J. Austral. Math. Soc. (Series A) 52 (1992), 261–284

OSCILLATIONS OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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(Received 21 MAY 1990)

Communicated by A. J. Pryde

Abstract

Consider the nth-order neutral differential equation

$$\text{(E)} \qquad \frac{d^n}{dt^n}[x(t) + \sum_{\mathscr{I}} p_i x(t-\tau_i)] + \delta \sum_{\mathscr{K}} q_k x(t-\sigma_k) = 0$$

where $n \geq 1$, $\delta = \pm 1$, $\mathscr{I}$, $\mathscr{K}$ are initial segments of natural numbers, p_i, τ_i, $\sigma_k \in \mathbb{R}$ and $q_k \geq 0$ for $i \in \mathscr{I}$ and $k \in \mathscr{K}$. Then a necessary and sufficient condition for the oscillation of all solutions of (E) is that its characteristic equation

$$\lambda^n + \lambda^n \sum_{\mathscr{I}} p_i e^{-\lambda\tau_i} + \delta \sum_{\mathscr{K}} q_k e^{-\lambda\sigma_k} = 0$$

has no real roots. The method of proof has the advantage that it results in easily verifiable sufficient conditions (in terms of the coefficients and the arguments only) for the oscillation of all solutions of Equation (E).

1991 *Mathematics subject classification* (*Amer. Math. Soc.*) primary 34 K 40; secondary 34 K 15, 34 C 10.

1. Introduction

Consider the nth-order neutral differential equation

$$\text{(E)} \qquad \frac{d^n}{dt^n}\left[x(t) + \sum_{\mathscr{I}} p_i x(t-\tau_i)\right] + \delta \sum_{\mathscr{K}} q_k x(t-\sigma_k) = 0$$

where $n \geq 1$, $\delta = \pm 1$, $\mathcal{J}$, $\mathcal{K}$ are initial segments of natural numbers, p_i, τ_i, $\sigma_k \in \mathbb{R}$ and $q_k \geq 0$ for $i \in \mathcal{J}$ and $k \in \mathcal{K}$. In the case where $\mathcal{J} = \varnothing$ Equation (E) reduces to the (nonneutral) equation

$$(\mathrm{E}_1) \qquad x^{(n)}(t) + \delta \sum_{\mathcal{K}} q_k x(t - \sigma_k) = 0,$$

while when $\mathcal{K} = \varnothing$ Equation (E) yields

$$(\mathrm{E}_2) \qquad \frac{d^n}{dt^n}\left[x(t) + \sum_{\mathcal{J}} p_i x(t - \tau_i)\right] = 0,$$

which admits a (nonoscillatory) solution of a polynomial form. Thus we assume that $\mathcal{K} \neq \varnothing$. When $p_i > 0$ or $p_i < 0$ for $i \in \mathcal{J}$ Equation (E) leads, respectively, to

$$\frac{d^n}{dt^n}\left[x(t) + \sum_{\mathcal{J}} p_i x(t - \tau_i)\right] + \delta \sum_{\mathcal{K}} q_k x(t - \sigma_k) = 0,$$

or

$$\frac{d^n}{dt^n}\left[x(t) - \sum_{\mathcal{J}} p_i (t - \tau_i)\right] + \delta \sum_{\mathcal{K}} q_k x(t - \sigma_k) = 0,$$

while in all other cases Equation (E) can be written in the form

$$(\mathrm{E}_3) \quad \frac{d^n}{dt^n}\left[x(t) + \sum_{I} p_i x(t - \tau_i) - \sum_{J} r_j x(t - \rho_j)\right] + \delta \sum_{\mathcal{K}} q_k x(t - \sigma_k) = 0,$$

where $p_i > 0$ and $r_j > 0$ for $i \in I$, $j \in J$. Observe that the former two equations are special cases of the latter one and therefore it suffices to study Equation (E_3).

It is easy to see (cf. [9, 11]) that in the case where

$$I_1 = \{i \in I : \tau_i < 0\} \subseteq I, \qquad J_1 = \{j \in J : \rho_j < 0\} \subseteq J$$

are nonempty, by taking

$$\tau = \max_{i \in I_1} |\tau_i| \quad \text{and} \quad \rho = \max_{j \in J_1} |\rho_j|$$

Equation (3) leads to an equation of the same form with $\tau_i > 0$ and $\rho_j > 0$ for $i \in I$ and $j \in J$. So in the sequel we will assume $\tau_i > 0$ and $\rho_j > 0$ for $i \in I$ and $j \in J$. Finally, because $\sigma_k \in \mathbb{R}$, (E_3) can be written in the following form

$$(1) \qquad \frac{d^n}{dt^n}\left[x(t) + \sum_{I} p_i x(t - \tau_i) - \sum_{J} r_j x(t - \rho_j)\right] + \delta\left(\sum_{K_1} q_k x(t - \sigma_k) + \sum_{K_2} \hat{q}_k x(t + \hat{\sigma}_k)\right) = 0$$

where I, J, K_1, K_2 are initial segments of natural numbers, p_i, τ_i, r_j, ρ_j, q_k, $\hat{q}_k \in (0, \infty)$ and σ_k, $\hat{\sigma}_k \in [0, \infty)$ for $i \in I$, $j \in J$, $k \in K_1 \cup K_2$. Note that when $\delta = -1$ and $K_2 = \varnothing$ Equation (1) admits a nonoscillatory solution so we exclude this case.

Our aim in this paper is to obtain a necessary and sufficient condition under which all solutions of Equation (1) oscillate. Indeed, we prove that every solution of Equation (1) oscillates if and only if its characteristic equation (2)

$$F(\lambda) \equiv \lambda^n + \lambda^n \sum_I p_i e^{-\lambda\tau_i} - \lambda^n \sum_J r_j e^{-\lambda\rho_j} + \delta \left(\sum_{K_1} q_k e^{-\lambda\sigma_k} + \sum_{K_2} q_k e^{\lambda\sigma_k} \right) = 0$$

has no real roots. That is, the oscillatory character of the solutions is determined by the roots of the characteristic equation. This is in contrast with the fact that the stability character is not determined by the characteristic roots. Some of these differences as well as some applications of neutral differential equations are discussed in [2, 3, 4, 5, 6, 14, 15, 23, 24]. Especially, higher order neutral differential equations were encountered in the study of vibrating masses attached to an elastic bar and also as the Euler equations in some variational problems (see Hale [15, p. 7]).

Necessary and sufficient conditions (in terms of the characteristic equation) for the oscillation of all solutions of first order neutral differential equations have been obtained by Sficas and Stavroulakis [22], Grove, Ladas and Meimaridou [13], Kulenović, Ladas and Meimaridou [16], Grammatikopoulos, Sficas and Stavroulakis [10], Farrell [7], and Grammatikopoulos and Stavroulakis [11, 12]. Necessary and sufficient conditions for the oscillation of higher order equations have been obtained by Ladas, Sficas and Stavroulakis [18], Ladas, Partheniadis and Sficas [17], and Wang [25]. See also Arino and Győri [1].

It is to be emphasized that in all the above mentioned papers $\sigma_k \in \mathbb{R}^+$ while here $\sigma_k \in \mathbb{R}$. To the best of the authors' knowledge this is the only paper at the present time dealing with the oscillation of all solutions of Equation (E) where $\sigma_k \in \mathbb{R}$ for $k \in \mathscr{K}$.

Let $T = \max_{i,j,k}\{\tau_i, \rho_j, .\sigma_k, \hat{\sigma}_k\}$. By a solution of Equation (1) we mean a function $x \in C([t_0, -T, \infty), \mathbb{R})$ for some $t_0 \in \mathbb{R}$, such that

$$x(t) + \sum_I p_i x(t - \tau_i) - \sum_J r_j x(t - \rho_j)$$

is n-times continuously differentiable on $[t_0, \infty)$ and such that Equation (1) is satisfied for $t \geq t_0$.

As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory*, that is, if it is eventually positive or eventually negative.

In the sequel all functional inequalities that we write are assumed to hold eventually, that is for sufficiently large t.

In the case where I, J, K_1, K_2 are nonempty we can assume, without loss of generality, that

$$I = \{1, 2, \dots, l\}, \quad J = \{1, 2, \dots, m\},$$
$$K_1 = \{1, 2, \dots, n_1\}, \quad K_2 = \{1, 2, \dots, n_2\},$$
$$0 < \tau_1 < \cdots < \tau_l, \; 0 < \rho_1 < \cdots < \rho_m, \; 0 \le \sigma_1 < \cdots < \sigma_{n_1}, \; 0 \le \hat{\sigma}_1 < \cdots < \hat{\sigma}_{n_2}$$

and

$$\tau_i \neq \rho_j, \qquad i \in I, \; j \in J,$$

since otherwise the terms in Equation (1) can be abbreviated. Also for convenience we use the following notations

$$P = \sum_I p_i, \qquad R = \sum_J r_j, \qquad Q_1 = \sum_{K_1} q_k, \qquad Q_2 = \sum_{K_2} \hat{q}_k$$

and

$$Q = Q_1 + Q_2.$$

Note that, since $\delta = \pm 1$ and n is an odd or an even number, $(-1)^{n-1}\delta = \pm 1$. Thus, we consider Equation (1) in the following cases:

(i) $(-1)^{n-1}\delta = +1$ ($\delta = +1$, n odd or $\delta = -1$, n even);
(ii) $(-1)^{n-1}\delta = -1$ ($\delta = +1$, n even or $\delta = -1$, n odd).

2. Preliminaries

In this section we establish some useful lemmas which will be used in the proof of our main theorem (cf. [10, 11, 12]).

LEMMA 1. *Consider Equation* (1). *Then the following inequalities*

$$\max\{\rho_m, \sigma_{n_1}\} > \tau_l \quad \textit{in case (i)}, \tag{3}$$

$$\max\{\tau_l, \sigma_{n_1}\} > \rho_m \quad \textit{in case (ii)}, \tag{4}$$

and

$$m = \min_{\lambda \in \mathbf{R}} \delta F(\lambda) > 0 \tag{5}$$

are necessary conditions for the characteristic equation (2) *to have no real roots.*

PROOF. As $\delta F(0) = Q > 0$ and Equation (2) has no real roots it follows that $\delta F(\lambda) > 0$ for $\lambda \in \mathbb{R}$. Also observe that $\delta F(+\infty) = +\infty$. Thus, $\delta F(-\infty)$ must be positive or $+\infty$. But when $\max\{\rho_m, \rho_{n_1}\} \le \tau_l$ in case (i) and $\max\{\tau_l, \sigma_{n_1}\} \le \rho_m$ in case (ii) $\delta F(-\infty) = -\infty$, that is, Equation (2) has a real root. This is impossible and thus conditions (3) and (4) must hold. Finally, since Equation (2) has not real roots and $\delta F(-\infty) = \delta F(+\infty) = +\infty$ it follows that condition (5) holds. The proof of the lemma is complete.

From (5) it follows that for all $\lambda \in \mathbb{R}$

$$\delta\left(\lambda^n + \lambda^n \sum_I p_i e^{-\lambda\tau_i} - \lambda^n \sum_J r_j e^{-\lambda\rho_j}\right) + \sum_{K_1} q_k e^{-\lambda\sigma_k} + \sum_{K_2} q_k e^{\lambda\hat{\sigma}_k} \ge m,$$

which is equivalent to

$$(6)\quad \begin{cases} \lambda^n + \lambda^n \sum_I p_i e^{\lambda\tau_i} - \lambda^n \sum_J r_j e^{\lambda\rho_j} - \sum_{K_1} q_k e^{\lambda\sigma_k} - \sum_{K_2} \hat{q}_k^{-\lambda\hat{\sigma}_k} \le -m \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in case (i)} \\ -\lambda^n - \lambda^n \sum_I p_i e^{\lambda\tau_i} + \lambda^n \sum_J r_j e^{\lambda\rho_j} - \sum_{K_1} q_k e^{k\sigma_k} - \sum_{K_2} \hat{q}_k e^{-\lambda\hat{\sigma}_k} \le -m \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in case (ii)} \end{cases}$$

and to

$$(7)\quad \begin{cases} -\lambda^n - \lambda^n \sum_I p_i e^{-\lambda\tau_i} + \lambda^n \sum_J r_j e^{-\lambda\rho_j} - \sum_{K_1} q_k e^{-\lambda\sigma_k} - \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k} \le -m \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } \delta = +1 \\ \lambda^n + \lambda^n \sum_I p_i e^{-\lambda\tau_i} - \lambda^n \sum_J r_j e^{-\lambda\rho_j} - \sum_{K_1} q_k e^{-\lambda\sigma_k} - \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k} \le -m \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } \delta = -1. \end{cases}$$

LEMMA 2. *Let* $x(t)$ *be a solution of Equation* (1) *and let* a, b *and* c *be real numbers. Then each one of the functions*

$$x(t-a),\ \int_{t-b}^{t-c} x(s)\,ds, \quad \text{and} \quad \int_t^\infty x(s)\,ds$$

$$\text{for } x(t) \in L^1[t_0, \infty) \text{ and } \lim_{t\to\infty} x(t) = 0$$

is a solution of (1).

https://doi.org/10.1017/S1446788700034406 Published online by Cambridge University Press

PROOF. The conclusion follows easily and it is a consequence of the linearity of Equation (1) and its autonomous nature.

DEFINITION. Consider Equation (1). Then the set of all solutions of Equation (1) at least μ-times $(\mu \geq n)$ continuously differentiable and such that

$$(-1)^{\nu} w^{(\nu)}(t) > 0, \quad \nu = 0, 1, \ldots, \mu$$

and

$$\lim_{t\to\infty} w^{(\nu)}(t) = 0, \nu = 0, 1, \ldots, \mu - 1$$

is called Class I_{μ}, while the set of all solutions of Equation (1) at least μ-times continuously differentiable and such that

$$w^{(\nu)}(t) > 0, \ \nu = 0, 1, \ldots, \mu \text{ and } \lim_{t\to\infty} w^{(\nu)}(t) = +\infty, \ \nu = 0, 1, \ldots, \mu - 1$$

is called Class II_{μ}.

LEMMA 3. *Let* $x(t)$ *be a nonoscillatory solution of Equation* (1). *Then Equation* (1) *also has a nonoscillatory solution* $w(t)$ *such that either* $w(t) \in$ Class I_{2n} *or* $w(t) \in$ Class II_{2n}.

PROOF. Without loss of generality $x(t)$ can be considered eventually positive. Set

$$z(t) = \delta\left[x(t) + \sum_I p_i x(t-\tau_i) - \sum_J r_j x(t-\rho_j)\right]$$

and

$$w(t) = \delta\left[z(t) + \sum_I p_i z(t-\tau_i) - \sum_J r_j z(t-\rho_j)\right].$$

Then, by Lemma 2, $z(t)$ and $w(t)$ are both solutions of Equation (1) such that

$$z^{n}(t) = -\left(\sum_{K_1} q_k x(t-\sigma_k) + \sum_{K_2} \hat{q}_k x(t+\hat{\sigma}_k)\right) < 0, \tag{8}$$

$$w^{(n)}(t) = -\left(\sum_{K_1} q_k z(t-\sigma_k) + \sum_{K_2} \hat{q}_k z(t+\hat{\sigma}_k)\right) \tag{9}$$

and

$$w^{(2n)}(t) = -\left(\sum_{K_1} q_k z^{(n)}(t-\sigma_k) + \sum_{K_2} \hat{q}_k z^{(n)}(t+\hat{\sigma}_k)\right) > 0. \tag{10}$$

Thus, $z \in C^n[T_0, \infty)$, $w \in C^{2n}[t_0, \infty)$ and they are eventually strictly monotone functions. We have from (8) that $z^{(n-1)}(t)$ is strictly decreasing. So either

$$\lim_{t\to\infty} z^{(n-1)}(t) = -\infty \tag{11}$$

or

$$\lim_{t\to\infty} z^{(n-1)}(t) = L, \qquad L \in \mathbb{R}. \tag{12}$$

First assume that (11) holds. Then

$$\lim_{t\to\infty} w^{(2n-1)}(t) = -\lim_{t\to\infty}\left(\sum_{K_1} q_k z^{(n-1)}(t-\sigma_k) + \sum_{K_2} \hat{q}_k z^{(n-1)}(t+\hat{\sigma}_k)\right) = +\infty.$$

Thus

$$\lim_{t\to\infty} w^{(\nu)}(t) = +\infty, \qquad \nu = 0, 1, \dots, 2n-1,$$

which together with (10) imply that $w \in$ Class II_{2n}. Next assume that (12) holds. Then, integrating (8) over the interval $[t_1, \infty)$, we obtain

$$z^{(n-1)}(t_1) - L = \sum_{K_1} q_k \int_{t_1-\sigma_k}^{\infty} x(s)\,ds + \sum_{K_2} \hat{q}_k \int_{t_1+\hat{\sigma}_k}^{\infty} x(s)\,ds$$

which implies that $x \in L^1[t_1 - \sigma_{n_1}, +\infty)$ and so $z \in L^1[t_1 - \sigma_{n_1}, +\infty)$. Since $z(t)$ is strictly monotone it follows that

$$\lim_{t\to\infty} z(t) = 0$$

and therefore

$$\lim_{t\to\infty} z^{(\nu)}(t) = 0, \qquad \nu = 0, 1, \dots, n-1 \tag{13}$$

that is, $L = 0$. Thus we have established that $z^{(n)}(t) < 0$ and $\lim_{t\to\infty} z^{(n-1)}(t) = 0$. This implies that $z^{(n-1)}(t) > 0$ and hence

$$(-1)^{n-1} z(t) > 0.$$

In view of (9), we conclude that

$$(-1)^n w^{(n)}(t) > 0 \quad \text{and} \quad w(t) > 0.$$

Also, by (13), it follows that

$$\lim_{t\to\infty} w^{(\nu)}(t) = 0, \qquad \nu = 0, 1, \dots, 2n-1$$

which together with (10) imply

$$(-1)^\nu w^{(\nu)}(t) > 0, \qquad \nu = 0, 1, \dots, 2n.$$

https://doi.org/10.1017/S1446788700034406 Published online by Cambridge University Press

Thus $w(t) \in$ Class II_{2n} and the proof is complete.

LEMMA 4. *Assume that* (3) *and* (4) *hold. Then we have the following*:
(a) *let* $x(t) \in$ Class I_{2n}, *then there exists a solution* $w(t)$ *of Equation* (1) *which belongs to* Class I_{2n}, *such that the set*

$$\Lambda^+(w) = \{\lambda > 0 \colon (-1)^{n-1} w^{(n)}(t) + \lambda^n w(t) \le 0\} \ne \varnothing;$$

(b) *let* $x(t) \in$ Class II_{2n}, *then there exists a solution* $w(t)$ *of Equation* (1) *which belongs to class* II_{2n}, *such that the set*

$$\Lambda^-(w) = \{\lambda > 0 \colon -w^{(n)}(t) + \lambda^n w(t) \le 0\} \ne \varnothing.$$

PROOF. (a) Let $x(t) \in$ Class I_{2n}. Set

$$w(t) = (-1)^{n-1}\delta\left[x(t) + \sum_I p_i x(t-\tau_i) - \sum_J r_j x(t-\rho_j)\right]. \tag{14}$$

It is easy to see that $w(t)$ is a solution of Equation (1) which belongs to Class I_{1n} and that

$$(-1)^{n-1} w^{(n)}(t) + \sum_{K_1} q_k x(t-\sigma_k) + \sum_{K_2} q_k x(t+\hat{\sigma}_k) = 0. \tag{15}$$

Since (3) and (4) hold, if we set

$$\alpha = \begin{cases} \tau_l & \text{in case (i)} \\ \rho_m & \text{in case (ii)} \end{cases} \quad \text{and} \quad \beta = \begin{cases} \max\{\rho_m, \sigma_{n_1}\} & \text{in case (i)} \\ \max\{\tau_l, \sigma_{n_1}\} & \text{in case (ii)} \end{cases} \tag{16}$$

then we see that

$$\sigma_{n_1} > \alpha \quad \text{or} \quad \beta > \alpha \ge \sigma_{n_1}.$$

Thus we examine the following:

1) $\sigma_{n_1} > \alpha$. Since $x(t)$ is positive and decreasing, (14) yields

$$0 < w(t) < A_1 x(t-\alpha) < A_1 x(t-\sigma_{n_1}),$$

where

$$A_1 = \begin{cases} 1+P & \text{in case (i)} \\ R & \text{in case (ii)}. \end{cases} \tag{17}$$

Thus

$$\begin{aligned} 0 &= (-1)^{n-1} w^{(n)}(t) + \sum_{K_1} q_k x(t-\sigma_k) + \sum_{K_2} \hat{q}_k x(t+\hat{\sigma}_k) \\ &> (-1)^{n-1} w^{(n)}(t) + q_{n_1} x(t-\sigma_{n_1}) > (-1)^{n-1} w^{(n)}(t) + \frac{1}{A_1} q_{n_1} w(t), \end{aligned}$$

which says that

$$0 < \left(\frac{1}{A_1} q_{n_1}\right)^{1/n} \in \Lambda^+(w), \text{ that is, } \Lambda^+(w) \neq \varnothing.$$

2) $\beta > \alpha \geq \sigma_{n_1}$. First assume $\beta > \alpha > \sigma_{n_1}$. As before, (14) yields

$$0 < w(t) < A_1 x(t-\alpha) \tag{18}$$

where A_1 is as in (17). Also, from (14), since x, $w \in$ Class I_{2n}, we find that

$$x(t-\alpha) > A_2 x(t-\beta)$$

where

$$A_2 = \begin{cases} \dfrac{r_m}{1+P} & \text{in case (i)} \\ \dfrac{p_l}{R} & \text{in case (ii)}. \end{cases}$$

We have from the last inequality, that

$$x[t+(\beta-\alpha)] > A_2 x(t)$$

and by iteration, we obtain

$$x[t+\mu(\beta-\alpha)] > A_2^\mu x(t).$$

Note that there exists an integer $\mu > 0$ such that

$$\mu(\beta-\alpha) \geq \alpha - \sigma_{n_1} > 0.$$

We have using the above inequalities, for some $\mu \in \mathbb{N}$, that

$$\begin{aligned} 0 &= (-1)^{n-1} w^{(n)}(t) + \sum_{K_1} q_k x((t-\sigma_k) + \sum_{K_2} \hat{q}_k x(t+\hat{\sigma}_k) \\ &> (-1)^{n-1} w^{(n)}(t) + q_{n_1} x(t-\sigma_{n_1}) \\ &= (-1)^{n-1} w^{(n)}(t) + q_{n_1} x[t-\alpha+(\alpha-\sigma_{n_1})] \\ &\geq (-1)^{n-1} w^{(n)}(t) + q_{n_1} x[t-\alpha+\mu(\beta-\alpha)] \\ &> (-1)^{n-1} w^{(n)}(t) + q_{n_1} A_2^\mu x(t-\alpha) \\ &> (-1)^{n-1} w^{(n)}(t) + \frac{1}{A_1} q_{n_1} A_2^\mu w(t). \end{aligned}$$

This implies that

$$0 < \left(\frac{1}{A_1} q_{n_1} A_2^\mu\right)^{1/n} \in \Lambda^+(w), \text{ that is, } \Lambda^+(w) \neq \varnothing.$$

Next assume $\beta > \alpha = \sigma_{n_1}$. As before, we have

$$0 > (-1)^{n-1} w^{(n)}(t) + q_{n_1} x(t-\sigma_{n_1})$$

and, since $\alpha = \sigma_{n_1}$, the last inequality, in view of (18), yields

$$0 > (-1)^{n-1} w^{(n)}(t) + \frac{1}{A_1} q_{n_1} w(t),$$

which says that

$$0 > \left(\frac{1}{A_1} q_{n_1}\right)^{1/n} \in \Lambda^+(w), \text{ that is, } \Lambda^+(w) \neq \varnothing.$$

(b) Let $x(t) \in$ Class II_{2n}. Set

$$w(t) = -\delta \left[x(t) + \sum_I p_i x(t - \tau_i) - \sum_J r_j x(t - \rho_j) \right]. \tag{14'}$$

It is easy to see that $w(t)$ is a solution of Equation (1) which belongs to Class II_{2n} and that

$$-w^{(n)}(t) + \sum_{K_1} q_k x(t - \sigma_k) + \sum_{K_2} \hat{q}_k x(t + \hat{\sigma}_k) = 0. \tag{19}$$

Now, since $x(t)$ is positive and increasing, $(14')$ yields

$$0 < w(t) < A_3 x(t)$$

where

$$A_3 = \begin{cases} R & \text{when } \delta = +1 \\ 1 + P & \text{when } \delta = -1. \end{cases}$$

Thus

$$\begin{aligned} 0 &= -w^{(n)}(t) + \sum_{K_1} q_k x(t - \sigma_k) + \sum_{K_2} \hat{q}_k x(t + \hat{\sigma}_k) \\ &> -w^{(n)}(t) + Q_2 x(t) > -w^{(n)}(t) + \frac{1}{A_3} Q_2 w(t) \end{aligned}$$

which says that

$$0 < \left(\frac{1}{A_3} Q_2\right)^{1/n} \in \Lambda^-(w), \text{ that is, } \Lambda^-(w) \neq \varnothing.$$

The proof of the lemma is complete.

LEMMA 5. (a) *Let* $x(t) \in$ Class I_{2n} *for which the set* $\Lambda^+(x) \neq \varnothing$. *If for given* $\omega > 0$ *there exists* $M > 0$ *such that*

$$(-1)^\nu x^{(\nu)}(t) > M(-1)^\nu x^{(\nu)}(t - \omega), \qquad \nu = 0, 1, \ldots, n \tag{20}$$

then the positive number $\lambda_0 = (1/\omega)\log(1/M)$ *is an upper bound of* $\Lambda^+(x)$.

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(b) *Let* $x(t) \in$ Class II_{2n} *for which the set* $\Lambda^-(x) \neq 0$. *If for given* $\omega > 0$ *there exists* $M > 0$ *such that*

$$x^{(\nu)}(t) < Mx^{(\nu)}(t-\omega), \qquad \nu = 0, 1, \ldots, n \tag{20'}$$

then the positive number $\lambda_0 = (1/\omega)\log M$ *is an upper bound of* $\Lambda^-(x)$.

Proof. (a) Otherwise $\lambda_0 \in \Lambda^+(x)$ which means that

$$(-1)^{n-1}x^{(n)}(t) + \lambda_0^n x(t) \leq 0.$$

Set

$$y(t) = \sum_{\nu=0}^{n-1} \lambda_0^{n-\nu-1}(-1)^\nu x^{(\nu)}(t).$$

It is easy to see that $y(t)$ is a solution of Equation (1) such that

$$y(t) \in \text{Class } \mathrm{I}_{n+1} \supset \text{Class } \mathrm{I}_{2n},$$
$$\dot{y}(t) + \lambda_0 y(t) = (-1)^{n-1}x^{(n)}(t) + \lambda_0^n x(t) \leq 0$$

and, in view of (20),

$$y(t) > My(t-\omega).$$

Now the conclusion follows from [10, Lemma 5(a)].

(b) Otherwise $\lambda_0 \in \Lambda^-(x)$ which implies that

$$-x^{(n)}(t) + \lambda_0^n x(t) \leq 0.$$

Set

$$y(t) = \sum_{\nu=0}^{n-1} \lambda_0^{n-\nu-1} x^{(\nu)}(t).$$

It is easy to see that $y(t)$ is a solution of Equation (1) such that

$$y(t) \in \text{Class } \mathrm{II}_{n+1} \supset \text{Class } \mathrm{II}_{2n},$$
$$-\dot{y}(t) + \lambda_0 y(t) = -x^{(n)}(t) + \lambda_0^n x(t) \leq 0$$

and, in view of (20′),

$$y(t) < My(t-\omega).$$

The conclusion follows from [10, Lemma 5(b)].

Lemma 6. *Assume that* (3) *and* (4) *hold. Then we have the following*:

(a) *let* $x(t) \in$ Class I_{2n} *for which* $\Lambda^+(x) \neq \varnothing$, *then the set* $\Lambda^+(x)$ *has an upper bound which is independent of* x;

(b) *let* $x(t) \in$ Class II_{2n} *for which* $\Lambda^{-}(x) \neq \varnothing$, *then the set* $\Lambda^{-}(x)$ *has an upper bound which is independent of* x.

PROOF. (a) Let $x(t) \in$ Class I_{2n}. Set $w(t)$ as in (14), then $w(t) \in$ Class I_{2n} and (15) holds. Setting α and β as in (16), we examine the following:

1) $\sigma_{n_1} > \alpha$. By (14), we find that

$$(21) \qquad 0 < (-1)^{\nu} w^{(\nu)}(t) < A_1 (-1)^{\nu} x^{(\nu)}(t-\alpha), \qquad \nu = 0, 1, \ldots, n$$

where A_1 is given by (17). Also, from (15) we obtain

$$(-1^{\text{`}})^{n+\nu-1} w^{(n+\nu)}(t) + q_{n_1}(-1)^{\nu} x^{(\nu)}(t-\sigma_{n_1}) < 0, \qquad \nu = 0, 1, \ldots, n.$$

Integrating the last inequality over the interval $[t-\omega, t]$, where $\omega = (1/2n)(\sigma_{n_1} - \alpha)$, and using the fact that $(-1)^{\nu} x^{(\nu)}(t) > 0$ and decreasing, we find

$$(-1)^{n+\nu-1} w^{(n+\nu-1)}(t) > q_{n_1} \omega (-1)^{\nu} x^{\nu}(t - \sigma_{n_1} + \omega), \nu = 0, 1, \ldots, n.$$

Repeating this procedure $n-1$ more times, we finally obtain

$$(-1)^{\nu} w^{(\nu)}(t) > q_{n_1} \omega^{n} (-1)^{\nu} x^{(\nu)}(t - \sigma_{n_1} + n\omega), \qquad \nu = 0, 1, \ldots, n.$$

Combining the last inequality with (21), we find

$$(-1)^{\nu} x^{(\nu)}(t) > \frac{1}{A_1} q_{n_1} \left(\frac{1}{2n}(\sigma_{n_1} - \alpha)\right)^{n} (-1)^{\nu} x^{(\nu)} \left(t - \frac{1}{2}(\sigma_{n_1} + \alpha)\right), \qquad \nu = 0, 1, \ldots, n,$$

and, by Lemma 5(a), the positive number

$$\lambda_1 \equiv \frac{2}{\sigma_{n_1} + \alpha} \log \frac{A_1}{q_{n_1}} \left(\frac{2n}{\sigma_{n_1} - \alpha}\right)^{n}$$

is an upper bound of $\Lambda^{+}(x)$.

2) $\beta > \alpha \geq \sigma_n$. From (14), we obtain

$$(-1)^{\nu} x^{(\nu)}(t) > A_2 (-1)^{\nu} x^{(\nu)}(t-\omega), \qquad \nu = 0, 1, \ldots, n$$

where A_2 as in the proof of Lemma 4(a), and

$$\omega = \beta - \alpha = \begin{cases} \rho_m - \tau_l & \text{in case (i)} \\ \tau_l - \rho_m & \text{in case (ii)}. \end{cases}$$

Now, by Lemma 5(a), the positive number

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$$\lambda_2 \equiv \frac{1}{\omega} \log \frac{1}{A_2}$$

is an upper bound of $\Lambda^+(x)$.

(b) Let $x(t) \in \text{Class II}_{2n}$. Set $w(t)$ as in $(14')$, then $w(t) \in \text{Class II}_{2n}$ and (19) holds. First assume $\delta = +1$. Then from $(14')$, we have

$$w^{(\nu)}(t) > 0, \qquad \nu = 0, 1, \dots, n$$

which implies that

$$x^{(\nu)}(t) < Rx^{(\nu)}(t - \rho_1), \quad \nu = 0, 1, \dots, n.$$

So, by Lemma 5(b), the positive number

$$\lambda_3 \equiv \frac{1}{\rho_1} \log R$$

is an upper bound of $\Lambda^-(x)$.

Next assume that $\delta = -1$. Then from $(14')$ we have

$$w^{(\nu)}(t) < (1 + P)x^{(\nu)}(t), \qquad \nu = 0, 1, \dots, n, \tag{22}$$

and, from (19), we find

$$\begin{aligned} w^{(n+\nu)}(t) &= \sum_{K_1} q_k x^{(\nu)}(t - \sigma_k) + \sum_{K_2} \hat{q}_k x^{(\nu)}(t + \hat{\sigma}_k) \\ &> Q_2 x^{(\nu)}(t + \hat{\sigma}_1), \qquad \nu = 0, 1, \dots, n. \end{aligned}$$

Integrating the last inequality over the interval $[t - \omega, t]$, where $\omega = (1/2n)\hat{\sigma}_1$, we obtain

$$w^{(n+\nu-1)}(t) > Q_2 \omega x^{(\nu)}(t + \hat{\sigma}_1 - \omega), \qquad \nu = 0, 1, \dots, n.$$

Repeating this procedure $n - 1$ more times, we finally obtain

$$w^{(\nu)}(t) > Q_2 \omega^n x^{(\nu)}(t + \hat{\sigma}_1 - n\omega), \qquad \nu = 0, 1, \dots, n,$$

and combining with (22), we find

$$x^{(\nu)}(t) < \frac{1 + P}{Q_2} \left(\frac{2n}{\hat{\sigma}_1}\right)^n x^{(\nu)} \left(t - \frac{1}{2}\hat{\sigma}_1\right), \qquad \nu = 0, 1, \dots, n.$$

Thus, by Lemma 5(b), the positive number

$$\lambda_4 \equiv \frac{2}{\hat{\sigma}_1} \log \frac{1 + P}{Q_2} \left(\frac{2n}{\hat{\sigma}_1}\right)^n$$

is an upper bound of $\Lambda^-(x)$.

The proof of the Lemma is complete.

3. Main result

Our main result is the following

THEOREM. *Consider the nth-order neutral differential equation*

$$\frac{d^n}{dt^n}\left[x(t)+\sum_I p_i x(t-\tau_i)-\sum_J r_j x(t-\rho_j)\right] + \delta\left(\sum_{K_1} q_k x(t-\sigma_k)+\sum_{K_2}\hat{q}_k x(t+\hat{\sigma}_k)\right)=0 \tag{1}$$

where $n \geq 1$, $\delta = \pm 1$, I, J, K_1, K_2 are initial segments of natural numbers, p_i, τ_i, r_j, ρ_j, q_k, $\hat{q}_k \in (0, \infty)$ and σ_k, $\hat{\sigma}_k \in [0, \infty)$ for $i \in I$, $j \in J$, $k \in K_1 \cup K_2$. Then a necessary and sufficient condition for the oscillation of all solutions of Equation (1) *is that its characteristic equation*

$$\lambda^n+\lambda^n\sum_I p_i e^{-\lambda\tau i}-\lambda^n\sum_J r_j e^{-\lambda\rho_j}+\delta\left(\sum_{K_1} q_k e^{-\lambda\sigma_k}+\sum_{K_2}\hat{q}_k e^{\lambda\hat{\sigma}_k}\right)=0 \tag{2}$$

has not real roots.

PROOF. The theorem will be proved in the contrapositive form: there is a nonoscillatory solution of (1) if and only if the characteristic equation (2) has a real root. Assume first that (2) has a real root λ. Then (1) has the nonoscillatory solution $x(t) = e^{\lambda t}$.

Assume, conversely, that there is a nonoscillatory solution of (1) and, for the sake of contradiction, that Equation (2) has not real roots. Then by Lemma 3, Equation (1) has also a nonoscillatory solution $x(t)$ which belongs either to Class I_{2n} or to Class II_{2n}. Consider the following cases:

(a) $x(t) \in$ Class I_{2n}. For this solution $x(t)$, by Lemma 4(a), we can assume, without loss of generality, that $\Lambda^+(x) \neq \varnothing$. Let $\lambda_5 \in \Lambda^+(x)$. Also, by Lemma 6(a), there exists a positive number, say λ_0, such that $\Lambda^+(x)$ is bounded above by λ_0.

For $\lambda \in \Lambda^+(x)$ consider the functions

$$z(t) \equiv F_1 x = (-1)^{n-1}\delta\left[x(t)+\sum_I p_i x(t-\tau_i)-\sum_J r_j x(t-\rho_j)\right],$$

$$w(t) \equiv F_2 z = \sum_{\nu=1}^{n}\lambda^{n-\nu}[(-1)^\nu z^{(\nu)}(t)], \quad \text{(cf. [19])}$$

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and

$$(23)\qquad u(t) \equiv F_3 w = \begin{cases} H_w(t) + \lambda^n \sum_J r_j \int_{t-T}^{t-\rho_j} w(s)\,ds & \text{in case (i)} \\ H_w(t) + \lambda^n \int_{t-T}^{t} w(s)\,ds + \lambda^n \sum_I p_i \int_{t-T}^{t-\tau_i} w(s)\,ds & \\ & \text{in case (ii)}, \end{cases}$$

where

$$(24)\qquad H_w(t) = \delta \left[w(t) + \sum_I p_i w((t-\tau_i) - \sum_J r_j w(t-\rho_j) \right]^{(n-1)} \\ + \sum_{K_1} q_k \int_{t-T}^{t-\sigma_k} w(s)\,ds + \sum_{K_2} \hat{q}_k \int_{t-T}^{t+\hat{\sigma}_k} w(s)\,ds$$

and

$$T = \max\{\tau_l, \rho_m, \sigma_n, \hat{\sigma}_{n_2}\}.$$

It is easy to see that $z(t)$, $w(t)$ and $u(t)$ are solutions of Equation (1) and they belong to Class I_{2n}.

Since Equation (2) has no real roots, by Lemma 1, the inequalities (5) and (6) hold. We will show that $(\lambda^n + m_0)^{1/n} \in \Lambda^+(u)$ where $m_0 = m/N_1 > 0$ with

$$N_1 = \left(1 + P + R + \frac{1}{\lambda_5^n} Q\right) e^{\lambda_0 T}.$$

To this end, it suffices to show that

$$(-1)^{n-1} u^{(n)}(t) + (\lambda^n + m_0) u(t) \le 0.$$

Define

$$\varphi(t) = e^{\lambda t}[(-1)^{n-1} w^{(n-1)}(t)].$$

Then $\varphi(t) > 0$ and for $\lambda \in \Lambda^+(x)$ with $\lambda \ge \lambda_5$, we have

$$\begin{aligned} \dot{\varphi}(t) &= e^{\lambda t}[(-1)^{n-1} w^{(n)}(t) + \lambda(-1)^{n-1} w^{(n-1)}(t)] \\ &= e^{\lambda t}[(-1)^{2n-1} z^{(2n)}(t) + \lambda^n (-1)^n z^{(n)}(t)] \\ &= e^{\lambda t} \Bigg(\sum_{K_1} q_k [(-1)^{n-1} x^{(n)}(t-\sigma_k) + \lambda^n x(t-\sigma_k)] \\ &\qquad + \sum_{K_2} \hat{q}_k [(-1)^{n-1} x^{(n)}(t+\hat{\sigma}_k) + \lambda^n x(t+\hat{\sigma}_k)] \Bigg) \\ &\le 0, \end{aligned}$$

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that is, $\varphi(t)$ is decreasing. Since $w(t) \in \text{Class } \mathrm{I}_{2n}$ and

$$(-1)^{n-1} w^{(n-1)}(t) = e^{-\lambda t} \varphi(t), \tag{25}$$

we see that

$$\begin{aligned} w(t) &= \int_t^\infty \int_{t_1}^\infty \cdots \int_{t_{n-2}}^\infty (-1)^{n-1} w^{(n-1)}(t_{n-1})\, dt_{n-1}\, dt_{n-2} \cdots dt_1 \\ &\le \frac{1}{\lambda^{n-1}} e^{-\lambda t} \varphi(t) \end{aligned}$$

and therefore for any $\omega \le T$

$$\int_{t-T}^{t-\omega} w(s)\, ds \le \frac{1}{\lambda^n} e^{-\lambda t} \varphi(t-T)[e^{\lambda T} - e^{\lambda \omega}]. \tag{26}$$

Now, from (23), in view of (24), we obtain

$$u^{(n)}(t) = \begin{cases} -Qw^{(n-1)}(t-T) + \lambda^n \sum_J r_j [w^{(n-1)}(t-\rho_j) - w^{(n-1)}(t-T)] \\ \qquad \text{in case (i)} \\ -Qw^{(n-1)}(t-T) + \lambda^n [w^{(n-1)}(t) - w^{(n-1)}(t-T)] \\ \qquad + \lambda^n \sum_I p_i [w^{(n-1)}(t-\tau_i) - w^{(n-1)}(t-T)] \\ \qquad \text{in case(ii)} \end{cases}$$

and, in view of (25),

$$(-1)^{n-1} u^{(n)}(t) = \begin{cases} -Qe^{-\lambda(t-T)} \varphi(t-T) \\ \qquad + \lambda^n \sum_J r_j [e^{-\lambda(t-\rho_j)} \varphi(t-\rho_j) - e^{-\lambda(t-T)} \varphi(t-T)] \\ \qquad \text{in case (i)} \\ -Qe^{-\lambda(t-T)} \varphi(t-T) + \lambda^n [e^{-\lambda t} \varphi(t) - e^{-\lambda(t-T)} \varphi(t-T)] \\ \qquad + \lambda^n \sum_I p_i [e^{-\lambda(t-\tau_i)} \varphi(t-\tau_i) - e^{-\lambda(t-T)} \varphi(t-T)] \\ \qquad \text{in case (ii)}. \end{cases}$$

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Also, from (23), in view of (24), (25) and (26), we find that

$$u(t) \leq \frac{1}{\lambda^n} e^{-\lambda t} \varphi(t-T) \left[Q e^{\lambda T} - \sum_{K_1} q_k e^{\lambda \sigma_k} - \sum_{K_2} \hat{q}_k e^{-\lambda \hat{\sigma}_k} \right]$$

$$+ \begin{cases} e^{-\lambda t}\varphi(t) + \sum_I p_i e^{-\lambda(t-\tau_i)} \varphi(t-\tau_i) - \sum_J r_j e^{-\lambda(t-\rho_j)} \varphi(t-\rho_j) \\ \quad + e^{-\lambda t}\varphi(t-T) \left[R e^{\lambda T} - \sum_J r_j e^{\lambda \rho_j} \right] \quad \text{in case (i)} \\ -e^{-\lambda t}\varphi(t) - \sum_I p_i e^{-\lambda(t-\tau_i)} \varphi(t-\tau_i) + \sum_J r_j e^{-\lambda(t-\rho_j)} \varphi(t-\rho_j) \\ \quad + e^{-\lambda t}\varphi(t-T) \left[e^{\lambda T} - 1 + P e^{\lambda T} - \sum_I p_i e^{\lambda \tau_i} \right] \quad \text{in case (ii)}. \end{cases}$$

Finally, in view of (6), we obtain

$$(-1)^{n-1} u^{(n)}(t) + \lambda^n u(t)$$

$$\leq \begin{cases} e^{-\lambda t}\varphi(t-T) \left[\lambda^n + \lambda^n \sum_I p_i e^{\lambda \tau_i} - \lambda^n \sum_J r_j e^{\lambda \rho_j} - \sum_{K_1} q_k e^{\lambda \sigma_k} \right. \\ \qquad \left. - \sum_{K_2} \hat{q}_k e^{-\lambda \hat{\sigma}_k} \right] \quad \text{in case (i)} \\ e^{-\lambda t}\varphi(t-T) \left[-\lambda^n - \lambda^n \sum_I p_i e^{\lambda \tau_i} + \lambda^n \sum_J r_j e^{-\lambda \rho_j} - \sum_{K_1} q_k e^{\lambda \sigma_k} \right. \\ \qquad \left. - \sum_{K_2} \hat{q}_k^{-\lambda \hat{\sigma}_k} \right] \quad \text{in case (ii)} \end{cases}$$

$$\leq e^{-\lambda t}\varphi(t-T)(-m)$$

and consequently

$$\begin{aligned} (-1)^{n-1} u^{(n)}(t) + (\lambda^n + m_0) u(t) &= [(-1)^{n-1} u^{(n)}(t) + \lambda^n u(t)] + m_0 u(t) \\ &\leq e^{-\lambda t}\varphi(t-T)(-m + m_0 N_1) = 0 \end{aligned}$$

as required. Now set

$$x_0 \equiv x, \quad x_1 = Fx = F_3(F_2(F_1 x)) = u, \qquad x_2 = F x_1$$

and in general

$$x_\nu = F x_{\nu-1}, \qquad \nu = 1, 2, \ldots$$

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and observe that $x_r \in$ Class I_{2n} with $\Lambda^+(x_\nu) \neq \varnothing$, and for

$$\lambda \in \Lambda^+(x) \equiv \Lambda^+(x_0) \Rightarrow (\lambda^n + m_0)^{1/n} \in \Lambda^+(u) \equiv \Lambda^+(x_1)$$

and after ν steps $(\lambda^n + \nu m_0)^{1/n} \in \Lambda^+(x_\nu)$, $\nu = 1, 2, \ldots$ which is a contradiction since λ_0 is a common upper bound for all $\Lambda^+(x_\nu)$.

This completes the proof when $x(t) \in$ Class I_{2n}.

(b) $x(t) \in$ Class II_{2n}.. By Lemma 4(b) we can assume that $\Lambda^-(x) \neq \varnothing$. Let $\lambda_6 \in \Lambda^-(x)$. Also, by Lemma 6(b), there exists a positive number, say λ_0, such that $\Lambda^-(x)$ is bounded above by λ_0. For $\lambda \in \Lambda^-(x)$ consider the functions

$$z(t) = -\delta\left[x(t) + \sum_I p_i x(t-\tau_i) - \sum_J r_j x(t-\rho_j)\right],$$

$$w(t) = \sum_{\nu=1}^{n} \lambda^{n-\nu} z^{(\nu)}(t),$$

and

(23′)

$$u(t) = \begin{cases} \widetilde{H}_w(t) + \lambda^n \int_t^{t+T} w(s)\,ds + \lambda^n \sum_I p_i \int_{t-\tau_i}^{t+T} w(s)\,ds & \text{when } \delta = +1 \\ \widetilde{H}_w(t) + \lambda^n \sum_J r_j \int_{t-\rho_j}^{t+T} w(s)\,ds & \text{when } \delta = -1, \end{cases}$$

where

$$\text{(24′)} \qquad \widetilde{H}_w(t) = -\delta\left[w(t) + \sum_I p_i w(t-\tau_i) - \sum_J r_j w(t-\rho_j)\right]^{(n-1)} + \sum_{K_1} q_k \int_{t-\sigma_k}^{t+T} w(s)\,ds + \sum_{K_2} \hat{q}_k \int_{t+\hat{\sigma}_k}^{t+T} w(s)\,ds$$

and

$$T = \max\{\tau_l, \rho_m, \sigma_{n_1}, \hat{\sigma}_{n_2}\}.$$

It is easy to see that $z(t)$, $w(t)$ and $u(t)$ are solutions of Equation (1) and they belong to Class II_{2n}.

Since Equation (2) has no real roots, by Lemma 1, the inequalities (5) and (7) hold. We will show that

$$(\lambda^n + m_0)^{1/n} \in \Lambda^-(u),$$

where $m_0 = m/N_2 > 0$ with

$$N_2 = \begin{cases} \sum_J r_j e^{-\lambda_6 \rho_j} + \left[1 + P + \frac{Q}{\lambda_6^n}\right] e^{\lambda_0 T} & \text{when } \delta = +1 \\ 1 + \sum_I p_i e^{-\lambda_6 \tau_i} + \left[R + \frac{Q}{\lambda_6^n}\right] e^{\lambda_0^T} & \text{when } \delta = -1. \end{cases}$$

To this end, it suffices to show that

$$-u^{(n)}(t) + (\lambda^n + m_0)u(t) \leq 0.$$

Define

$$\varphi(t) = e^{-\lambda t} w^{(n-1)}(t).$$

Then $\varphi(t) > 0$ and for $\lambda \in \Lambda^-(x)$ with $\lambda \geq \lambda_6$, we have

$$\begin{aligned}\dot{\varphi}(t) &= -e^{-\lambda t}[-w^{(n)}(t) + \lambda w^{(n-1)}(t)] = -e^{-\lambda t}[-z^{(2n)}(t) + \lambda^n z^{(n)}(t)] \\ &= -e^{-\lambda t}\Bigg(\sum_{K_1} q_k[-x^{(n)}(t-\sigma_k) + \lambda^n x(t-\sigma_k)] \\ &\qquad\qquad + \sum_{K_2} \hat{q}_k[-x^{(n)}(t+\hat{\sigma}_k) + \lambda^n x(t+\hat{\sigma}_k)]\Bigg) \geq 0,\end{aligned}$$

that is, $\varphi(t)$ is increasing. Also

$$w^{(n-1)}(t) = e^{\lambda t}\varphi(t). \tag{25$'$}$$

Now, as in [25], we extend the definition of $w^{(\nu)}(t)$, $\nu = 0, 1, \ldots, n$ such that $w^{(\nu)}(t)$ are continuous positive and increasing on $(-\infty, \infty)$ and $\lim_{t\to-\infty} w^{(\nu)}(t) = 0$, $\nu = 0, 1, \ldots, n-1$. Then, in view of $(25')$,

$$\begin{aligned}w(t) &= \int_{-\infty}^{t}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_{n-2}} w^{(n-1)}(t_{n-1})\,dt_{n-1}\,dt_{n-2}\cdots dt_1 \\ &\leq \frac{1}{\lambda^{n-1}} e^{\lambda t}\varphi(t)\end{aligned}$$

and therefore for any $\omega \leq T$

$$\int_{t-\omega}^{t+T} w(s)\,ds \leq \frac{1}{\lambda^n} e^{\lambda t}\varphi(t+T)[e^{\lambda T} - e^{-\lambda\omega}]. \tag{26$'$}$$

Consequently from $(23')$, in view of $(24')$, $(25')$, $(26')$ and the fact that

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$\varphi(t)$ is increasing, as in case (a), we obtain

$$-u^{(n)}(t) + \lambda^n u(t)$$
$$\leq \begin{cases} e^{\lambda t}\varphi(t+T)\left[-\lambda^n - \lambda^n \sum_I p_i e^{-\lambda\tau_i} + \lambda^n \sum_J r_j e^{-\lambda\rho_j} - \sum_{K_1} q_k e^{-\lambda\sigma_k} \right. \\ \qquad \left. - \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k}\right] \text{ when } \delta = +1 \\ e^{\lambda t}\varphi(t+T)\left[\lambda^n + \lambda^n \sum_I p_i e^{-\lambda\tau_i} - \lambda^n \sum_J r_j e^{-\lambda\rho_j} - \sum_{K_1} q_k e^{-\lambda\sigma_k} \right. \\ \qquad \left. - \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k}\right] \text{ when } \delta = -1 \end{cases}$$

which, by (7), leads to

$$-u^{(n)}(t) + \lambda^n u(t) \leq e^{\lambda t}\varphi(t+T)(-m).$$

Finally,

$$\begin{aligned} -u^{(n)}(t) + (\lambda^n + m_0)u(t) &= [-u^{(n)}(t) + \lambda^n u(t)] + m_0 u(t) \\ &\leq e^{\lambda t}\varphi(t+T)(-m + m_0 N_2) = 0 \end{aligned}$$

as required. Now, as in case (a), we have a contradiction.

The proof of the theorem is complete.

4. Applications and examples

In this section we apply our main theorem and obtain some useful corollaries.

COROLLARY 1. *Consider the mixed type differential equation*

$$x^{(n)}(t) + \delta\left(\sum_{K_1} q_k x(t-\sigma_k) + \sum_{K_2} \hat{q}_k x(t+\hat{\sigma}_k)\right) = 0,$$

where $n \geq 1$, $\delta = \pm 1$, q_k, σ_k, $\hat{q}_k$, $\hat{\sigma}_k \in \mathbb{R}^+$ *for* $k \in K_1 \cup K_2$. *Then all solutions of this equation oscillate if and only if its characteristic equation*

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$$\lambda^n + \delta\left(\sum_{K_1} q_k e^{-\lambda\sigma_k} + \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k}\right) = 0$$

has no real roots.

Observed that in the case of the mixed type equations (cf. [20])

$$x^{(n)}(t) + \sum_{K_1} q_k x(t-\sigma_k) - \sum_{K_2} \hat{q}_k x(t+\sigma_k) = 0$$

and

$$x^{(n)}(t) - \sum_{K_1} q_k x(t-\sigma_k) + \sum_{K_2} \hat{q}_k x(t+\hat{\sigma}_k) = 0,$$

their characteristic equations are respectively

$$f(\lambda) \equiv \lambda^n + \sum_{K_1} q_k e^{-\lambda\sigma_k} - \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k} = 0$$

and

$$g(\lambda) \equiv \lambda^n - \sum_{K_1} q_k e^{-\lambda\sigma_k} + \sum_{K_2} \hat{q}_k e^{\lambda\hat{\sigma}_k} = 0,$$

and it holds

$$f(+\infty)f(-\infty) < 0 \quad \text{and } g(+\infty)g(-\infty) < 0.$$

Therefore equations of the above forms always admit nonoscillatory solutions.

The method of proof which we used to establish our main result is short (cf. [7]) and also has the advantage that it results in easily verifiable sufficient conditions for the oscillation of solutions to Equation (1). Indeed, this is derived by comparing elements of the set $\Lambda^+(x)$ (respectively $\Lambda^-(x)$) in each case. Observe that we found points λ_a and λ_b such that $\lambda_a \in \Lambda^+(x)$ (respectively $\lambda_a \in \Lambda^-(x)$), while λ_b is an upper bound of $\Lambda^+(x)$ (respectively $\Lambda^-(x)$). Thus, if we assume

$$\lambda_a \geq \lambda_b,$$

we are led to a contradiction. Utilizing this idea, we can obtain several sufficient conditions (in terms of the coefficients and the arguments only) for the oscillation of solutions of Equation (1). The advantage of working with these sufficient conditions rather than the characteristic equation (2) directly is that the said conditions are explicit, while determining whether or not a real root to Equation (2) exists may be quite a problem in itself. Thus, using Lemmas 4 and 6, one can draw a number of corollaries. We confine ourselves to the following:

COROLLARY 2. *Consider Equation* (1). *Then any one of the following two conditions imply that Equation* (1) *has no* (*nonoscillatory*) *solutions of*

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Class I_{2n}:

(27)
$$\left(\frac{1}{1+P}q_{n_1}\right)^{1/n} > \frac{1}{\sigma_{n_1}+\tau_l}\log\frac{1+P}{q_{n_1}}\left(\frac{2n}{\sigma_{n_1}-\tau_l}\right)^n \quad \textit{when } \sigma_{n_1} > \tau_l \textit{ in case (i)}$$

or

(28)
$$\left(\frac{1}{R}q_{n_1}\right)^{1/n} > \frac{1}{\tau_l - \rho_m}\log\frac{R}{P_l} \quad \textit{when } \tau_l > \rho_m = \sigma_n \textit{ in case (ii)}.$$

COROLLARY 3. *Consider Equation* (1). *Then any one of the following two conditions imply that Equation* (1) *has no (nonoscillatory) solutions of* Class II_{2n}:

(29)
$$\left(\frac{1}{R}Q_2\right)^{1/n} > \frac{1}{\rho_1}\log R \quad \textit{when } \delta = +1$$

or

(30)
$$\left(\frac{1}{1+P}Q_2\right)^{1/n} > \frac{2}{\hat{\sigma}_1}\log\frac{1+P}{Q_2}\left(\frac{2n}{\hat{\sigma}_1}\right)^n \quad \textit{when } \delta = -1$$

REMARK 1. Observe that if there exists a bounded nonoscillatory solution of Equation (1), then Class I_{2n} is not empty.

REMARK 2. It is clear that when Equation (1) has no (nonoscillatory) solutions of Class I_{2n} and Class II_{2n}, then all solutions of (1) oscillate.

EXAMPLE 1. Consider the third order neutral differential equation

(31)
$$\frac{d^3}{dt^3}\left[x(t) + 2x(t-2\pi) - \frac{21}{10}x\left(t-\frac{\pi}{2}\right)\right] - 3x\left(t-\frac{\pi}{2}\right) - \frac{21}{10}x(t+2\pi) = 0.$$

Observe that the conditions (28) and (30) are satisfied. Therefore by Corollaries 2, 3 and Remark 2, all solutions oscillate. For example, sint and cost are oscillatory solutions of Equation (31).

EXAMPLE 2. For the second order neutral equation

(32)
$$\frac{d^2}{dt^2}[x(t) + x(t-8\pi) - 30x(t-\pi)] + x(t-\pi) + 33x(t+2\pi) = 0$$

condition (28) is satisfied. Therefore, by Corollary 2, Equation (32) has no (nonoscillatory) solutions of Class I_{2n} and, by Remark 1, *all bounded solutions* oscillate. For example, sint and cost are bounded oscillatory solutions of (32). Note that condition (29) is not satisfied. Thus, in this example, we can not conclude that all solutions oscillate.

EXAMPLE 3. Consider the mixed differential equation

(33)
$$x^{(4)}(t) + 5e^{2\pi}x(t-2\pi) + e^{-\pi}x(t+\pi) = 0.$$

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Observe that this equation has no (nonoscillatory) solutions of Class I_{2n} and Class II_{2n} and therefore all solutions oscillate. For example, $e^t \sin t$ and $e^t \cos t$ are (non-bounded) oscillatory solutions of Equation (33).

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https://doi.org/10.1017/S1446788700034406 Published online by Cambridge University Press