ A^* & 0\end{bmatrix} = \begin{bmatrix}L^2A^* & 0\\ L^2A^* & 0\end{bmatrix} \subseteq \mathscr{R}(B).$$

Since $\mathscr{R}$ is one-sided hereditary and M is a left ideal of $\mathscr{R}(B)$, $M \in \mathscr{R}$; and $N=\begin{bmatrix}0 & 0\\ L^2A^* & 0\end{bmatrix}$ is an ideal of M, so $M/N \simeq L^2A^* \in \mathscr{R}$. Because L^2A^* is an ideal of A, $L^2A^* \subseteq \mathscr{R}(A)$. Hence $L^2 \subseteq \mathscr{R}(A)$.

Let $\bar{A}=A/\mathscr{R}(A)$ and $\bar{L}=(L+\mathscr{R}(A))/\mathscr{R}(A)$. Then $\bar{L}+\bar{L}\bar{A}$ is an ideal of $\bar{A}$, and $\bar{L}$ is an ideal of $\bar{L}+\bar{L}\bar{A}$ because $L^2 \subseteq \mathscr{R}(A)$. Since $\mathscr{R}(\bar{A})=(0)$, $\mathscr{R}(\bar{L}+\bar{L}\bar{A})=(0)$ [2, Corollary 2 to Theorem 47]. However, since $\mathscr{R}(L)=L$, $\mathscr{R}(\bar{L})=\bar{L}$. Thus $\bar{L}=(\bar{0})$ and so $L \subseteq \mathscr{R}(A)$ as required.

Similarly, if I is a right ideal of A and $\mathscr{R}(I)=I$, then $I \subseteq \mathscr{R}(A)$.

If a radical class $\mathscr{R}$ satisfies the conclusion of Anderson's Theorem, then $\mathscr{R}(eAe) \subseteq e\mathscr{R}(A)e$ for every ring A and idempotent $e \in A$. Even when $\mathscr{R}$ is strong this inclusion may be strict: if F is a field and $e=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, then

$$\mathscr{N}_g\left(e\begin{bmatrix}F & F\\ F & F\end{bmatrix}e\right) = (0)$$

but

$$e\mathscr{N}_g\left(\begin{bmatrix}F & F\\ F & F\end{bmatrix}\right)e = \begin{bmatrix}F & 0\\ 0 & 0\end{bmatrix}$$

where $\mathscr{N}_g$ is the generalized nil radical class [2, 7.8]. However, if $\mathscr{R}$ is one-sided hereditary it is clear that for every ring A and idempotent $e \in A$, $e\mathscr{R}(A)e \in \mathscr{R}$. Thus we obtain the following corollary.

Corollary. *Let $\mathscr{R}$ be a one-sided hereditary radical class. The following are equivalent.*

(i) *$\mathscr{R}$ is a strong radical class,*

(ii) *eAe is $\mathscr{R}$-semisimple whenever A is $\mathscr{R}$-semisimple and $e=e^2 \in A$,*

(iii) *$\mathscr{R}(eAe)=e\mathscr{R}(A)e$ for all rings A and all idempotents $e \in A$.*

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Since the nil radical class $\mathcal{N}$ is one-sided hereditary we may take $\mathcal{R}=\mathcal{N}$ in the above corollary to see that the Koethe conjecture ($\mathcal{N}$ is strong) is equivalent to (ii) and (iii) above.

Finally we remark that one-sided hereditary radical classes are not scarce: the lower radical class determined by any one-sided hereditary class of rings is a one-sided hereditary radical class [4].

References

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