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FINITE RANK OPERATORS AND FUNCTIONAL CALCULUS ON HILBERT MODULES OVER ABELIAN C^*-ALGEBRAS

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ABSTRACT. We consider the problem: If K is a compact normal operator on a Hilbert module E, and $f \in C_0(\operatorname{Sp} K)$ is a function which is zero in a neighbourhood of the origin, is $f(K)$ of finite rank? We show that this is the case if the underlying C^*-algebra is abelian, and that the range of $f(K)$ is contained in a finitely generated projective submodule of E.

1**. Introduction.** If we alter the definition of a Hilbert space by formally replacing the field of scalars by a C^*-algebra, we obtain a *Hilbert module*. It is possible to define generalizations of the compact operators and the finite-rank operators for Hilbert modules, but generally there is no satisfactory way to diagonalize a self-adjoint compact operator on a Hilbert module. Manuilov [Man] has recently shown that a generalized form of diagonalization is possible for strictly positive compact operators on the standard module over a small class of von Neumann algebras, but it is known that diagonalization fails even in the case of modules over abelian C^*-algebras.

In this paper we attempt to generalize the following fundamental result to the Hilbert module setting:

THEOREM 1. *If K is a compact normal operator on Hilbert space, and $f \in C_0(\mathbb{C})$ is a function which is zero in a neighbourhood of zero, then $f(K)$ is of finite rank.*

We show that the above statement holds if the Hilbert space is replaced by a Hilbert module over $C_0(X)$, and in fact we prove that $f(K)$ has image contained in a finitely generated module.

2**.** $C_0(X)$**-Hilbert modules.** The algebra of adjointable operators on a Hilbert module is labeled $\mathcal{L}$, the ideal of compact operators (generated by the finite-rank operators in the norm topology) is labeled $\mathcal{K}$. The algebra $C_b(X)$ is the algebra of bounded continuous functions over some locally compact Hausdorff space X, and $C_c(X)$ is the subalgebra of compactly supported functions.

Now, recall the definition and elementary properties of the *s-numbers* of a compact operator on Hilbert space [GK]:

DEFINITION 2. The *s-numbers* of a bounded compact operator $L \in \mathcal{K}_{\mathbb{C}}$ are

$$s_i(L) := \min\{\|L - P\| : \dim \operatorname{Im}_{\mathbb{C}} P < i\}.$$

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i) the s-numbers of any compact operator form a non-increasing sequence, converging to zero.

ii) the dimension of the range of an operator $L \in K_{\mathbb{C}}$ is given by the cardinality of the set of nonzero s-numbers of L.

iii) each s-number is a continuous function from $K_{\mathbb{C}}$ to $\mathbb{R}$.

iv) if $L \in K_{\mathbb{C}}$ is normal, the set of s-numbers coincides with the set $\{|\lambda_i| : \lambda_i \in \operatorname{sp} L\}$.

For convenience in exposition, we give a special name to the continuous functions $f: \mathbb{C} \longrightarrow \mathbb{R}$ that are radially symmetrical, nondecreasing in the radial direction, and zero in a neighbourhood of zero.

DEFINITION 3. A *cone function* $c_t: \mathbb{C} \longrightarrow \mathbb{R}$ is of the form $c_t(z) = f(|z|)$, where $f: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous non-decreasing function with zero set $[0, t]$, $t > 0$.

REMARK. Given a cone function c_t and a normal compact operator K, it is a consequence of i) and iv) that $s_i\big(c_t(K)\big) = c_t\big(s_i(K)\big)$.

We may now state the main result concerning cone functions and families of compact operators:

PROPOSITION 4. *Let $c_t: \mathbb{C} \longrightarrow \mathbb{C}$ be a cone function. If $K: X \longrightarrow K_{\mathbb{C}}$ is normal at every point, norm-continuous, and decays to zero at infinity, then the range of $c_t(K)$ is contained in an algebraically finitely generated submodule of $H_{C_0(X)}$.*

PROOF. Let c_r be a cone function such that $c_t \circ c_r = c_t$. We first show that only finitely many of the s-numbers of $c_r(K): X \longrightarrow K_{\mathbb{C}}$ are not identically zero. Let $\{s_i: X \longrightarrow \mathbb{R}\}_{i\in\mathbb{N}}$ be the s-numbers of $K: X \longrightarrow K_{\mathbb{C}}$, and let $Y \subset X$ be a compact set such that $\|K(x)\| < r$ outside Y, so $c_r(K)$ is supported within Y. It is sufficient to show that there are only finitely many $i \in \mathbb{N}$ such that $s_i(y) \geq r$ for some $y \in Y$. Let $P_i := \{y \in Y : s_i(y) \geq r\}$, and let $\{P_{n_k}\}$ be the set of nonempty P_i. Since $s_i(y)$ is a non-increasing sequence with respect to i, $P_{i+1} \subseteq P_i$, and any finite intersection of the closed sets $\{P_{n_k}\}$ is nonempty. The finite intersection property of compact spaces shows that $\bigcap P_{n_k} \neq \emptyset$, so there is a point $y_0 \in Y$ which is covered by all the nonempty P_i. On the other hand, since the sequence $s_i(y_0)$ converges to zero, there can be only finitely many P_i covering the point y_0. We conclude that $c_r(K): X \longrightarrow K_{\mathbb{C}}$ has only finitely many s-numbers not identically zero, so is of uniformly finite rank.

Let F denote $c_r(K): X \longrightarrow K_{\mathbb{C}}$. Since F is compactly supported and $c_t(F) = c_t(K)$, we may establish the lemma by proving that every point $y \in Y$ has an open neighbourhood G such that the image of $c_t(F)$ restricted to the standard module over G is algebraically finitely generated, which we do by finding a compact projection P which is the identity on the image of $c_t(F)$ over G.

Assign $y \in Y$ and let $\{q_i: X \longrightarrow \mathbb{R}\}_1^N$ be the set of nonzero s-numbers of F. The functions $\{q_i\}_1^N$ can be divided into two disjoint sets, according to whether or not $q_i(y) = 0$. Since $\big(q_i(y)\big)$ is a non-increasing sequence, there is an M such that $\{q_i\}_1^M = \{q_i : q_i(y) \neq 0\}$ and $\{q_i\}_{M+1}^N = \{q_i : q_i(y) = 0\}$. Let the zero set of c_t be $[0, t]$. Both sets of q_i's are finite, so there is an open neighbourhood G of y and an open interval $I = (\delta, \delta')$ with $\delta' \leq t$,

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such that $s_i(x) \notin I$ for any $x \in G$ and $i \in 1 \cdots N$. Because of this gap in the spectrum we can define a compact projection P (over G) by $c_\delta(F)$ where c_δ is equal to zero on $(0, \delta)$ and one on (δ', ∞). Then $Pc_t(F) = c_t(F) = c_t(K)$, and the image of $c_t(K)$ over $H_{C(G)}$ is algebraically finitely generated. By the compactness of Y, it is sufficient to consider finitely many G_i, and then the generators over each G_i can be extended (by zero) to X by multiplying with the appropriate element of a subordinate partition of unity. ∎

The standard Hilbert module over $C_0(X)$ has an alternative description in terms of families of operators on ℓ^2:

PROPOSITION 5. *The standard Hilbert $C_0(X)$-module $H_{C_0(X)}$ can be identified with the space of norm-continuous functions $C_0(X, \ell^2)$. The bounded adjointable operators $\mathcal{L}_{C_0(X)}$ on E can be identified with the algebra of norm-bounded strong-$*$ continuous functions $C_{qb}(X, \mathcal{L}_{\mathbb{C}})$, and the compact operators $\mathcal{K}_{C_0(X)}$ on E can be identified with the algebra of norm-continuous functions $C_0(X, \mathcal{K}_{\mathbb{C}})$.*

In other words, this module is given by the sections of the trivial bundle $X \times \ell^2$. Because of Kuiper's theorem on the contractability of the unitary operators, there is little reason to consider nontrivial bundles with fibre ℓ^2 and base X. For a proof of the theorem, see [APT], also see [Skan].

The last two propositions can be combined to prove the following result:

THEOREM 6. *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a continuous function that is zero on some neighbourhood of zero, then f maps the normal compact operators to the compactly supported normal finite-rank operators.*

PROOF. Choose a normal compact operator K. The given function f can be written as a product with a cone function c_t, so that $fc_t = f$, and it is sufficient to show that $c_t(K)$ is finite-rank.

Proposition 5 implies that the family defined by K, $\boldsymbol{\pi}K$ is in $C_0(X, \mathcal{K}_{\mathbb{C}})$, and satisfies the hypothesis of Proposition 4. So the image E of the self-adjoint compact operator $c_t(K)$ is an algebraically finitely generated module, over $C_0(X)$. But since $c_t(K)$ will be zero outside a compact set, we may as well assume that $C_0(X)$ is unital. It is known [Wegg, 15.4.8] that an algebraically finitely generated Hilbert module over a unital C^*-algebra A is isomorphic to a complemented submodule of A^n, and of course the identity map on A^n is finite-rank, so the identity map on E extends to a (compactly supported) finite-rank operator on $H_{C_0(X)}$. Thus the composition with $c_t(K)$ is finite-rank, and $c_t(K)$ is finite-rank, as was to be shown. ∎

One might hope that the above result could be generalized to "pointwise finite-rank" operators, as follows:

CONJECTURE 7. *If X is a compact Hausdorff space, and K is a norm-continuous family of compact operators over X, then the rank of $K(x)$ is uniformly bounded if and only if K is of finite rank as an operator on the Hilbert module $H_{C(X)}$.*

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However, this conjecture is false, and we give an example of a family of compact operators which are of finite rank at every point but do not give a finite-rank operator on the relevant Hilbert module. We need a lemma about the structure of complex projective spaces.

LEMMA 8. *Let L_n be the standard line bundle over $\mathbb{C}P(n)$. If K is a compact operator of rank n when regarded as a Hilbert module operator on $\Gamma(L_n)$ over $C\big(\mathbb{C}P(n)\big)$, there is a point at which $K(x)$ is zero.*

We give a topological proof.

PROOF. The given operator K can be written as a finite sum

$$\sum_{1}^{n} p_k \langle q_k, \cdot \rangle$$

where the p_k are continuous sections of L_n. Let E be the vector bundle given by the direct sum of n copies of L_n. Then $p_1 \oplus \cdots \oplus p_n$ is a section of E, and we show that the top Chern class of E is nonzero, implying that every section of E must vanish somewhere, and hence that there is a point at which all the p_k vanish simultaneously.

By the Whitney product formula, the top Chern class of E is just $c_1(L_n)^n$. But it is well-known that $-c_1(L_n) \in H^2\big(\mathbb{C}P(n)\big)$ generates the cohomology of $\mathbb{C}P(n)$, and that $H^{2n}\big(\mathbb{C}P(n)\big)$ is nontrivial (*e.g.* see [BT]). Hence $c_1(L_n)^n$ cannot be zero. ∎

Now, we define a Hilbert module H over the one-point compactification of the disjoint union

$$X := \coprod_{1}^{\infty} \mathbb{C}P(n)$$

by the direct sum of the spaces of sections of the line bundles L_n. This is a countably generated Hilbert module, and hence [Kas1] it is isomorphic to a complemented submodule of the standard Hilbert module over $C(X^+)$. If we define an operator K that acts on $\Gamma(L_n)$ by scalar multiplication with $1/n$, and is zero otherwise, then K gives a family of rank 1 operators over X^+. Since $K(x)$ does not vanish at any point in any $\mathbb{C}P(n)$, the lemma shows that K cannot be of finite rank as a Hilbert module operator, but by Proposition 5, K is a compact Hilbert module operator. Thus we have a counterexample to the conjecture, showing that Theorem 6 is, in a sense, the best we can hope for.

If Theorem 4 is combined with Proposition 5, we have the desired result:

COROLLARY 9. *Let A be an abelian C^*-algebra, let H be a Hilbert A-module, and let K be any compact normal operator on H. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a continuous function that is zero on some neighbourhood of zero, then $f(K)$ and $f(K)^*$ map H into a algebraically finitely generated submodule.*

Corollary 9 may be false for Hilbert modules over more general algebras.

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