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SEMI-COMPACTNESS WITH RESPECT TO A EUCLIDEAN CONE

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1. Introduction. Our motivation for this note originates with consideration of a subset A of Euclidean n-space, R^n, which contains only part of its boundary. The part contained is that part of the closure of A which cannot be "bettered" within A with respect to the preference associated with a fixed closed convex cone Γ. Here b is preferred to a if and only if $a - b \in \Gamma$; if, for instance, Γ is the non-negative orthant of R^n, this preference is ordinary vector inequality. We will see in § 4 that obtaining these partially closed sets can often be a matter of relaxing continuity conditions to semi-continuity, and therefore we call them Γ *semi-closed* sets. We are further concerned with partial boundedness in the following sense: When the convex hull of $A \subset R^n$ is unbounded at most in directions which are contained in the fixed cone Γ, we say A is Γ *semi-bounded.* These concepts are formalized in § 2.

The usefulness of these notions in asserting existence of constrained extrema is evident. For example, suppose we wish to choose q such that $(F_1(q), \ldots, F_{n-1}(q)) \in N \subset R_{n-1}$ and such that subject to this $F_n(q)$ is maximized. To assert existence of such a q it is relevant for the range of $(F_1, \ldots, F_n)$ to contain the "upper" part of its boundary, not necessarily all of its boundary, and that this range be partially bounded, i.e., that this range be Γ semi-closed and Γ semi-bounded, where $\Gamma = R^n \cap \{(0, \ldots, 0, y) : y \leqq 0\}$.

Applications of Γ semi-closedness and Γ semi-boundedness to existence of constrained extrema of F of the form $F(q) = \int_T f(t, q(t))d\mu t \in R^n$, with fixed T, μ and f, and related literature, are discussed for this half-line Γ in [**5**], and for more general Γ in [**4**].

To relate Γ semi-closedness to semi-continuity, we recall the criterion that a real-valued function g on a topological space is upper semi-continuous if and only if $\{t : g(t) \geqq a\}$ is closed for each real a. Permitting g to be R^n-valued, in § 4 we replace the inequality with the Γ preference cited above and require that $\{t : a - g(t) \in \Gamma\}$, i.e., $g^{-1}(a - \Gamma)$, be closed for each $a \in R^n$. This condition generalizes ordinary semi-continuity, but does not reduce to continuity when $\Gamma = \{0\}$; we call it *weak* Γ *semi-continuity* of g. The condition may be strengthened to define Γ semi-continuity of g by requiring $g^{-1}(C - \Gamma)$ to be closed whenever $C \subset R^n$ and $C - \Gamma$ is closed; then $\{0\}$ semi-continuity coincides with continuity.

In Theorem 2.16 of [**4**], it is shown that the range of a Γ semi-continuous function on a compact space is Γ semi-closed. The proof suggests the usefulness

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of weakening the Heine-Borel property to pertain only to open coverings by sets of the form $R^n\backslash(C - \Gamma)$. This condition we call Γ *semi-compactness*. The Heine-Borel theorem states that a closed bounded subset of R^n is compact. Theorem 2.10 below, our main result, generalizes this statement in terms of the "semi" concepts, however semi-boundedness must be strengthened, as shown by Example 2.11. Theorem 2.1 generalizes the converse of the Heine-Borel theorem.

Our definition of semi-closedness originated with Olech [**1**; **2**], who called it *lower-closedness*, and who has discussed its applications to control theory. Unfortunately [**1**] is not easily accessible; [**2**] reviews results without proof.

We are indebted to others for personally communicated proofs of some of our conjectures in this development. Mr. David H. Wagner proved Theorem 2.16 of [**4**] and did so in a way which suggested the concept of Γ semi-compactness and which is the essence of the proof of semi-closedness in Theorems 2.1 and 2.2 below. He also contributed the first example in 2.3. Professor Victor Klee proved Lemma 2.7 below, which is of independent interest. Our realization that $\mathscr{T}_\Gamma$ (§ 2) is a topology arose from questions by Professor Harry W. McLaughlin.

We now proceed with our formal development. The successive sections treat semi-compactness, semi-boundaries, and semi-continuity.

Throughout this paper, Γ *is a closed convex subcone of* R^n. That Γ is a *cone* means $r\gamma \in \Gamma$ whenever $\gamma \in \Gamma$ and $0 \leqq r \in R$.

We denote the usual inner product in R^n by $x \cdot y$, and the Euclidean norm by $\|\ \|$. Suppose $A, B \subset R^n$ and $a \in R^n$. Then $A + B$, $A - B$, $a + A$, etc., refer to the obvious vector set sums. We denote the convex hull of A by co A, the closure of A by cl A, and the interior of A (R^n topology) by int A. By $\bar{A}$ we mean $R^n\backslash A$.

2. Γ semi-compactness.

In this section we develop a generalization of the Heine-Borel theorem and its converse, Theorems 2.10 (our main result) and 2.1 respectively. We begin with the underlying definitions.

Suppose $A \subset R^n$ is convex. The *asymptotic cone* of A (often called the *characteristic cone* of A), in symbols $\mathscr{A}(A)$, is defined by

$$\mathscr{A}(A) = R^n \cap \{\gamma : A + \gamma \subset A\}, \quad \text{when } A \neq \emptyset.$$

We agree that $\mathscr{A}(\emptyset) = \{0\}$. Always $\mathscr{A}(A)$ is a convex cone and if A is closed, so is $\mathscr{A}(A)$. If $\mathscr{A}(\text{cl } A) = \{0\}$, A is bounded. If $A \subset B \subset R^n$, $\mathscr{A}(A) \subset \mathscr{A}(\text{cl co } B)$. These and other properties of asymptotic cones are given in Chapter 8 of [**3**] (where they are called *recession cones*) and in Lemma 2.2 of [**4**].

Suppose $A \subset R^n$. We say A is Γ *semi-closed* if cl $A \subset A + \Gamma$ and Γ *semi-bounded* if $\mathscr{A}(\text{cl co } A) \subset \Gamma$. We say A is Γ *semi-compact* if every open covering of A by sets of the form $\overline{C - \Gamma}$ has a finite subcovering, i.e., whenever I is a set, $C_i \subset R^n$ and $C_i - \Gamma$ is closed for $i \in I$, and $A \subset \bigcup_{i\in I} \overline{C_i - \Gamma}$, there exists a finite set $J \subset I$ such that $A \subset \bigcup_{i\in J} \overline{C_i - \Gamma}$. When $\Gamma = \{0\}$, these terms reduce to their usual meaning without the prefix "semi." We also say A

is *strongly* Γ *semi-bounded* if $\mathscr{A}(\text{cl co } A) \subset \{0\} \cup \text{int } \Gamma$ and *weakly* Γ *semi-compact* if every open covering of A by sets of the form $\overline{a - \Gamma}$, $a \in R^n$, has a finite subcovering. Examples will appear below.

An alternative approach to Γ semi-compactness is to define

$$\mathscr{T}_\Gamma = \{\overline{C - \Gamma} : C \subset R^n \text{ and } C - \Gamma \text{ is closed}\}.$$

Then $\mathscr{T}_\Gamma$ is a topology over R^n. However, if $\Gamma \neq \{0\}$, $\mathscr{T}_\Gamma$ is not a very interesting topology, since it does not satisfy the T_1 separation axiom. If Γ does not contain a line, $\mathscr{T}_\Gamma$ is a T_0 space, i.e., for $x, y \in R^n$ with $x \neq y$, there exists $U \in \mathscr{T}_\Gamma$ such that $[x \in U \text{ and } y \notin U]$ or $[x \notin U \text{ and } y \in U]$. If Γ contains a line, $\mathscr{T}_\Gamma$ is not even T_0. In any event, Γ semi-compactness coincides with $\mathscr{T}_\Gamma$ compactness. Accordingly, an infinite Γ semi-compact set has a $\mathscr{T}_\Gamma$ accumulation point; when $\Gamma = \{0\}$, this reduces to the Bolzano-Weierstrass theorem. However, Γ semi-closedness and Γ semi-boundedness do not seem to relate directly to $\mathscr{T}_\Gamma$.

THEOREM 2.1. *If $A \subset R^n$ is Γ semi-compact, then A is Γ semi-closed*† *and Γ semi-bounded.*

Proof. To show A is Γ semi-closed, suppose $a \in \text{cl } A$ and $a \notin A + \Gamma$. For $r > 0$, let $C_r = R^n \cap \{z : ||z - a|| \leqq r\}$. Since Γ is closed,

$$\bigcap_{r>0} (C_r - \Gamma) = a - \Gamma.$$

Since $(a - \Gamma) \cap A = \emptyset$, we have

$$A \subset \overline{\bigcap_{r>0} (C_r - \Gamma)} = \bigcup_{r>0} \overline{C_r - \Gamma}.$$

Since A is Γ semi-compact and the covering is nested, there exists $r_0 > 0$ such that $A \subset \overline{C_{r_0} - \Gamma} \subset \overline{C_{r_0}}$, contrary to $a \in \text{cl } A$. Hence A is Γ semi-closed.

Suppose $\gamma \in \mathscr{A}(\text{cl co } A)$ and $\gamma \notin \Gamma$. Let $b \in A$. Since Γ is convex and closed we may choose a closed half-space H with 0 in its boundary such that $\gamma \notin H \supset \Gamma$. Take $w \in R^n$ such that $H = R^n \cap \{z : w \cdot z \leqq 0\}$. Then $w \cdot \gamma > 0$. Define the closed sets

$$D_r = b + r\gamma - H \quad \text{for } r > 0.$$

To see that

$$A \subset \bigcup_{r>0} \overline{D_r - \Gamma} = \bigcup_{r>0} \overline{D_r},$$

let $c \in A$ and choose $s > \max\{0, [w \cdot c - w \cdot b]/w \cdot \gamma\}$; then $w \cdot [b + s\gamma - c] > 0$, so $c \notin D_s$.

Since A is Γ semi-compact, there exists $r_1 > 0$ such that $A \subset \overline{D_{r_1}}$. Since

†For this much Γ could be an arbitrary closed set such that $0 \in \Gamma \subset R^n$.

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$\gamma \in \mathscr{A}(\text{cl co } A)$, we have $b + 2r_1\gamma \in \text{cl co } A \subset \text{cl } \overline{D_{r_1}} = b + r_1\gamma + H$, contrary to $\gamma \notin H$. Therefore $\gamma \in \Gamma$.

THEOREM 2.2. *If $A \subset R^n$ is weakly Γ semi-compact and* $\text{int } \Gamma \neq \emptyset$, *then A is Γ semi-closed.*

Proof. Let $\gamma \in \text{int } \Gamma$ and $a \in \text{cl } A$ and suppose $A \cap (a - \Gamma) = \emptyset$. For $b \in A$, letting s be the distance from b to $a - \Gamma$, we have $s > 0$ (since Γ is closed) and $b \notin a + [\frac{1}{2} s\gamma/||\gamma||] - \Gamma$. Thus $\{\overline{a + r\gamma - \Gamma} : r > 0\}$ is a nested open covering of A. Hence there exists $r_0 > 0$ such that $A \subset \overline{a + r_0\gamma - \Gamma}$. But $a \in \text{int } (a + r_0\gamma - \Gamma)$, so we have contradicted $a \in \text{cl } A$.

Example 2.3. We may not omit "$\text{int } \Gamma \neq \emptyset$" in Theorem 2.2: Let $n = 2$, $A = \{(x, y) : x^2 + y^2 \leqq 1 \text{ and } x > 0\}$ and $\Gamma = \{(0, y) : y \geqq 0\}$ (due to David Wagner). Also we may not conclude in Theorem 2.2 that A is Γ semi-bounded: Let $n = 2$, $A = \{(x, y) : x = -y\}$ and $\Gamma = \{(x, y) : x \geqq 0, y \geqq 0\}$.

LEMMA 2.4. *If $A \subset R^n$ is bounded and Γ semi-closed, then A is Γ semi-compact.*

Proof. Suppose $C_i \subset R^n$ and $\overline{C_i - \Gamma}$ is open for $i \in I$ and $A \subset \bigcup_{i \in I} \overline{C_i - \Gamma}$. Since $\text{cl } A \subset A + \Gamma$, it follows that $\text{cl } A \subset \bigcup_{i \in I} \overline{C_i - \Gamma}$. Since $\text{cl } A$ is compact it has a finite subcovering, which also covers A.

LEMMA 2.5. *If $A \subset R^n$ is Γ semi-closed, $C \subset R^n$, and $C - \Gamma$ is closed, then $A \cap (C - \Gamma)$ is Γ semi-closed.*

Proof. We have

$$\text{cl } [A \cap (C - \Gamma)] \subset \text{cl } A \cap (C - \Gamma) \subset (A + \Gamma) \cap (C - \Gamma) \subset [A \cap (C - \Gamma)] + \Gamma.$$

LEMMA 2.6. *Suppose $a \in R^n$ and Δ is a closed subcone of $\{0\} \cup$* int Γ. *Then $(a + \Delta)\backslash\Gamma$ is bounded.*

Proof. It suffices to show that there exists $r \geqq 0$ such that

$$\{a + \beta : \beta \in \Delta \text{ and } ||\beta|| \geqq r\} \subset \Gamma.$$

Choose r_0 such that $0 < r_0 < 1$ and

$$[\delta \in R^n, \gamma \in \Delta, ||\delta|| = ||\gamma|| = 1, \text{ and } ||\delta - \gamma|| \leqq \sqrt{2}\, r_0] \text{ implies } \delta \in \Gamma. \tag{2.1}$$

Let $r = ||a||/r_0$. Suppose $\beta \in \Delta$ and $||\beta|| \geqq r$. We may assume $a \neq 0$. Let

$$\alpha = \frac{\beta}{||\beta||} + \frac{r_0 a}{||a||}.$$

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Since $0 < r_0 < 1, \alpha \neq 0$. We have

$$\left\| \frac{\alpha}{\|\alpha\|} - \frac{\beta}{\|\beta\|} \right\|^2 = 2 - \frac{2\alpha \cdot \beta}{\|\alpha\| \, \|\beta\|} = 2 + \frac{1}{\|\alpha\|} \left[\left\| \alpha - \frac{\beta}{\|\beta\|} \right\|^2 - \alpha \cdot \alpha - 1 \right]$$

$$(2.2) \qquad = \frac{1}{\|\alpha\|} [2\|\alpha\| + r_0^2 - \|\alpha\|^2 - 1]$$

$$= \frac{1}{\|\alpha\|} [2\|\alpha\|(1 - r_0^2) - \|\alpha\|^2 - (1 - r_0^2) + 2\|\alpha\|r_0^2] \leqq 2r_0^2.$$

It follows from (2.1) and (2.2) that $\alpha/\|\alpha\| \in \Gamma$. Hence,

$$a + \frac{r}{\|\beta\|}\beta = a + \frac{\|a\|}{\|\beta\| r_0}\beta = \frac{\|a\|}{r_0}\alpha \in \Gamma.$$

Since $\|\beta\| \geqq r$, $\beta - [r\beta/\|\beta\|] \in \Delta \subset \Gamma$. Since Γ is a convex cone,

$$a + \beta = \left[a + \frac{r\beta}{\|\beta\|} + \beta - \frac{r\beta}{\|\beta\|} \right] \in \Gamma + \Gamma = \Gamma.$$

LEMMA 2.7 (proved by Klee). *Suppose $A \subset R^n$ is strongly Γ semi-bounded, and $\gamma \in \operatorname{int} \Gamma$. Then there exists $r \geqq 0$ such that $A \subset -r\gamma + \Gamma$.*

Proof. Suppose the conclusion fails. Then there exist, for $i = 1, 2, \ldots$, $r_i \geqq 0$ and $a_i \in A$ such that

$$(2.3) \qquad a_i + r_i\gamma \notin \Gamma$$

and such that $r_i \to \infty$. We may assume without loss of generality that either $a_i \to a \in R^n$ or $\|a_i\| \to \infty$ and $a_i/\|a_i\| \to u \in R^n$. In the first case $a_i/r_i \to 0$ and since $\gamma \in \operatorname{int} \Gamma$ we have for all sufficiently large i,

$$a_i/r_i + \gamma \in \Gamma,$$

whence $a_i + r_i\gamma \in r_i\Gamma \subset \Gamma$, contrary to (2.3). In the second case, $u \in \mathscr{A}(\text{cl co } A)\backslash\{0\} \subset \operatorname{int} \Gamma$, whence for all sufficiently large i we have $a_i/\|a_i\| \in \Gamma$ and

$$\frac{a_i}{\|a_i\|} + \frac{r_i}{\|a_i\|}\gamma \in \Gamma + \Gamma = \Gamma,$$

so that $a_i + r_i\gamma \in \|a_i\|\Gamma \subset \Gamma$, and again (2.3) is contradicted.

Example 2.8. We may not omit "strongly" in Lemma 2.7, even if we require $\operatorname{int} \Gamma \neq \emptyset$ and we weaken the conclusion to assert that $A \subset b + \Gamma$ for some $b \in R^n$: Let $n = 2$, $A = \{(x, y) : x = y^2 \text{ or } y = x^2\}$, and $\Gamma = \mathscr{A}(\text{co } A)$ $(= \{(x, y) : x \geqq 0, y \geqq 0\})$.

LEMMA 2.9. *Suppose A, $C \subset R^n$, A is strongly Γ semi-bounded, and $A \cap (C - \Gamma)$ is unbounded. Then $A \subset C - \Gamma$.*

Proof. We may choose a closed convex subcone Δ of R^n such that

$$\mathscr{A}(\text{cl co }[A \cap (C - \Gamma)]) \subset \mathscr{A}(\text{cl co } A) \subset \{0\} \cup \text{int } \Delta \subset \Delta \subset \{0\} \cup \text{int } \Gamma.$$

Suppose $a \in A$. Since $A \cap (C - \Gamma)$ is unbounded, int $\Delta \neq \emptyset$ and we may apply Lemma 2.7 to choose $b \in R^n$ such that $A \cap (C - \Gamma) \subset b + \Delta$. By Lemma 2.6, $(b - a + \Delta)\backslash\Gamma$ is bounded, hence so is $(b + \Delta)\backslash(a + \Gamma)$, and hence so is $[A \cap (C - \Gamma)]\backslash(a + \Gamma)$. Since $A \cap (C - \Gamma)$ is unbounded,

$$[A \cap (C - \Gamma)] \cap (a + \Gamma) \neq \emptyset.$$

Thus, $(a + \Gamma) \cap (C - \Gamma) \neq \phi$, so $a \in C - \Gamma$.

THEOREM 2.10. *Suppose $A \subset R^n$ is Γ semi-closed and strongly Γ semi-bounded. Then A is Γ semi-compact.*

Proof. Suppose $C_i \subset R^n$ and $C_i - \Gamma$ is closed for $i \in I$ and $A \subset \cup_{i \in I} \overline{C_i - \Gamma}$. For $i \in I$, by Lemma 2.9, $A \cap \overline{C_i - \Gamma} = \emptyset$ if $A \cap (C_i - \Gamma)$ is unbounded. Hence for some $j \in I$, $A \cap (C_j - \Gamma)$ is bounded. By Lemmas 2.5 and 2.4, $A \cap (C_j - \Gamma)$ is Γ semi-compact, so for some finite subset J of $I\backslash\{j\}$,

$$A \cap (C_j - \Gamma) \subset \bigcup_{i \in J} \overline{C_i - \Gamma},$$

whence $A \subset \cup_{i \in J \cup \{j\}} \overline{C_i - \Gamma}$.

Example 2.11. We may not omit "strongly" in Theorem 2.10 even when int $\Gamma \neq \emptyset$: Let

$$n = 2, \quad A = \{(x, y) : y \geqq x^2\},$$
$$\Gamma = \{(x, y) : x \geqq 0 \text{ and } y \geqq 0\},$$

and

$$C_r = \{(x, y) : y \in R, x \leqq r\} \text{ for } r \in R.$$

3. Γ semi-boundaries. We now formalize the concept of Γ semi-boundary and, as foretold in § 1, relate it to Γ semi-closedness.

For $A \subset R^n$ we define the Γ *semi-boundary* of A to be

$$R^n \cap \{a : (a - \Gamma) \cap \text{cl } A = \{a\}\},$$

unless $\Gamma = \{0\}$ in which case it is defined as the boundary of A. In § 2 of [**4**], this concept is compared with Yu's [**6**] set of "cone extreme" points.

Theorems 3.1 and 3.2 hold without assuming that Γ is closed (see [**4**]), although then the proof of Theorem 3.2 (i) is somewhat harder.

THEOREM 3.1. *If $A \subset R^n$ is Γ semi-closed, A contains its Γ semi-boundary.*

THEOREM 3.2. *Suppose $A \subset R^n$, Γ does not contain a line, and $(-\Gamma) \cap \mathscr{A}(\text{cl co } A) = \{0\}$. Then*

(i) *if $\Gamma \neq \{0\}$ and $a \in \text{cl } A$, there is a b in the Γ semi-boundary of A such that $a - b \in \Gamma$;*

(ii) *if A contains its Γ semi-boundary, A is Γ semi-closed;*

(iii) *if $C_i \subset R^n$ for $i \in I$ and $\bigcup_{i\in I} \overline{C_i - \Gamma}$ contains the Γ semi-boundary of A, it also contains* cl *A;*

(iv) *if A is Γ semi-compact, so is the Γ semi-boundary of A.*

Proof. Conclusions (i) and (ii) are given as Lemmas 2.8, 2.9, and 2.10 in [**4**] ((ii) follows from (i)), (iii) follows from (i), and (iv) follows from (iii).

4. Γ semi-continuity. We conclude by defining Γ semi-continuous functions in such a way that it is obvious that they map compact sets onto Γ semi-compact sets. Theorems 2.1 and 2.2 make Theorem 4.1 meaningful.

If f maps a topological space into R^n, we say f is Γ *semi-continuous* if $f^{-1}(A - \Gamma)$ is closed whenever $A \subset R^n$ and $A - \Gamma$ is closed, i.e., if f is $\mathscr{T}_\Gamma$ continuous. We say f is *weakly* Γ *semi-continuous* if $f^{-1}(a - \Gamma)$ is closed for each $a \in R^n$.

THEOREM 4.1. *Suppose T is a compact space and $f: T \to R^n$ is (weakly) Γ semi-continuous. Then $f(T)$ is (weakly) Γ semi-compact.*

Proof. The unbracketed statement holds since f is $\mathscr{T}_\Gamma$ continuous. Proof of the bracketed statement is similar to the well-known proof for continuous maps.

THEOREM 4.2. *Suppose T is a topological space, $f: T \to R^n$, and $\gamma \cdot f$ is upper semi-continuous for each $\gamma \in \Gamma^p$, where Γ^p is $\{\delta : \delta \cdot \delta' \leqq 0$ for $\delta' \in \Gamma\}$, the polar cone of Γ. Then f is weakly Γ semi-continuous.*

Proof. By Theorem 14.1 of [**3**], $\Gamma^{pp} = \Gamma$, so for $a \in R^n$ we have

$$f^{-1}(a - \Gamma) = \{t : \gamma \cdot f(t) \geqq \gamma \cdot a \text{ for } \gamma \in \Gamma^p\} = \bigcap_{\gamma \in \Gamma^p} \{t : \gamma \cdot f(t) \geqq \gamma \cdot a\},$$

hence $f^{-1}(a - \Gamma)$ is closed.

THEOREM 4.3. *Suppose S and T are topological spaces, $g: S \to T$ is continuous, $f: T \to R^n$ is Γ semi-continuous, $h: R^n \to R^k$ is linear, and $h^{-1}(h(\Gamma)) = \Gamma$. Then $f \circ g$ is Γ semi-continuous and $h \circ f$ is $h(\Gamma)$ semi-continuous.*

REFERENCES

1. C. Olech, *Integrals of set valued functions and linear optimal control problems*, IFAC, Fourth Congress of the International Federation of Automatic Control, Warsaw, June 16–21 (1969), 22–35.

2. ——— *Integrals of set-valued functions and linear optimal control problems*, Proceedings of the Colloquium on Optimal Control Theory, Brussels (1969), 109–125.

3. R. T. Rockafellar, *Convex analysis* (Princeton University Press, 1970).

4. D. H. Wagner, *Integral of a set-valued function with semi-closed values*, Journal of Mathematical Analysis and Applications, *55* (1976), 616–633.

5. D. H. Wagner and L. D. Stone, *Necessity and existence results on constrained optimization of separable functionals by a multiplier rule*, SIAM Journal on Control *12* (1974), 356–372.

6. P. L. Yu, *Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives*, J. Optimization Theory Appl. *14* (1974), 319–377.

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