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ADDITIVE FUNCTIONAL INEQUALITIES AND DERIVATIONS ON HILBERT C^*-MODULES

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Abstract. In this paper we investigate the following functional inequality

$$\|f(x-y-z)-f(x-y+z)+f(y)+f(z)\| \le \|f(x+y-z)-f(x)\|$$

in Banach spaces, and employing the above inequality we prove the generalized Hyers–Ulam stability of derivations in Hilbert C^*-modules.

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1. Introduction and preliminaries. A classical question in the theory of functional equations is the following: 'When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$, must be close to an exact solution of $\mathcal{E}$?' If the problem has a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [**31**] in 1940 and solved in the next year by Hyers [**11**] for the Cauchy functional equation. It gave rise to the stability theory for functional equations. In 1950, the result of Hyers [**11**] was extended by Aoki [**3**] by considering the unbounded Cauchy differences. In 1978, Rassias [**27**] proved that the additive mapping T, obtained by Hyers [**11**] or Aoki [**3**], is linear if, in addition, for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$. Găvruta [**7**] generalized Rassias' result. Following the techniques of the proof of the corollary of Hyers [**11**], we observed that Hyers introduced (in 1941) the Hyers continuity condition about the continuity of mapping for each fixed, and then he proved homogeneity of degree one and therefore the famous linearity. This condition has been assumed till now through the complete Hyers direct method to prove linearity for generalized Hyers–Ulam stability problem forms (see [**15**]). Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [**1**, **4**, **5**, **14**, **16–22**, **26**, **28**, **29**]).

Rassias [**24**] following the spirit of the innovative approach of Hyers [**11**], Aoki [**3**] and Rassias [**27**] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [**23**, **25**] for a number of other new results). Gilányi [**9**] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [**30**]. Fechner [**6**] and Gilányi [**10**] proved the generalized Hyers–Ulam stability of the functional inequality (1.1).

Hilbert C^*-modules provide a natural generalization of Hilbert spaces arising when the field of scalars C is replaced by an arbitrary C^*-algebra. This generalization was introduced by Kaplansky in [**13**] (see also [**2**, **8**]).

DEFINITION 1.1. A *pre-Hilbert A-module* is a (right) A-module $\mathcal{M}$ equipped with a sesquilinear form $\langle ., . \rangle : \mathcal{M} \times \mathcal{M} \to A$ with the following properties:

1. $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$.
2. $\langle x, x \rangle = 0$ implies that $x = 0$.
3. $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$.
4. $\langle x, ya \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$ and any $a \in A$.

The map $\langle ., . \rangle$ is called an *A-valued inner product.*

DEFINITION 1.2. A pre-Hilbert A-module $\mathcal{M}$ is called a Hilbert C^*-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

DEFINITION 1.3. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a derivation on the Hilbert C^*-module $\mathcal{M}$ if it satisfies the condition $d(\langle x, y \rangle z) = \langle d(x), y \rangle z + \langle x, d(y) \rangle z + \langle x, y \rangle d(z)$ for every $x, y, z \in \mathcal{M}$.

In this paper we investigate an $\mathbb{C}$-linear mapping associated with the following functional inequality:

$$\|f(x - y - z) - f(x - y + z) + f(y) + f(z)\| \leq \|f(x + y - z) - f(x)\|, \tag{1.2}$$

and prove the generalized Hyers–Ulam stability of $\mathbb{C}$-linear mappings in Banach spaces associated with the functional inequality (1.2). These results are applied to investigate derivations in Hilbert C^*-modules and to prove the generalized Hyers–Ulam stability of derivations in Hilbert C^*-modules.

Throughout this paper X is a Banach space, Y is a Banach space with norm $\|.\|_Y$ and $\mathcal{M}$ denotes a Hilbert C^*-module with norm $\|.\|$.

2. Functional inequalities in Banach spaces.

LEMMA 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\|\mu f(x - y - z) - f(\mu x - y + z) + \mu f(y) + f(z)\|_Y \leq \|f(x + y - \mu z) - f(x)\|_Y \tag{2.1}$$

for all $x, y, z \in X$ and all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Then f is $\mathbb{C}$-linear.

Proof. Letting $x = y = z = 0$ and $\mu = 1$ in (2.1), we get

$$\|f(0)\|_Y \leq 0.$$

So $f(0) = 0$. Letting $\mu = 1$ and $z = y$ in (2.1), we have

$$\|f(x - 2y) - f(x) + 2f(y)\|_Y \leq 0$$

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for all $x, y \in X$. Hence,

$$f(x-2y)=f(x)-2f(y) \tag{2.2}$$

for all $x, y \in X$. By replacing x by $x+y$ in (2.2), we have

$$f(x-y)=f(x+y)-2f(y) \tag{2.3}$$

for all $x, y \in X$. Letting $x=0$ in (2.2), we get $f(-y)=-f(y)$ for all $y \in X$, therefore f is an odd function. Interchanging x and y in (2.3), we have

$$-f(x-y)=f(x+y)-2f(x) \tag{2.4}$$

for all $x, y \in X$. Adding (2.3) and (2.4), we conclude that f is additive. Letting $y=z=0$ in (2.1), we get

$$f(\mu x)=\mu f(x) \tag{2.5}$$

for all $x \in X$ and all $\mu \in \mathbb{T}^1$.

Now let $\mu \in \mathbb{C}$ and K be a natural number greater than $4|\mu|$. Then $|\frac{\mu}{K}| < \frac{1}{4} < 1-\frac{2}{3}=\frac{1}{3}$. By Theorem 1 of [**12**] there exist three numbers μ_1, μ_2 and $\mu_3 \in \mathbb{T}^1$ such that $3(\frac{\mu}{K})=\mu_1+\mu_2+\mu_3$. So by additivity of f and (2.5)

$$\begin{aligned} f(\mu x) &= \frac{K}{3} f\left(3\frac{\mu}{K}x\right) \\ &= \frac{K}{3} f(\mu_1 x+\mu_2 x+\mu_3 x) \\ &= \frac{K}{3}(f(\mu_1 x)+f(\mu_2 x)+f(\mu_3 x)) \\ &= \frac{K}{3}(\mu_1+\mu_2+\mu_3) f(x) \\ &= \mu f(x) \end{aligned}$$

for all $x \in X$. Therefore, $f: X \to Y$ is $\mathbb{C}$-linear. □

Now we prove the generalized Hyers–Ulam stability of $\mathbb{C}$-linear mappings in Banach spaces.

THEOREM 2.1. *Let $f: X \to Y$ be a mapping for which there exists a control function $\varphi: X^3 \to [0,\infty)$ such that*

$$\lim_{k\to\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right)=0, \tag{2.6}$$

$$\widetilde{\varphi}(x,y,y)=\sum_{k=0}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{y}{2^k}\right) < \infty \tag{2.7}$$

and

$$\begin{aligned} &\|\mu f(x-y-z)-f(\mu x-y+z)+\mu f(y)+f(z)\|_Y \\ &\qquad \leq \|f(x+y-\mu z)-f(x)\|_Y+\varphi(x,y,z) \end{aligned} \tag{2.8}$$

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for all $x, y, z \in X$ *and all* $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. *Then there exists a unique* $\mathbb{C}$*-linear mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\|_Y \le \widetilde{\varphi}\left(x, \frac{x}{2}, \frac{x}{2}\right) \tag{2.9}$$

for all $x \in X$.

Proof. It follows from (2.6) and (2.8) that $f(0) = 0$. Letting $\mu = 1$ and $z = y$ in (2.8), we get

$$\|f(x - 2y) - f(x) + 2f(y)\|_Y \le \varphi(x, y, y) \tag{2.10}$$

for all $x \in X$. Replacing x and y by $2x$ and x in (2.10) respectively, we have

$$\|f(2x) - 2f(x)\|_Y \le \varphi(2x, x, x)$$

for all $x \in X$. So

$$\|f(x) - 2f\left(\frac{x}{2}\right)\|_Y \le \varphi\left(x, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence,

$$\begin{aligned}\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\|_Y &\le \sum_{j=l}^{m-1} \|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\|_Y \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{2.11}$$

for all non-negative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.9).

It follows from (2.6) and (2.8) that

$$\begin{aligned}&\|\mu L(x - y - z) - L(\mu x - y + z) + \mu L(y) + L(z)\|_Y \\ &= \lim_{n\to\infty} 2^n \|\mu f\left(\frac{x - y - z}{2^n}\right) - f\left(\frac{\mu x - y + z}{2^n}\right) + \mu f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\|_Y \\ &\le \lim_{n\to\infty} 2^n \|f\left(\frac{x + y - \mu z}{2^n}\right) - f\left(\frac{x}{2^n}\right)\|_Y + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &= \|L(x + y - \mu z) - L(x)\|_Y\end{aligned}$$

for all $x, y, z \in X$ and all $\mu \in \mathbb{T}^1$. So

$$\|\mu L(x - y - z) - L(\mu x - y + z) + \mu L(y) + L(z)\|_Y \le \|L(x + y - \mu z) - L(x)\|_Y$$

for all $x, y, z \in X$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.1, the mapping $L : X \to Y$ is $\mathbb{C}$-linear.

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Now let $L' : X \to Y$ be another $\mathbb{C}$-linear mapping satisfying (2.9). Then we have

$$\begin{aligned}\|L(x) - L'(x)\|_Y &= 2^n \left\| L\left(\frac{x}{2^n}\right) - L'\left(\frac{x}{2^n}\right) \right\|_Y \\ &\leq 2^n \left(\left\| L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y + \left\| L'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y \right) \\ &\leq 2^{n+1} \widetilde{\varphi}\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) = 2\sum_{k=n}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $L(x) = L'(x)$ for all $x \in X$. This proves the uniqueness of L. Thus, the mapping $L : X \to Y$ is a unique $\mathbb{C}$-linear mapping satisfying (2.9). □

COROLLARY 2.3. *Let $\theta \geq 0$ and $\{p_i\}_{i=1}^3$ be real numbers such that $p_i > 1$ for all $i = 1, 2, 3$. Assume that a mapping $f : X \to Y$ satisfying*

$$\begin{aligned}&\|\mu f(x - y - z) - f(\mu x - y + z) + \mu f(y) + f(z)\|_Y \\ &\qquad \leq \|f(x + y - \mu z) - f(x)\|_Y + \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})\end{aligned} \tag{2.12}$$

for all $x, y, z \in X$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique $\mathbb{C}$-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \theta \left(\frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1} + \frac{1}{2^{p_2} - 2}\|x\|^{p_2} + \frac{1}{2^{p_3} - 2}\|x\|^{p_3} \right)$$

for all $x \in X$.

Proof. It is clear from (2.12) that $f(0) = 0$. By taking $\varphi(x, y, z) := \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})$ in Theorem 2.1, we get the desired result. □

COROLLARY 2.4. *Let $\theta \geq 0$ and $\{p_i\}_{i=1}^3$ be positive real numbers such that $p_1 + p_2 + p_3 > 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|\mu f(x - y - z) - f(\mu x - y + z) + \mu f(y) + f(z)\|_Y \\ &\qquad \leq \|f(x + y - \mu z) - f(x)\|_Y + \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3})\end{aligned} \tag{2.13}$$

for all $x, y, z \in \mathcal{M}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique $\mathbb{C}$-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \theta \left(\frac{2^{p_1}}{2^{p_1+p_2+p_3} - 2} \right) \|x\|^{p_1+p_2+p_3}$$

for all $x \in X$.

Proof. It follows from (2.13) that $f(0) = 0$. By defining $\varphi(x, y, z) := \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3})$, and applying Theorem 2.1, we get the desired result. □

3. Stability of derivations in Hilbert C^*-modules. Now we prove the generalized Hyers–Ulam stability of derivations in Hilbert C^*-modules.

THEOREM 3.1. *Let $f : \mathcal{M} \to \mathcal{M}$ be a mapping for which there exist a control function $\varphi : \mathcal{M}^3 \to [0, \infty)$ satisfying (2.6), (2.7) and*

$$\|\mu f(x-y-z) - f(\mu x - y + z) + \mu f(y) + f(z)\| \\ \leq \|f(x+y-\mu z) - f(x)\| + \varphi(x,y,z) \tag{3.1}$$

and

$$\|f(<x,y>z) - <f(x),y>z - <x,f(y)>z - <x,y>f(z)\| \leq \varphi(x,y,z) \tag{3.2}$$

for all $x, y, z \in \mathcal{M}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique derivation $d : \mathcal{M} \to \mathcal{M}$ such that

$$\|f(x) - d(x)\| \leq \widetilde{\varphi}\left(x, \frac{x}{2}, \frac{x}{2}\right) \tag{3.3}$$

for all $x \in \mathcal{M}$.

Proof. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $d : \mathcal{M} \to \mathcal{M}$ satisfying (3.3). The mapping $d : \mathcal{M} \to \mathcal{M}$ is given by

$$d(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathcal{M}$.

By the assumption, we have

$$\begin{aligned}&\|d(<x,y>z) - <d(x),y>z - <x,d(y)>z - <x,y>d(z)\| \\ &\leq \lim_{k\to\infty} 2^k \Big\| f\left(<\frac{x}{2^k}, \frac{y}{2^k}>\frac{z}{2^k}\right) - <f\left(\frac{x}{2^k}\right), \frac{y}{2^k}>\frac{z}{2^k} \\ &\qquad - <\frac{x}{2^k}, f\left(\frac{y}{2^k}\right)>\frac{z}{2^k} - <\frac{x}{2^k}, \frac{y}{2^k}>f\left(\frac{z}{2^k}\right)\Big\| \\ &\qquad\qquad \leq \lim_{k\to\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{M}$. So

$$d(<x,y>z) = <d(x),y>z + <x,d(y)>z + <x,y>d(z)$$

for all $x, y, z \in \mathcal{M}$. Therefore, the mapping $d : \mathcal{M} \to \mathcal{M}$ is a derivation on $\mathcal{M}$. □

COROLLARY 3.2. *Let $\theta \geq 0$ and $\{p_i\}_{i=1}^3$ be real numbers such that $p_i > 1$ for all $i = 1, 2, 3$. Assume that a mapping $f : \mathcal{M} \to \mathcal{M}$ satisfying*

$$\|\mu f(x-y-z) - f(\mu x - y + z) + \mu f(y) + f(z)\|_Y \\ \leq \|f(x+y-\mu z) - f(x)\|_Y + \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})$$

and

$$\|f(<x,y>z) - <f(x),y>z - <x,f(y)>z - <x,y>f(z)\| \\ \leq \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})$$

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for all $x, y, z \in \mathcal{M}$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique derivation* $d : \mathcal{M} \to \mathcal{M}$ *such that*

$$\|f(x) - d(x)\|_Y \leq \theta \left(\frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1} + \frac{1}{2^{p_2} - 2}\|x\|^{p_2} + \frac{1}{2^{p_3} - 2}\|x\|^{p_3} \right)$$

for all $x \in \mathcal{M}$.

Proof. The result follows from Theorem 3.1. □

COROLLARY 3.3. *Let* $\theta \geq 0$ *and* $\{p_i\}_{i=1}^3$ *be real numbers such that* $p_1 + p_2 + p_3 > 1$. *Assume that a mapping* $f : \mathcal{M} \to \mathcal{M}$ *satisfying*

$$\begin{aligned} &\|\mu f(x - y - z) - f(\mu x - y + z) + \mu f(y) + f(z)\|_Y \\ &\qquad \leq \|f(x + y - \mu z) - f(x)\|_Y + \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}) \end{aligned}$$

and

$$\|f(<x, y> z) - <f(x), y> z - <x, f(y)> z - <x, y> f(z)\| \leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3})$$

for all $x, y, z \in \mathcal{M}$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique derivation* $d : \mathcal{M} \to \mathcal{M}$ *such that*

$$\|f(x) - d(x)\|_Y \leq \theta \left(\frac{2^{p_1}}{2^{p_1+p_2+p_3} - 2} \right) \|x\|^{p_1+p_2+p_3}$$

for all $x \in \mathcal{M}$.

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