J. Austral. Math. Soc. (*Series A*) 31 (1981), 262–268

NILPOTENT AND SEMI-n-ABELIAN GROUPS

G. KOWOL

(Received 14 August 1980; revised 9 October 1980)

Communicated by H. Lausch

Abstract

A group G is called semi-n-abelian, if for every $g \in G$ there exists at least one $a(g) \in G$–which depends only on g–such that $(gh)^n = a^{-1}(g)g^nh^na(g)$ for all $h \in G$; a group G is called n-abelian, if $a(g) = e$ for all $g \in G$. According to Durbin the following holds for n-abelian groups: If G is n-abelian for at least 3 consecutive integers, then G is n-abelian for all integers and these groups are exactly the abelian groups. In this paper this problem is generalized to the semi-n-abelian case: If a finite group G is semi-n-abelian for at least 4 consecutive integers then G is semi-n-abelian for all integers and these groups are exactly the nilpotent groups, where the Sylow-2-subgroup is abelian, the Sylow-3-subgroup is any element of the Levi-variety ($[[g, h], h] = e \ \forall g, h \in G$) and the Sylow-$p$-subgroup ($p > 3$) is of class $\leqslant 2$. As a consequence we get a description of all finite (3-)groups, which are elements of the Levi-variety.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 20 D 15; secondary 20 E 10.

1. According to Kowol (1977) a group G is called semi-n-abelian, if there exists to every $g \in G$ at least one element $a(g) \in G$, depending only on g, such that $(gh)^n = a^{-1}(g)g^nh^na(g)$ holds for all $h \in G$. These groups are closely connected with nilpotent groups of class $\leqslant 2$; for instance we have (see Satz 9 of Kowol (1977)): A finite semi-n-abelian group whose order is relatively prime to $n(n^2 - 1)$ is nilpotent of class $\leqslant 2$. Conversely (see Bemerkung 2 of Kowol (1977)) the following holds: A nilpotent group of class $\leqslant 2$ and of odd order is semi-n-abelian for every $n \in \mathbf{Z}$.

In this paper we transfer a question of Durbin (1967) concerning n-abelian groups (that are semi-n-abelian groups with $a(g) = e$ for all $g \in G$) to the case of semi-n-abelian groups: Which finite groups are semi-n-abelian for all $n \in \mathbf{Z}$ and what is the minimal number k, such that G semi-n-abelian for k consecutive

https://doi.org/10.1017/S1446788700019406 Published online by Cambridge University Press

numbers implies G is semi-n-abelian for all $n \in \mathbf{Z}$? This question is answered in the n-abelian case in Durbin (1967): Exactly the abelian groups are n-abelian for every $n \in \mathbf{Z}$ and $k = 3$. In the case treated here we give the following answer: A finite group G is semi-n-abelian for all $n \in \mathbf{Z}$ if and only if G is nilpotent with Sylow-2-subgroup abelian, Sylow-3-subgroup an element of the Levi-variety (that means $[[g, h], h] = e \ \forall\, g, h \in G$) and Sylow-$p$-subgroup ($p > 3$) of class $\leqslant 2$. For the minimal number k we derive $k = 4$; $k \leqslant 3$ is impossible, because every group of exponent n is semi-$(n-1)$-, semi-n- and semi-$(n+1)$-abelian, evidently (see Lemma 3a of Kowol (1977)).

As a consequence we derive a description of all finite groups, in particular finite 3-groups, which are elements of the Levi-variety.

All groups considered in this paper are assumed to be finite.

2. We state the following lemma:

LEMMA 1. *If G is semi-m-abelian and semi-n-abelian then G is semi-mn-abelian too.*

PROOF. Evident.

We start with the case G is a 2-group.

THEOREM 1. *Let $o(G) = 2^a$. Then the following properties are equivalent*:

i) *G is semi-n-abelian for 4 consecutive integers.*

ii) *G is abelian.*

iii) *G is semi-2-abelian.*

iv) *G is semi-n-abelian for every $n \in \mathbf{Z}$.*

PROOF. ii) $\Leftrightarrow$ iii) is part of Lemma 11, c in Kowol (1977).

i) $\Rightarrow$ ii) We use induction on the order $o(G)$ of G. Since property i) is hereditary to homomorphic images (see Bemerkung 10 in Kowol (1977)) we can assume that all non-trivial ones are abelian. This implies $o(G') \leqslant 2$ and $c(G) \leqslant 2$ $-c(G)$ denoting the class of G. Thus we can apply Hilfssatz III.1.3a of Huppert (1967) and obtain $[g^2, h] = e$ for all $g, h \in G$, which means $g^2 \in Z(G)$ for all $g \in G$. Now among the 4 consecutive integers for which G is semi-n-abelian there exists exactly one, denoted by m, which fulfills $m \equiv 2 \pmod 4$, that is $m = 4s + 2$. Taking into account that $g^2 \in Z(G)$ for all $g \in G$ the definition of a semi-m-abelian group gives the equality $(gh)^m = g^m h^m$. Using

https://doi.org/10.1017/S1446788700019406 Published online by Cambridge University Press

finally Hilfssatz III.1.3b of Huppert (1967) we derive

$$(gh)^2 = (gh)^{4s+2}(gh)^{-4s} = g^{4s+2}h^{4s+2}\left(g^{4s}h^{4s}[h, g]^{\binom{4s}{2}}\right)^{-1} = g^2h^2,$$

thus G is abelian and ii) holds.

ii) $\Rightarrow$ iv) $\Rightarrow$ i) is evident.

The next case we want to treat of is $o(G) = 3^a$. For this we need the following.

LEMMA 2. *Let $o(G) = 3^a$. If G is semi-2-abelian then $c(G) \leqslant 3$.*

PROOF. We can assume $\exp G > 3$. Since 2 is a primitive root mod $\exp G$, G is semi-5-abelian too, because of Lemma 1. The proof of Satz 19 in Kowol (1977) then implies the regularity of G. Thus we have G' is abelian by Satz III.10.3b in Huppert (1967). According to Lemma 5 of Kowol (1977) the property of being semi-2-abelian is hereditary to subgroups of G. Therefore we can assume using an induction argument that all proper subgroups of G have class $\leqslant 3$. We distinguish three cases:

a) G is generated by at least 4 elements. Then each subgroup of G which is generated by 3 elements has class $\leqslant 3$, therefore $c(G) \leqslant 3$ by Satz III.6.10 of Huppert (1967).

b) The minimal number of generators of G is exactly 3. Since G' is abelian we can apply Theorem 1.3 of Gupta (1965) in this case and we again have $c(G) \leqslant 3$.

c) $G = \langle a, b\rangle$. The regularity of G then implies G' cyclic (see Satz III.10.3b of Huppert (1967)) thus $K_3(G) := [G', G] \subseteq (G')^3$ since $(G')^3$ is the only maximal subgroup of G'.

On the other hand Lemma 4d of Kowol (1977) with $n = 2$ implies $[G^3, G] \subseteq C_G(G^2) = Z(G)$. Because of the regularity of G we have $[G^3, G] = (G')^3$ (Satz III.10.8c of Huppert (1967)) which together with the above result yields $K_3(G) \subseteq (G')^3 = [G^3, G] \subseteq Z(G)$ which means $c(G) \leqslant 3$.

THEOREM 2. *Let $o(G) = 3^a$. Then the following properties are equivalent:*

i) *G is semi-n-abelian for 4 consecutive integers.*

ii) *G is semi-2-abelian.*

iii) *G is semi-3-abelian and $c(G) \leqslant 3$.*

iv) *G is a homomorphic image of a subgroup of the direct product $P \times H$ where P is a finite group of exponent 3 and H is a finite 3-group with $c(H) = 2$.*

v) *G is semi-n-abelian for every integer n.*

vi) $[[g, h], h] = e \ \forall\, g, h \in G$.

PROOF. i) $\Rightarrow$ ii) Let G be semi-n-abelian for $n \in I = \{i, i+1, i+2, i+3\}$. Then there exists an element $k \in I$ with $k - 1 \in I$ and $k \equiv 2 \pmod 3$. If $k \equiv 2, 5 \pmod 9$ then k is a primitive root mod 3^n for all $n \in \mathbf{N}$ (see for example Satz 43 of Scholz-Schönberg (1973)), in particular for the modulus $\exp G = 3^b$. If on the other hand $k \equiv 8 \pmod 9$, then by Lemma 1 G is semi-$k(k-1)$-abelian too and we have $k(k-1) \equiv 8 \cdot 7 \equiv 2 \pmod 9$, which means $k(k-1)$ is a primitive root mod $\exp G$. Combining both results we have shown the existence of an integer m such that G is semi-m-abelian and m is a primitive root mod $\exp G$. Therefore there exists a natural number r with $m^r \equiv 2 \pmod{\exp G}$; now Lemma 1 implies that G is semi-m^r-abelian hence semi-2-abelian.

ii) $\Rightarrow$ iii) By Lemma 2 we know that $c(G) \leqslant 3$. Now G is semi-2-abelian thus we have by Lemma 5 of Kowol (1977): $(g^2h)^2 = g^3h^2g$ which is equivalent to

(1) $$hg^2h = gh^2g \ \forall\, g, h \in G.$$

We calculate $(g^2h)^3$ using (1) twice:

$$\begin{aligned}(g^2h)^3 &= (g^2h)^2(g^2h) = g^3h^2gg^2h = g^3(h^2g^2h^2)h^{-2}gh\\ &= g^3gh^4(gh^{-2}g)h = g^4h^4h^{-1}g^2h^{-1}h = g^4h^3g^2\end{aligned}$$

which is equivalent to G is semi-3-abelian by Lemma 5 of Kowol (1977).

iii) $\Rightarrow$ iv) Essentially this is Satz 13 of Kowol (1977)–there it was shown that G is an element of var $\bar{P} \cup$ var $\bar{H}$, where $\bar{P}$ is a finite group of exponent 3 and $\bar{H}$ is a finite 3-group with $c(\bar{H}) = 2$ (the finiteness of $\bar{P}$, $\bar{H}$ is not stated explicitly but follows from the proof). Since var $\bar{P} \cup$ var $\bar{H} = \operatorname{var}(\bar{P} \times \bar{H})$, this means that G is a homomorphic image of a subgroup of the infinite direct product of $\bar{P} \times \bar{H}$. By Lemma 4.3 of Higman (1959) it suffices to take only finite direct products of $\bar{P} \times \bar{H}$, but these always are of the form $P \times H$, where P is a finite group of exponent 3 and H is a finite group of class 2, thus iv) holds.

iv) $\Rightarrow$ v) By Lemma 3a of Kowol (1977) finite groups of exponent 3 are semi-n-abelian for all $n \in \mathbf{Z}$, since there can occur only the cases: semi-0-, semi-1- and semi-(-1)-abelian. On the other hand according to Bemerkung 2 of Kowol (1977) all finite 3-groups of nilpotence class 2 are semi-n-abelian for all $n \in \mathbf{Z}$ too. Thus P and H (in the notation of condition iv)) satisfy the law $(g^2h)^n = g^{n+1}h^n g^{n-1} \ \forall\, g, h \in G, \forall\, n \in \mathbf{Z}$ (Lemma 5 of Kowol (1977)). Since G lies in the variety generated by P and H and since $o(G)$ is odd we derive using Lemma 5 of Kowol (1977) once more that G is semi-n-abelian for all $n \in \mathbf{Z}$, hence v).

v) $\Rightarrow$ i) is trivial.

iv) $\Rightarrow$ vi) According to Satz III.6.6 of Huppert (1967) P fulfills condition vi) and so does H, evidently, therefore we have

$$G \in \left\{K, \left[\left[g, h\right], h\right] = e \ \forall\, g, h \in K, g^{\exp G} = e, \forall\, g \in K\right\} = \mathfrak{L}$$

that means G is element of the Levi-variety $\mathfrak{L}$.

https://doi.org/10.1017/S1446788700019406 Published online by Cambridge University Press

vi) $\Rightarrow$ iii) If G is an element of the Levi-variety $\mathfrak{L}$, then Satz III.6.5 of Huppert (1967) first implies $c(G) \leqslant 3$. On the other hand using Hilfssatz III.6.4 of Huppert (1967) we know that $[g, h]$ commutes with g (and h)

(2) $$[[g, h], g] = e \ \forall g, h \in G.$$

Applying this we get

$$\begin{aligned}(g^2h)^3 &= g^2(hg^2)hg^2h = g^2g^2h[h, g^2]hg^2h \\ &= g^4hh(g^2h)[h, g^2] = g^4h^2hg^2[g^2, h][h, g^2] \\ &= g^4h^3g^2,\end{aligned}$$

which means that G is semi-3-abelian (Lemma 5 of Kowol (1977)).

Note. a) Condition iv) can also be used to rephrase the theorem in terms of varieties of groups: for example we have: the variety generated by all finite semi-2-abelian 3-groups is the join of the variety generated by all finite groups of exponent 3 and the variety generated by all finite 3-groups of class 2.

b) Condition vi) of the theorem yields other already known equivalences:

vii) *Conjugate elements of G commute* (see Huppert (1967), Hilfssatz III.6.4).

viii) $[[g, h], g] = e \ \forall g, h \in G$ (see also Levi-v.d. Waerden (1932)).

Theorem 2 can be used to give a description of all finite groups satisfying the law $[[g, h], h] = e$–it seems that this characterization has not appeared in the literature yet.

Corollary. *Let G be a finite group. G satisfies the law $[[g, h], h] = e$ if and only if G is nilpotent such that the class of every Sylow-p-subgroup is < 2 for $p \neq 3$ and the Sylow-3-subgroup is a homomorphic image of a subgroup of $P \times H$ where P is a finite group of exponent 3 and H is a finite 3-group of class 2.*

Proof. The result follows immediately from Satz III.6.5 of Huppert (1967) and Theorem 2.

We now turn to the general case; here G_p denotes as usually a Sylow-p-subgroup of G.

Theorem 3. *For a finite group G the following properties are equivalent:*

i) *G is semi-n-abelian for 4 consecutive integers.*

ii) *G is nilpotent with G_2 abelian, G_3 a homomorphic image of a subgroup of $P \times H$, where P is a finite group of exponent 3 and H is a finite 3-group of class $c(H) = 2$ and G_p has nilpotency class $c(G_p) \leqslant 2$ for $p > 3$.*

https://doi.org/10.1017/S1446788700019406 Published online by Cambridge University Press

iii) *G is semi-2-abelian.*
iv) *G is semi-n-abelian for all integers n.*

PROOF. i) $\Rightarrow$ ii) Let G be semi-n-abelian for $n \in I = \{i, i+1, i+2, i+3\}$. First we claim that for $(p, 6) = 1$ the Sylow-p-subgroup G_p is a direct factor of G and has class $\leqslant 2$. This statement follows using Satz 7 of Kowol (1977), since one cannot have $p | n(n^2-1)$ for all $n \in I$ ($(p, 6) = 1$), thus $G = G_p \times G_{p'}$. Now condition i) is hereditary to homomorphic images so we get by induction $G = G_{\{2,3\}} \times G_{\{2,3\}'}$ where $G_{\{2,3\}'}$ satisfies condition ii) (as usual G_π denotes a Hall π-subgroup of G and π' is the set of all primes not in π but dividing $o(G)$).

Now let G be a group with $o(G) = 2^a 3^b$ and let G be semi-n-abelian for all $n \in I$. G is solvable, and we may assume $a > 0$, $b > 0$. First we claim that G is nilpotent. To prove this we assume indirectly that all homomorphic images of G are nilpotent but G itself is not. Then by well-known results of Ore (see also Huppert (1967), Satz II.3.2 and Satz II.3.3) we have: there exists exactly one minimal normal subgroup N of G, with $o(N) = p^c$ and $C_G(N) = N$, and if U is a maximal, non-normal subgroup of G, then $G = N \cdot U$, $N \cap U = E$ and U does not possess any non-trivial normal subgroup of order p^d. In our case this last condition yields $o(N) = 2^a$ or $o(N) = 3^b$.

1) $o(N) = 2^a$. Choose $g \in G$ $(g \neq e)$ with $g^3 = e$ and $n \in I$ with $n \equiv 1 \pmod 3$ then we get using Lemma 4c of Kowol (1977)

$$(g^2h)^n = g^{n+1}h^n g^{n-1} = g^2h^n = g^{2n}h^n$$

and thus

$$(hg^2)^n = (g^{-2}g^2hg^2)^n = g^{-2}(g^2h)^n g^2 = h^n g^2 = h^n g^{2n}.$$

Now Baer (1951/52), p. 173, Folgerung 2 implies $g^{2n} = g^2 \in C_G(\langle G^{n-1} \rangle)$. Assume $G^{n-1} \neq E$; then we have $N \subseteq \langle G^{n-1} \rangle$ and therefore $g^2 \in C_G(N) = N$, but $g^3 = e$, $g \neq e$. Therefore it follows $\exp G | (n-1)$. If $n \in I$, then either $n+1 \in I$ or $n-2 \in I$, thus we have that G is semi-(± 2)-abelian, which by Lemma 3 of Kowol (1977) implies that G is semi-2-abelian, too. But Satz 7 of Kowol (1977) yields the nilpotence of G, contrary to the assumption.

2) $o(N) = 3^b$. In this case we have $o(U) = 2^a$ and $G/N \simeq U$, which implies U semi-n-abelian for all $n \in I$, too. Theorem 1 yields that U is abelian and therefore $G' \subseteq N$ and G' is nilpotent. Corollary 2 in Baer (1957), p. 159 gives $U/\mathrm{Core}_G\, U$ is cyclic and since N is the only minimal normal subgroup of G we get $\mathrm{Core}_G\, U = E$ $(o(\mathrm{Core}_G\, U) | 2^a)$ and thus U is cyclic itself. Assume that $\exp U > 2$. Then we choose an element $g \in U$, $g \neq e$, with $g^4 = e$ and $n \in I$ with $n \equiv 1 \pmod 4$. As in 1) above we obtain $g^2 \in C_G(N) = N$ or $\exp G | (n-1)$ which in both cases gives a contradiction.

Therefore we have $\exp U = 2$ and since U is cyclic we get $o(U) = 2$ and $o(G) = 2 \cdot 3^b$. According to Scott (1964), 7.2.15 G is supersolvable, in particular $o(N) = 3$ and $o(G) = 6$. Since G is not nilpotent $G \cong S_3$. But it is easy to see that S_3 never is semi-n-abelian for n even, thus we have a contradiction.

Having proved the nilpotency of G the further results in ii) now follow from Theorems 1 and 2.

ii) $\Rightarrow$ iii), iv) This follows immediately from Theorem 1, 2 and Bemerkung 2 in Kowol (1977), noting that direct products of semi-n-abelian groups are semi-n-abelian again.

iv) $\Rightarrow$ i) is trivial.

iii) $\Rightarrow$ ii) If G is semi-2-abelian, Satz 7 of Kowol (1977) implies the nilpotency of G and $c(G_p) \leqslant 2$ for $p > 3$. The remaining part of ii) follows from Theorems 1 and 2.

References

R. Baer (1957), "Classes of finite groups and their properties", *Illinois J. Math.* **1**, 115–187.

R. Baer (1951/52), "Endlichkeitskriterien für Kommutatorgruppen", *Math. Ann.* **124**, 161–177.

J. R. Durbin (1967), "Commutativity and n-abelian groups", *Math. Z.* **98**, 89–92.

Ch. K. Gupta (1965), "A bound for the class of certain nilpotent groups", *J. Austral. Math. Soc.* **5**, 506–511.

G. Higman (1959), "Some remarks on varieties of groups", *Quart. J. Math. Oxford* (2)10, 165–178.

B. Huppert (1967), *Endliche Gruppen I* (Springer, Berlin-Heidelberg-New York).

G. Kowol (1977), "Fast-n-abelsche Gruppen", *Arch. Math.* **29**, 55–66.

F. W. Levi and B. L. van der Waerden (1932), "Über eine besondere Klasse von Gruppen", *Abh. Math. Sem. Univ. Hamburg* **9**, 154–158.

A. Scholz and B. Schönberg (1973), "*Einführung in die Zahlentheorie*" (W. de Gruyter, Berlin-New York).

W. R. Scott (1964), *Group theory* (Prentice Hall, Englewood Cliffs).

Department of Mathematics
University of Vienna
1090 Vienna
Austria