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Property T and Amenable Transformation Group C^*-algebras

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Abstract. It is well known that a discrete group that is both amenable and has Kazhdan's Property T must be finite. In this note we generalize this statement to the case of transformation groups. We show that if G is a discrete amenable group acting on a compact Hausdorff space X, then the transformation group C^*-algebra $C^*(X,G)$ has Property T if and only if both X and G are finite. Our approach does not rely on the use of tracial states on $C^*(X,G)$.

1 Introduction

Property T for groups was first introduced by Kazhdan in [4]. It has since been found to be extremely useful in many other contexts. In particular, Bekka introduced Property T to C^*-algebras in [2]. Property T for groups represents the opposite of amenability. A discrete group that is both amenable and has Property T must be finite. Replacing amenability with nuclearity, one would expect a similar relationship to exist in the context of C^*-algebras. In fact, the relationship between nuclearity and Property T has been studied in [1–3, 5]. Brown showed in [3] that a nuclear C^*-algebra with a faithful tracial state must be finite dimensional. We also know that if φ is a tracial state on a nuclear C^*-algebra A, then the corresponding GNS representation of A on the Hilbert space $\mathcal{H}_\varphi$ must be completely atomic [2].

The aim of this note is to show that if G is a discrete amenable group acting on a compact Hausdorff space X, then the transformation group C^*-algebra $C^*(X,G)$ has Property T if and only if both X and G are finite. The main result is given in Proposition 2.5. The existing literature on C^*-algebras and Property T is based on the use of tracial states. In contrast, our approach completely avoids the use of tracial states and the corresponding high powered machinery. In fact, the existence of tracial states on $C^*(X,G)$ will follow from our main result. We note that while tracial states on $C^*(X,G)$ always exist, the existence of a faithful tracial state is not necessarily guaranteed (see Remark 2.6).

2 Background and Results

Suppose that X is a compact Hausdorff space and that G is a discrete group acting on X. We denote the action of $t \in G$ on $x \in X$ by xt. The C^*-algebra associated with the pair (X,G) is called the transformation group C^*-algebra and is denoted

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by $C^*(X,G)$. Recall that $C^*(X,G)$ is the norm closure of the linear span of the set $\{fV_s : f \in C(X), s \in G\}$.

Definition 2.1 A covariant representation of (X,G) consists of a pair (π,U), where $\pi\colon C(X) \to \mathcal{B}(\mathcal{H})$ is a C^*-representation and $U\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary representation such that

$$U(t)\pi(f)U(t^{-1}) = \pi(f_t),$$

for all $f \in C(X)$ and $t \in G$, where $f_t(x) = f(xt)$.

Given a covariant representation (π,U) of (X,G) we can construct a corresponding representation $\pi \times U$ of $C^*(X,G)$ by $(\pi \times U)(\sum f_j V_{s_j}) = \sum \pi(f_j)U(s_j)$.

We define a Hilbert bimodule over a C^*-algebra A to be a Hilbert space $\mathcal{H}$ carrying a pair of commuting representations, one of A and one of its opposite algebra A^{op}. We denote the action by

$$\xi \mapsto a\xi b$$

for all $\xi \in \mathcal{H}$, $a \in A$, and $b \in A^{\mathrm{op}}$.

Definition 2.2 A C^*-algebra A has Property T if every bimodule with almost central vectors has a central vector; *i.e.*, if $\mathcal{H}$ is a bimodule and there exist unit vectors $\xi_n \in \mathcal{H}$ such that $\|a\xi_n - \xi_n a\| \to 0$ for all $a \in A$, then there exists a unit vector $\xi \in \mathcal{H}$ such that $a\xi = \xi a$ for all $a \in A$.

Theorem 2.3 *Suppose that G is a discrete amenable group acting on a compact Hausdorff space X. If the transformation group C^*-algebra $C^*(X,G)$ has Property T, then G is finite.*

Proof Let $x_0 \in X$. Define a left covariant representation of (X,G) on $L^2(G \times G)$ by

$$\big(\pi(f)\zeta\big)(s,t) = f(x_0 t)\zeta(s,t) \quad \text{and} \quad \big(U(r)\zeta\big)(s,t) = \zeta(s,tr)$$

for all $\zeta \in L^2(G\times G)$, $f \in C(X)$, and $s,t,r \in G$. Similarly, define a right covariant representation on $L^2(G \times G)$ by

$$\big(\zeta\pi(f)\big)(s,t) = f(x_0 s)\zeta(s,t) \quad \text{and} \quad \big(\zeta U(r)\big)(s,t) = \zeta(sr^{-1},t).$$

Then the Hilbert space $L^2(G \times G)$ is a $C^*(X,G)$-bimodule for the action

$$\zeta \mapsto (\pi \times U)(a)\zeta(\pi \times U)(b)$$

for all $\zeta \in L^2(G\times G)$ and $a,b \in C^*(X,G)$. Let λ denote the right regular representation of G on $L^2(G)$. Since G is amenable, there exists a sequence of vectors $\xi_n \in L^2(G)$ such that $\|\lambda(r)\xi_n - \xi_n\| \to 0$ for all $r \in G$. Let $\triangle = \{(s,s) : s \in G\}$ be the diagonal of $G \times G$. Define a sequence of unit vectors $\zeta_n \in L^2(G\times G)$ by $\zeta_n(s,t) = \chi_\triangle(s,t)\xi_n(s)$ for all $s,t \in G$. Given $f \in C(X)$, $r \in G$, and ζ_n, we have

$$\big(\pi(f)U(r)\zeta_n\big)(s,t) = \begin{cases} f(x_0 s r^{-1})\xi_n(s), & s = tr,\\ 0, & s \neq tr,\end{cases}$$

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and

$$\big(\zeta_n\pi(f)U(r)\big)(s,t)=\begin{cases} f(x_0sr^{-1})\xi_n(sr^{-1}), & s=tr,\\ 0, & s\neq tr,\end{cases}$$

for all $s,t\in G$. Then

$$\begin{aligned}\|\pi(f)U(r)\zeta_n-\zeta_n\pi(f)U(r)\|^2&=\sum_s|f(x_0sr^{-1})\xi_n(s)-f(x_0sr^{-1})\xi_n(sr^{-1})|^2\\ &\leq\sum_s M^2|\xi_n(s)-\xi_n(sr^{-1})|^2\\ &=M^2\|\lambda(r^{-1})\xi_n-\xi_n\|^2,\end{aligned}$$

where $M=\sup_x|f(x)|$. It follows that ζ_n's are almost central vectors. Since $C^*(X,G)$ has Property T, there exists a unit vector $\zeta\in L^2(G\times G)$ that is a $C^*(X,G)$-central vector. In particular, $(U(r)\zeta)(s,t)=(\zeta U(r))(s,t)$ for all $s,t,r\in G$. Then $\zeta(s,t)=\zeta(sr,tr)$ for all $s,t,r\in G$. Let $s_0,t_0\in G$ such that $\zeta(s_0,t_0)\neq 0$. Then we have

$$\|\zeta\|^2=\sum_{s,t}|\zeta(s,t)|^2\geq\sum_r|\zeta(s_0r,t_0r)|^2=\sum_r|\zeta(s_0,t_0)|^2.$$

It follows that $|G|\leq\frac{1}{|\zeta(s_0,t_0)|^2}$. ∎

Let μ be a Borel measure on X and let $\mathcal{H}_\mu=L^2(X,\mu)\otimes L^2(G)$. Define a left covariant representation (π,U) of (X,G) on the space $\mathcal{H}_\mu\otimes\mathcal{H}_\mu$ by

$$\big(\pi(f)\zeta\big)((x,s),(y,t))=f(yt)\zeta\big((x,s),(y,t)\big)$$

and

$$\big(U(r)\zeta\big)\big((x,s),(y,t)\big)=\zeta\big((x,s),(y,tr)\big)$$

for all $\zeta\in\mathcal{H}_\mu\otimes\mathcal{H}_\mu$, $f\in C(X)$, $s,t,r\in G$, and $x,y\in X$. Similarly, define a right covariant representation on $\mathcal{H}_\mu\otimes\mathcal{H}_\mu$ by

$$\big(\zeta\pi(f))((x,s),(y,t)\big)=f(xs)\zeta\big((x,s),(y,t)\big)$$

and

$$\big(\zeta U(r))((x,s),(y,t)\big)=\zeta\big((x,sr^{-1}),(y,t)\big).$$

Then $\mathcal{H}_\mu\otimes\mathcal{H}_\mu$ is a $C^*(X,G)$-bimodule. We say that $\mathcal{H}_\mu\otimes\mathcal{H}_\mu$ is the bimodule induced from the measure μ.

Lemma 2.4 *Let G be a finite group acting on a second countable compact Hausdorff space X. Suppose that μ is a finite Borel measure on X. Then the induced bimodule $\mathcal{H}_\mu\otimes\mathcal{H}_\mu$ has almost central unit vectors.*

Proof Since X is compact, there exists a point $x_0\in X$ such that $\mu(W)>0$ for all open sets containing x_0. Let d be the metric on X and let $B(x,\delta)$ denote the open ball with radius $\delta>0$ centred at $x\in X$. For each $n\geq 1$, define $E_n=\bigcap_{r\in G}B(x_0r,1/n)r^{-1}$. Let $f\in C(X)$ and $\epsilon>0$ be given. Since f is uniformly continuous on X, there exists $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ for all $x,y\in X$ with $d(x,y)<\delta$. It follows that

$$|f(xr)-f(yr)|<\epsilon \tag{2.1}$$

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for all $x, y \in E_n$, $r \in G$, $n > \frac{2}{\delta}$. Let $\triangle = \{(s, s) : s \in G\}$ be the diagonal of $G \times G$. Define a sequence of unit vectors in $\mathcal{H}_\mu \times \mathcal{H}_\mu$ by

$$\zeta_n\big((x, s), (y, t)\big) = \frac{1}{|G|^{1/2}\mu(E_n)} \chi_\triangle(s, t)\chi_{E_n \otimes E_n}(x, y).$$

Then for each $n \geq 1$,

$$\|\pi(f)U(r)\zeta_n - \zeta_n\pi(f)U(r)\| < M,$$

where $M = \sup\{|f(yt) - f(xt)| : x, y \in E_n, t \in G\}$. It follows from equation (2.1) that $\|\pi(f)U(r)\zeta_n - \zeta_n\pi(f)U(r)\| \to 0$ for all $f \in C(X)$, $r \in G$. ∎

Proposition 2.5 *Suppose that G is a discrete amenable group acting on a second countable compact Hausdorff space X. Then the transformation group C^*-algebra $C^*(X, G)$ has Property T if and only if both G and X are finite.*

Proof Suppose that G and X are finite. Then $C^*(X, G)$ is a finite dimensional C^*-algebra and has Property T. Now assume that $C^*(X, G)$ has Property T. It follows from Theorem 2.3 that G must be finite. Suppose for contradiction that X is infinite. Let μ be a nonatomic measure on X and let $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ be the $C^*(X, G)$-bimodule induced from μ. Since $C^*(X, G)$ has Property T, by Lemma 2.4, there exists a $C^*(X, G)$-central unit vector $\xi \in \mathcal{H}_\mu \otimes \mathcal{H}_\mu$. Then for every $f \in C(X)$ and $s, t \in G$ we have

$$f(yt)\xi\big((x, s), (y, t)\big) = f(xs)\xi\big((x, s), (y, t)\big) \tag{2.2}$$

for almost all $x, y \in X$. Let $s_0, t_0 \in G$ such that $\|\xi((\,\cdot\,, s_0), (\,\cdot\,, t_0))\|_{L^2(X \times X)}$ is nonzero. It follows from equation (2.2) that the measure of the set $\{(x, xs_0t_0^{-1}) : x \in X\} \subseteq X \times X$ must be nonzero. Since μ is a nonatomic measure this is a contradiction. ∎

Tracial states on $C^*(X, G)$ correspond to G-invariant probability measures on X. It is known that if G is an amenable group acting on a compact Hausdorff space X, then there exists a G-invariant probability measure on X. We can use Theorem 2.3 to give an alternative proof of the above fact. Indeed, if G is finite, then it is clear that $C^*(X, G)$ has a tracial state. Suppose that G is infinite. Then, by Theorem 2.3, $C^*(X, G)$ does not have Property T. On the other hand, it was shown in [2] that every C^*-algebra without a tracial state must have Property T. It follows that $C^*(X, G)$ has a tracial state.

Remark 2.6 Although G-invariant probability measures on X always exist, an invariant probability measure with full support is not guaranteed. Let $\mathbb{Z}$ be the group of integers and let X be the interval $[0, 1]$. Define the action of $\mathbb{Z}$ on X via the homeomorphism $\sigma(x) = x^2$. Suppose that μ is a G-invariant probability measure on X. Then $\mu([0.5, 1]) = \mu(\sigma([0.5, 1])) = \mu([0.25, 1])$. It follows that $\mu((0.25, 0.5)) = 0$.

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