ON THE PRODUCT OF TWO POWER SERIES

H. DAVENPORT AND G. PÓLYA

We consider the product of two power series with positive coefficients:

$$(\Sigma u_n x^n)(\Sigma v_n x^n) = \Sigma w_n x^n.$$

What conditions will ensure that the coefficients w_n shall be either (i) monotonic, or (ii) logarithmically convex? By the latter, we mean that $w_n^2 \leq w_{n-1}w_{n+1}$ for $n = 1, 2, \ldots$. In investigating this question, which was suggested by a special example, we have found it convenient to express the conditions in terms of the ratios of u_n, v_n to certain binomial coefficients, rather than in terms of u_n, v_n themselves.

We introduce α and β such that

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 1 \tag{1}$$

and let

$$\alpha_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1\,.\,2\ldots n}, \quad \beta_n = \frac{\beta(\beta+1)\ldots(\beta+n-1)}{1\,.\,2\ldots n} \tag{2}$$

for $n \geq 1$; $\alpha_0 = \beta_0 = 1$. Let

$$a_n = u_n/\alpha_n, \quad b_n = v_n/\beta_n$$

so that a_n and b_n are positive, and

$$w_n = \alpha_0 a_0 \beta_n b_n + \alpha_1 a_1 \beta_{n-1} b_{n-1} + \ldots + \alpha_n a_n \beta_0 b_0. \tag{3}$$

We have been led to the following very elementary results, which appear, however, to be new.

THEOREM 1. *If a_n and b_n are both monotonic increasing, so is w_n, and if a_n and b_n are both monotonic decreasing, so is w_n.*

THEOREM 2. *If a_n and b_n are both logarithmically convex, so is w_n.*

We prove these theorems in **1** and **2**, and add some general remarks concerning them in **3**. In **4** we apply them to the special example from which our investigation started. In **5** we mention the integral analogues.

1. The proof of Theorem 1 may be decomposed into two steps, the first of which is concerned only with properties of the binomial coefficients.

Put

$$\begin{cases} p_0 = \alpha_0\beta_n, & p_1 = \alpha_1\beta_{n-1}, \ldots, & p_n = \alpha_n\beta_0 \\ q_0 = \alpha_0\beta_{n+1}, & q_1 = \alpha_1\beta_n, \ldots, & q_{n+1} = \alpha_{n+1}\beta_0. \end{cases} \tag{4}$$

Then we assert that

$$p_0 + p_1 + \ldots + p_n = q_0 + q_1 + \ldots + q_{n+1} = 1, \tag{5}$$

and

$$q_0 < p_0 < q_0 + q_1 < p_0 + p_1 < \ldots < q_0 + q_1 + \ldots + q_n < p_0 + p_1 + \ldots p_n. \tag{6}$$

Thus we assert that the successive partial sums of the two sequences $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$ separate each other. If we imagine each sequence represented

Received January 29, 1948.

https://doi.org/10.4153/CJM-1949-001-1 Published online by Cambridge University Press

by a row of blocks, the two rows will have a form similar to that of two neighbouring rows of tiles in a wall, and we can express the property in question by saying that the two sequences are "tilewise ordered."

Of the two results (5) and (6), the former is immediate, since, by (2),

$$\sum_0^\infty \alpha_n x^n = (1-x)^{-\alpha}, \quad \sum_0^\infty \beta_n x^n = (1-x)^{-\beta}$$

and so, by (1),

$$\sum_0^\infty (\alpha_0\beta_n + \ldots + \alpha_n\beta_0)x^n = (1-x)^{-1} = \sum_0^\infty x^n.$$

To prove (6), we observe that, by (1) and (2), the α_n and β_n are monotonic decreasing, whence

$$\begin{aligned} q_0 + q_1 + \ldots + q_k &= \alpha_0\beta_{n+1} + \alpha_1\beta_n + \ldots + \alpha_k\beta_{n+1-k} \\ &< \alpha_0\beta_n + \alpha_1\beta_{n-1} + \ldots + \alpha_k\beta_{n-k} \\ &= p_0 + p_1 + \ldots + p_k. \end{aligned}$$

Similarly

$$\begin{aligned} q_{n+1} + q_n + \ldots + q_{k+1} &= \alpha_{n+1}\beta_0 + \alpha_n\beta_1 + \ldots + \alpha_{k+1}\beta_{n-k} \\ &< \alpha_n\beta_0 + \alpha_{n-1}\beta_1 + \ldots + \alpha_k\beta_{n-k} \\ &= p_n + p_{n-1} + \ldots + p_k. \end{aligned}$$

In view of (5), this implies that

$$q_0 + q_1 + \ldots + q_k > p_0 + p_1 + \ldots + p_{k-1},$$

and the proof of (6) is complete.

For the second step in the proof of Theorem 1, we introduce symbols for the successive differences of the terms in (6). We put

$$r_0 = q_0,\ r'_0 = p_0 - q_0,\ r_1 = (q_0 + q_1) - p_0,\ r'_1 = (p_0 + p_1) - (q_0 + q_1), \ldots$$
$$r_n = (q_0 + \ldots + q_n) - (p_0 + \ldots + p_{n-1}),\ r'_n = q_{n+1}.$$

All these numbers are positive, and we have

$$p_0 = r_0 + r'_0,\ p_1 = r_1 + r'_1, \ldots, p_n = r_n + r'_n,$$
$$q_0 = r_0,\ q_1 = r'_0 + r_1, \ldots, q_n = r'_{n-1} + r_n,\ q_{n+1} = r'_n.$$

Hence, by (3) and (4),

$$w_n = r_0 a_0 b_n + r'_0 a_0 b_n + r_1 a_1 b_{n-1} + \ldots + r_n a_n b_0 + r'_n a_n b_0,$$
$$w_{n+1} = r_0 a_0 b_{n+1} + r'_0 a_1 b_n + r_1 a_1 b_n + \ldots + r_n a_n b_1 + r'_n a_{n+1} b_0.$$

These expressions render Theorem 1 immediate, on comparison of corresponding terms.

2. To prove Theorem 2, we use the following lemma:

LEMMA. *Let W_n be defined by*

$$W_n = a_0 b_n + \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \ldots + a_n b_0. \tag{7}$$

Then, if a_n and b_n are positive and logarithmically convex, so is W_n.

Proof. The desired result $W_n^2 \leqq W_{n-1}W_{n+1}$ holds for $n = 1$ since

$$\begin{aligned} W_0 W_2 - W_1^2 &= a_0 b_0 (a_0 b_2 + 2a_1 b_1 + a_2 b_0) - (a_0 b_1 + a_1 b_0)^2 \\ &= a_0^2 (b_0 b_2 - b_1^2) + b_0^2 (a_0 a_2 - a_1^2) \geq 0. \end{aligned}$$

We prove it for general n by induction.

https://doi.org/10.4153/CJM-1949-001-1 Published online by Cambridge University Press

By the well-known property

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

of the binomial coefficients, we have, for $n \geq 1$,

$$W_n = W'_{n-1} + W''_{n-1},$$

where W'_{n-1} is formed with the sequences $a_1, a_2, \ldots$ and $b_0, b_1, \ldots$ and W''_{n-1} is formed with the sequences $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$. By the hypothesis of the induction, applied to the two former sequences, we have

$$(W'_{n-1})^2 \leq W'_{n-2}W'_n,$$

and similarly

$$(W''_{n-1})^2 \leq W''_{n-2}W''_n.$$

By the inequality of the arithmetic and geometric means, it follows that

$$2\ W'_{n-1}W''_{n-1} \leq 2\ \{W'_{n-2}W'_nW''_{n-2}W''_n\}^{\frac{1}{2}} \leq W'_{n-2}W''_n + W''_{n-2}W'_n.$$

Hence, using again the hypothesis of the induction, we obtain

$$W_n^2 = (W'_{n-1} + W''_{n-1})^2 \leq W'_{n-2}W'_n + W'_{n-2}W''_n + W''_{n-2}W'_n + W''_{n-2}W''_n = W_{n-1}W_{n+1}.$$

This proves the Lemma.

An immediate corollary to the Lemma is that the same conclusion holds for $W_n(\lambda, \mu)$ defined by

(8) $$W_n(\lambda, \mu) = a_0b_n\mu^n + \binom{n}{1} a_1\lambda b_{n-1}\mu^{n-1} + \ldots + a_n\lambda^n b_0,$$

where λ, μ are any two positive numbers.

We can now prove Theorem 2 as follows. By (1) and (2), we have

$$\alpha_m\beta_{n-m} = \binom{n}{m} \frac{\Gamma(\alpha+m)\Gamma(\beta+n-m)}{n!\,\Gamma(\alpha)\Gamma(\beta)}$$
$$= \binom{n}{m} \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 t^{\alpha+m-1}(1-t)^{\beta+n-m-1}dt.$$

Substituting in (3), and using the notation of (8), we obtain

$$w_n = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}W_n(t, 1-t)\, dt.$$

Since $W_n(t, 1-t)$ is logarithmically convex for each t, it follows from the inequality of Schwarz that w_n is, since

$$\Gamma(\alpha)\Gamma(\beta)w_n \leq \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\{W_{n-1}(t, 1-t)W_{n+1}(t, 1-t)\}^{\frac{1}{2}}\,dt$$
$$\leq \left\{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}W_{n-1}(t, 1-t)dt\right\}^{\frac{1}{2}} \cdot \left\{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}W_{n+1}(t, 1-t)dt\right\}^{\frac{1}{2}}$$
$$= (\Gamma(\alpha)\Gamma(\beta)w_{n-1}\Gamma(\alpha)\Gamma(\beta)w_{n+1})^{\frac{1}{2}}.$$

This proves Theorem 2.

https://doi.org/10.4153/CJM-1949-001-1 Published online by Cambridge University Press

3. The two theorems proved above have a certain resemblance to the following simple but useful theorem of Kaluza.[1]

If the a_n are positive and logarithmically convex, and

$$(a_0 + a_1x + a_2x^2 + \ldots)^{-1} = b_0 - b_1x - b_2x^2 - \ldots,$$

then all the b_n are positive.

All three theorems give conditions which ensure that a power series, derived from given power series by multiplication or division, shall have some simple property.

There is one class of power series to which our theorems can readily be applied. Suppose $\phi(t)$ is positive and integrable in the interval $(0, h)$, and let

$$\int_0^h \phi(t)(1 - xt)^{-\alpha}dt = \Sigma \alpha_n a_n x^n. \tag{9}$$

Then

$$a_n = \int_0^h \phi(t)t^n dt,$$

and the a_n, being the successive moments of a positive function, are logarithmically convex.

4. The particular problem from which our investigation started was that of showing that

$$\left[\int_0^1 (1 + u^4 - 2xu^2)^{-\frac{1}{2}}du\right]^{-2} + \left[\int_0^1 (1 + u^4 + 2xu^2)^{-\frac{1}{2}}du\right]^{-2} \tag{10}$$

decreases steadily as x increases from 0 to 1.

(It can be shown that the expression (10) represents $(2r_0\Lambda/\pi)^2$, where r_0 denotes the inner conformal radius of a rectangle with respect to its centre, and Λ denotes the principal frequency of vibration of a membrane with the rectangle as its boundary. The product $r_0\Lambda$ depends on the shape but not on the size of the rectangle, and the parameter x specifies this shape. As x increases from 0 to 1, the ratio of the two sides of the rectangle increases steadily from 1 to infinity. Our assertion concerning (10) means that the product $r_0\Lambda$ decreases steadily in this process.)

By the change of variable

$$2u^2/(1 + u^4) = t$$

the first integral in (10) is transformed into an integral $I(x)$ of the type (9), with $h = 1$ and $\alpha = 1/2$. Theorem 2, applied to this integral, tells us that the coefficients of the power series for $I^2(x)$ are logarithmically convex. From Kaluza's theorem, it follows that the expression (10) has the form

$$2b_0 - 2b_2x^2 - 2b_4x^4 - \ldots$$

with positive b_n. This obviously decreases as x increases.

We should perhaps observe that instead of using Theorem 2 in the above argument, we can use the following *ad hoc* argument. We have

$$I^2(x) = \int_0^1\int_0^1 \phi(t)\phi(t')(1 - xt)^{-\frac{1}{2}}(1 - xt')^{-\frac{1}{2}}\,dtdt'.$$

[1]*Math. Zeit.*, vol. 28 (1928), 161-170.

Let

$$(1 - xt)^{-\frac{1}{2}}(1 - xt')^{-\frac{1}{2}} = \Sigma A_n(t, t')x^n;$$

then

$$\mathrm{I}^2(x) = \Sigma c_n x^n,$$

where

$$c_n = \int_0^1 \int_0^1 \phi(t)\phi(t')A_n(t, t')dtdt'.$$

If we prove that $A_n(t, t')$ is logarithmically convex, for fixed t, t', it will follow that c_n is logarithmically convex, as desired. In fact, it is easily seen that

$$A_n(t, t') = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (t \cos^2\theta + t' \sin^2\theta)^n d\theta,$$

and this is obviously logarithmically convex.

5. For the sake of completeness, we mention the integral analogues of Theorems 1 and 2, although they are less interesting.

Suppose that $f(x)$ and $g(x)$ are positive and integrable for $x \geq 0$, and bounded in any finite interval. We retain (1) and put

$$h(x) = \int_0^x t^{\alpha-1}f(t)(x - t)^{\beta-1}g(x - t)dt.$$

THEOREM 3. *If $f(x)$ and $g(x)$ are both monotonic increasing, so is $h(x)$, and if $f(x)$ and $g(x)$ are both monotonic decreasing, so is $h(x)$.*

THEOREM 4. *If $f(x)$ and $g(x)$ are both logarithmically convex, so is $h(x)$.*

We say that $f(x)$ is logarithmically convex, if for $x \geq d > 0$,

$$f^2(x) \leq f(x - d)f(x + d).$$

By changing the variable of integration and using (1), we obtain

$$h(x) = \int_0^1 u^{\alpha-1}(1 - u)^{\beta-1} f(ux)\, g((1 - u)x)\, du,$$

and this representation of $h(x)$ renders Theorem 3 obvious. By the hypothesis of Theorem 4 and Schwarz's inequality,

$$\begin{aligned} h(x) &\leq \int_0^1 u^{\alpha-1}(1 - u)^{\beta-1}\{f(u[x - d])\, f(u[x + d])\}^{\frac{1}{2}} \\ &\qquad \cdot \{g([1 - u][x - d])\, g([1 - u][x + d])\}^{\frac{1}{2}} du \\ &\leq \left\{\int_0^1 u^{\alpha-1}(1 - u)^{\beta-1} f(u[x - d])\, g([1 - u][x - d])du\right\}^{\frac{1}{2}} \\ &\qquad \cdot \left\{\int_0^1 u^{\alpha-1}(1 - u)^{\beta-1} f(u[x + d])\, g([1 - u][x + d])du\right\}^{\frac{1}{2}} \\ &= \{h(x - d)\, h(x + d)\}^{\frac{1}{2}}. \end{aligned}$$

This proves Theorem 4.

University College, London
Stanford University