ON THE APPROXIMATION OF THE TOTAL AMOUNT OF CLAIMS

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Several "short cut" methods exist to approximate the total amount of claims ($= \chi$) of an insurance collective. The classical one is the normal approximation

$$F(\chi) \approx \Phi\left(\frac{\chi - \bar{\chi}}{\sigma_\chi}\right) \tag{1}$$

where $\bar{\chi}$ and σ_χ are the mean value and standard deviation of χ. Φ is the normal distribution function.

It is well-known that the normal approximation gives acceptable accuracy only when the volume of risk business is fairly large and the distribution of the amounts of the individual claims is not "too dangerous", i.e. not too heterogeneous (cf. fig. 2).

One way to improve the normal approximation is the so called NP-method, which provides for the standardized variable $z = \frac{\chi - \bar{\chi}}{\sigma_\chi}$ a correction Δz

$$F(\chi) \approx \Phi(z + \Delta z) = \Phi\left[\sqrt{\left(\frac{9}{\gamma_1^2} + 1 + \frac{6(\chi - P)}{\gamma_1 \sigma_\chi}\right)} - \frac{3}{\gamma_1}\right] \tag{2}$$

where

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} \tag{3}$$

is the skewness of the distribution $F(\chi)$. Another variant ($NP3$) of the NP-method also makes use of the moment μ_4, but, in the following, we limit our discussion mainly to the variant (2) ($= NP2$).

If Δz is small, a simpler formula

$$F(\chi) \approx \Phi\left(z - \frac{\gamma_1}{6}(z^2 - 1)\right) \tag{2a}$$

is available (cf. fig. 2).

Another approximation was introduced by Bohman and Esscher (1963). It is based on the incomplete gamma function

$$F(z) \approx \frac{1}{\Gamma(\alpha)} \int_{0}^{\alpha + z\sqrt{\alpha}} e^{-y}\, y^{\alpha-1}\, dy \tag{4}$$

where $\alpha = 4/\gamma_1^2$.

Experiments have been made with both formulae (2) and (4); they have been applied to various F functions, from which the exact (or at least controlled) values are otherwise known. It has been proved that the accuracy is satisfactory provided that the distribution F is not very "dangerous". The skewness γ_1 can be used as an indicator for the character of the distribution to be treated.

Hilary Seal has recently criticized the *NP*-method (1977). He claims that the gamma-approximation is superior and goes as far as to say, "surely here is a case for disregarding the *NP*-method altogether" (1977). His conclusions seem to be based on numerical comparisons obtained from the seven distributions given in his paper. Seal also claims that the superiority of the gamma-approximation does not seem to depend on the size of γ_1.

Since Seal's statements contradict a large body of existing numerical experience, we have picked some other comparisons in the following tables and figures in order to get a more adequate mapping for the structure of accuracy. The gamma- and *NP*-numbers are computed for the purpose by means of a special computer program. The results are compared with the "exact" values obtained by Bohman and Esscher or by Hovinen (1964). The numbers are only examples taken from a large collection of material and are chosen to illustrate the situation for varying distributions and sizes of skewness (table 1). The values of z range from 2 . . . 6 which is of interest when the approximations are applied in risk theory and in different kinds of solvency tests.

In the table the columns *FA*, *NP* and *GA* are denotations of the function 1-F calculated by means of the "exact" method, the *NP*-method and the gamma-method.

The "exact" data concerning the distributions *EXP*, Life A, Life B and Non-industrial Fire are taken from the report of Bohman-Esscher (1963). The variable t is the expected number of claims and

TABLE I

1 Distribution t/k	2 Skewness γ_1	3 z	4 p	5 FA x 10^p	6 ΔFA x 10^p	7 $NP2$ x 10^p	8 GA x 10^p	9 $NP3$ x 10^p	10 $\Delta_1 NP2$ %	11 $\Delta_1 GA$ %	12 $\Delta_1 NP3$ %	13 $\Delta_2 NP2$ %	14 $\Delta_2 GA$ %
Exp 1000/ ∞	.0671	2	5	2453	80	2454	2453	2453	0	0	0	0	0
		3	5	177	8	177	177	177	0	0	0	0	0
F Mo	.1570	2	4	264	32	269	268	.	0	0	.	2	2
Exp 100/ ∞	.2122	2	5	2815	83	2827	2816	2814	0	0	0	0	0
		3	5	282	11	285	285	282	0	0	0	+1	+1
F Ind	.3879	2	4	301	34	325	321	.	0	0	.	+8	+7
		3	4	43	14	44	44	.	0	0	.	+2	+2
Life B 10000/20	.4543	2	5	3350	84	3409	3349	3352	0	0	0	+2	0
		3	5	498	15	503	499	500	0	0	0	+1	0
Life A 1000/ ∞	.5570	2	5	3566	85	3640	3549	3551	0	0	0	+2	0
		3	5	589	17	606	598	590	0	0	0	+3	+1
		4	5	75	2	78	80	77	+1.2	+3.9	0	+4	+7
Life B 1000/20	.7749	2	5	3950	84	4104	3921	3935	+1.8	0	0	+4	—1
		3	5	820	20	840	813	831	0	0	0	+2	—1
		4	5	145	4	144	145	154	0	0	+3.4	—1	0
		6	6	32	2	27	34	41	—8.2	0	+22	—14	+5
Non Ind 1000/20	.8115	2	5	3968	82	4179	3977	4444	+3.2	0	0	+5	+1
		3	5	892	20	881	849	920	0	—2.6	+1.0	—1	—5
		4	5	177	4	157	158	105	—9.3	—8.5	+1.1	—12	—11
Non Ind 1000/ ∞	1.2139	2	5	4523	70	4938	4481	4497	+7.6	0	0	+9	—1
		3	5	1401	26	1348	1234	1387	—2.0	—10	0	—4	—12
		4	5	352	7	333	319	428	—3.3	—7.3	+20	—5	—9
		6	6	219	5	164	190	422	—23	—11	+90	—25	—13
Life B 1000/ ∞	1.2159	2	5	4569	90	4941	4483	4531	+6.2	0	0	+8	—2
		3	5	1258	24	1350	1236	1201	+5.4	0	—2.6	+7	—2
		4	5	281	6	334	320	291	+17	+12	+1.4	+19	+14
		6	6	115	3	165	191	142	+41	+64	+21	+44	+66
Life A 100/20	1.5286	2	5	5154	77	5464	4747	4812	+4.5	—6.4	—5.1	+6	—8
		3	5	1596	30	1721	1502	1548	+6.0	—4.0	—1.1	+8	—6
		4	5	444	8	507	462	488	+12	+2.2	+8.1	+14	+4
		6	6	333	7	380	414	471	+12	+22	+39	+14	+24
Life A 100/ ∞	1.7615	2	5	5555	66	5821	4884	4996	+3.6	—11	—8.9	+5	—12
		3	5	1859	35	1997	1676	1688	+5.5	—8.0	—7.3	+7	—10
		4	5	519	9	651	568	557	+24	+7.7	+5.6	+25	+9
		6	6	477	10	619	638	585	+28	+32	+21	+30	+34
F Mo	1.8564	2	4	411	40	596	493	.	+35	+10	.	+45	+20
		3	4	169	26	211	174	.	+9.3	0	.	+25	+3
F Mo	2.7318	2	4	550	46	707	505	.	+20	0	.	+29	—8
		3	4	275	34	309	217	.	+0.1	—8.6	.	+12	—21
		4	4	116	22	133	96	.	0	0	.	+15	—18
Life B 100/20	3.4504	2	5	3171	38	7805	4908	5215	+145	+54	+63	+146	+55
		3	5	2022	6	3827	2342	2551	+89	+16	+26	+89	+16
		4	5	1842	1	1867	1156	1285	+1.3	—37	—30	+1	—37
		6	6	2798	57	4372	2981	3475	+54	+5	+22	+56	+7
Non Ind 100/ ∞	3.8385	2	5	3450	41	8152	4783	4827	+135	+37	+39	+136	+39
		3	5	1709	13	4195	2383	3016	+145	+39	+76	+145	+39
		4	5	893	8	2156	1232	1966	+141	+37	+119	+141	+38
		6	6	3780	16	5647	3510	9079	+49	—7	+140	+49	—7

Δ_1 and Δ_2 are computed from unrounded figures.

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k the Polya-parameter. The Finnish industrial Fire distributions (F/Ind) and those of the Finnish Third Party Motor Insurance (*F/Mo*) are taken from the paper of Kauppi and Ojantakanen (1969); these figures were originally calculated by Hovinen by means of the Monte Carlo method. $NP3$-values, which also make use of the moment μ_4 are computed by L. Kauppi.

The deviation between the "exact" values and the approximated values depends on the inaccuracy of both of these figures. Bohman and Esscher have given the maximum deviation in their report and it is shown in our table (col. ΔFA). For the Monte Carlo method double standard deviation is used.

An important difference between the Finnish distributions and the Swedish ones (and Seal used the latter) is that Hovinen had the level of the net retention as a special variable. Thus it was possible to get risk distributions which are similar to those which the companies have, in practice, on their own retention in regard to a conventional reinsurance. This variable is reflected in the size of γ_1 which is shown in the tables. Unfortunately, however, the Monte Carlo method leaves the confidence intervals rather long.

The figures are arranged in increasing order of γ_1.

The deviations are computed in two ways. Δ_1 is the relative deviation of the respective values from the interval $FA \pm \Delta FA$ as a percentage of FA. If e.g. $| NP - FA | < \Delta FA$ the deviation is $= 0$. This figure gives an obvious lower limit for the error of the approximation. Δ_2 is the relative deviation from the "midpoint" values FA, e.g. $\Delta_2 NP2 = \dfrac{NP2 - FA}{FA} \times 100$. These figures also contain the effect of the original ΔFA and therefore they give a somewhat exaggerated gauge for the inaccuracy of the approximations. The difference between the two comparison methods is not significant as can be seen from the table.

From the table one can see that as long as $\gamma_1 < 0.5$ the accuracy is within the limits of the deviations (ΔFA) of the original figures for both approximations.

When $\gamma_1 > 3$ and $z = 3$ or 4 the NP-values can show more than 100% deviations whereas GA-values are within the limits — 37 ... + 39%. When $z = 6$ the largest NP-deviation is 54% and the largest GA-deviation 64%.

One can further observe

		Maximum deviation in %			
		NP		*GA*	
		Δ_1	(Δ_2)	Δ_1	(Δ_2)
when $\gamma_1 < 1$ and	$z = 3$	0	(3)	3	(5)
$\gamma_1 < 1$	$z = 4$	9	(12)	9	(11)
$\gamma_1 < 2$	$z = 3$	9	(25)	10	(12)
$\gamma_1 < 2$	$z = 4$	24	(25)	12	(14)
$\gamma_1 < 2$	$z = 6$	41	(44)	64	(66)

The *NP*3-variant does not seem to give any reliable improvement over the other approximations. It fails especially for large values of z, which is an expected feature, because the terms including z^3 increase so much that the formulae drastically decrease in accuracy.

On the basis of the results we still have good reason to claim that the *NP*-method is able to give very satisfactory accuracy provided that the distribution is not very dangerous, i.e. as long as γ_1 does not exceed 1 or (depending on the accuracy demanded) 2. This fact is already known (cf. Kauppi & Ojantakanen (1969) and Pesonen (1969)). On the other hand within these limits we cannot find any significant difference between the gamma- and the *NP*-method. Seal's conclusion that the comparison in question would be independent of γ_1 is clearly incorrect.

On the other hand, when there is a big increase in the value of γ_1 neither method is sufficiently reliable. This fact has already been stated by Bohman and Esscher concerning the gamma-method (1963, p. 207) and by Kauppi and Ojantakanen (1969). Because both methods are based only on the three lowest moments, which are equated with the corresponding moments of the distribution to be approximated, one cannot expect any great accuracy in these cases. The mutual deviations of the exact values show that no approximating function can exist which depends on γ_1 only and which would be able to approximate for all of them in a satisfactory way.

To further illustrate the point, we have computed a comparison of the gamma- and *NP*-values and given it graphically. When $\gamma_1 < 1.5$ (or $\gamma_1 < 2$ if inaccuracy up to 30% is still to be tolerated) the methods do not differ very much, which confirms the con-

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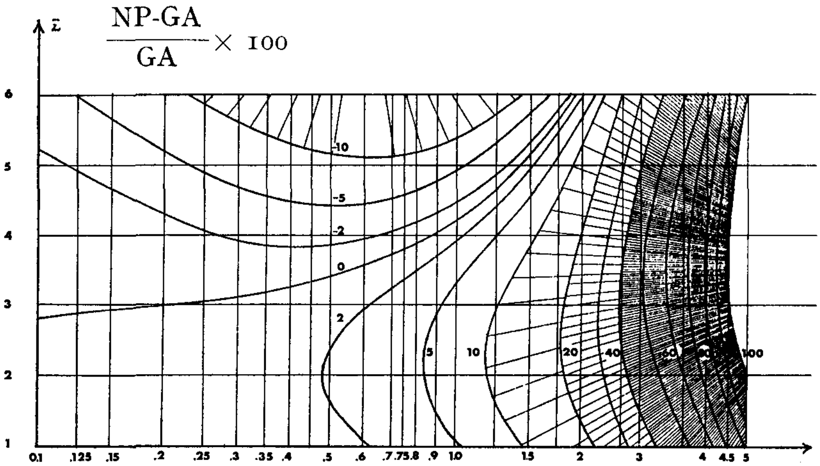

fig. 1 Skewness γ_1

clusions mentioned above. Neither method can claim to be "superior", if factually they give the same results.

A further observation still to be made is that $NP2$ gives large positive deviations for large values of γ_1. It is obvious that the approximation can be improved by introducing an extra negative correction for Δz, but this idea cannot be discussed here.

The Swedish distributions in table 1 and in the whole of Seal's material call into doubt one important point, namely that the fluctuation of the basic probabilities is assumed to obey Polya-distribution. In this particular case the distribution function $F \to \Gamma$ when $t \to \infty$ (Bohman & Esscher, p. 190). When the very same Γ is used as an approximating function, one has reason to ask whether it is fitting only in this rather special case. What would the result be if the basic probabilities have any other distribution U? It is well known that $F \to U$ when $t \to \infty$. If U deviates from Γ, certainly at least for large values of t, Γ-approximation cannot behave as well as now in the Polya-case. Unfortunately for the time being suitable exact values are not available for varying U-distributions. We tried the Finnish distributions but the simulation inaccuracy limits their fitness for use. When $k = \infty$, then $F \to \Phi$, and consequently the Γ-approximation cannot benefit its asymptotic behaviour in this case. It may be symptomatic that just in these cases Γ-approximation shows (for moderate γ_1) the worst results, e.g. Life B 1000/∞. However, our material is too limited to make any reliable conclusions. Any way it is an open question as to how the approximations suit for various kinds of U-distributions.

It is neither necessary nor meaningful to discuss which of the methods is "superior" for large γ_1-values, because cases exist where both of them deviate considerably from exact values. The irregularities of the observed deviations, and especially what was said about U- distribution, make it evident that the available material is too limited for any reliable conclusions to be drawn and obviously more computation or direct estimation methods will be needed. Our conclusion is rather that neither of the approximations is safe enough, when γ_1 exceeds a reasonable limit, say 1 or 2.

The astonishing divergence of opinion between Seal and ourselves may depend on the fact that Seal has selected from the large body of material available only the most dangerous cases, where, in fact, nobody had expected any adequacy for the approximations in question. Part of Seal's conclusion, at least concerning the independence of the size of γ_1 may also depend on a misprint in his tabular values (which appeared in the original paper, too).

In practice most insurance companies have an adequate reinsurance. Hence the business on the company's own retention is fairly homogeneous, i.e. not "dangerous". Then the size of γ_1 seems to be so small that we are certainly in the "safe" area. Our experience is that γ_1 is mostly of the order of .1 or .2, seldom more than .5 if the company has a conventional reinsurance. The approximations in question are, in fact, intended for just such cases.

One of the merits of the NP-method is that it is a natural enlargement of the normal approximation. The latter is well-known and much used in various statistical works. It is still useful as a first approximation in risk theory and its many applications providing, of course, that the user is well aware of its merits and weaknesses. It facilitates many considerations; e.g. it makes it possible in an illustrative way to use the variable z or the standard deviation σ_χ as a measure of stability which helps considerably the exploration of even very complicated problems. Often it is possible to get analytical equations which can at least give a rough illustration of the interdependence of the various variables of risk theory. So guidance for more accurate investigations can be found in a very much easier way than computing e.g. many dimensional tables by means of exact or more accurate methods. In fig. 2 a small illustration is given as to how NP-values deviate from the normal approximation. As long as the skewness is small, the deviations do not prevent the applicability of N-approximation as long as the demand of accuracy is not high, e.g. if only the order of magnitude is needed. When the skewness increases, the NP-correction is often easily introduced.

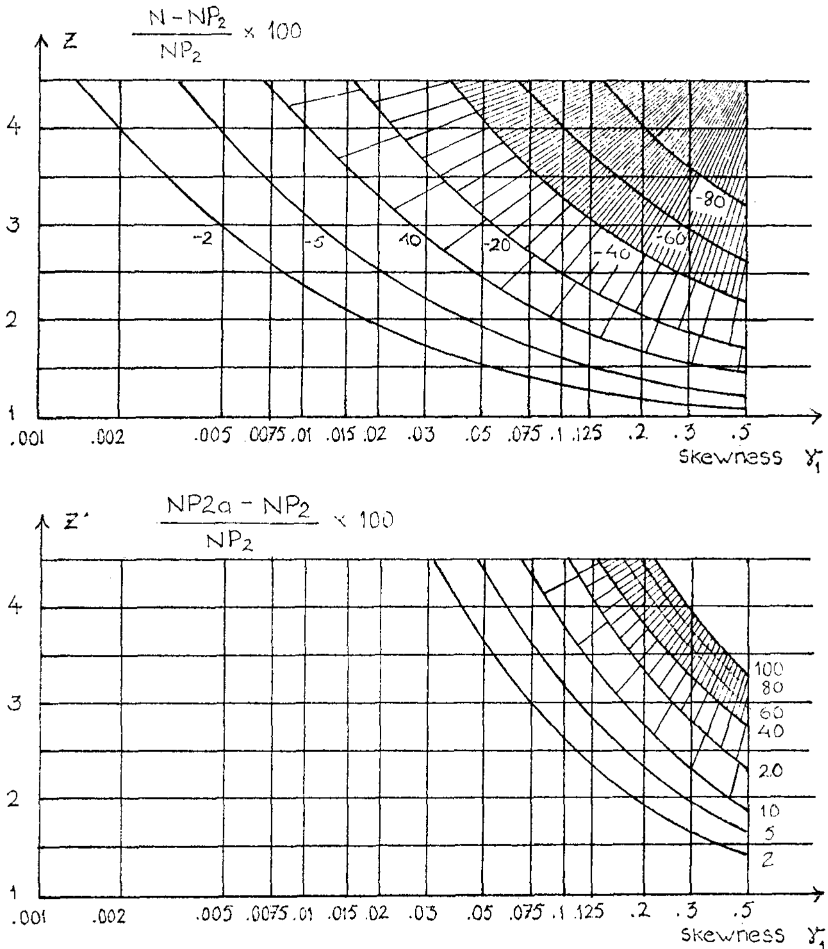

Fig 2. The deviation of the normal approximation N from the $NP2$ values, as well as the deviation of the simplified formula (2a) from the $NP2$ formula (2).

These are the reasons why the author of this paper is rather reluctant to disregard completely the *NP*-method and even the *N*-method at least until such time as new methods are found which really are superior in their range of applicability and in their "handiness" in dealing with problems such as outlined above. On the other hand it is, of course, very laudable that several alternative methods exist each of which has its own merits.

Finally, we would do well to keep in mind the fact that the answer to the approximation problem of the *F*-function is still very much open. The present methods are based on the basic provision that an approximating function is to be found, which has two or three lowest moments equal with the original distribution. Several approximating functions are suggested and have been experimented with. We have only referred to two of them here. Beard has also suggested the use of four moments, which leads to the Pearson-system of curves. An interesting problem and one that remains to a large extent open is that of finding the most fitting type of approximating function and of mapping the domain of its applicability. Nowadays, when complicated functions and formulae are easily programmed by computer we can expect more knowledge on this topic to be available in the near future.

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