Can. J. Math., Vol. XL, No. 4, 1988, pp. 865-988

SEMICONTINUITY AND MULTIPLIERS OF C^*-ALGEBRAS

LAWRENCE G. BROWN

1. Introduction. In **[5]** C. Akemann and G. Pedersen defined four concepts of semicontinuity for elements of A^{**}, the enveloping W^*-algebra of a C^*-algebra A. For three of these the associated classes of lower semicontinuous elements are $\overline{A_{sa}^m}$, $\widetilde{A}_{sa}^m$, and $(\widetilde{A}_{sa}^m)^-$ (notation explained in Section 2), and we will call these the classes of *strongly lsc*, *middle lsc*, and *weakly lsc* elements, respectively. There are three corresponding concepts of continuity: The strongly continuous elements are the elements of A itself, the middle continuous elements are the multipliers of A, and the weakly continuous elements are the quasi-multipliers of A. It is natural to ask the following questions, each of which is three-fold.

(Q1) Is every lsc element the limit of a monotone increasing net of continuous elements?

(Q2) Is every positive lsc element the limit of an increasing net of positive continuous elements?

(Q3) If $h \geqq k$, where h is lsc and k is usc, does there exist a continuous x such that $h \geqq x \geqq k$?

We give affirmative answers to (Q1) and (Q2) for separable A in the strong and weak cases. For the middle case the answer to (Q1) is trivially yes and the answer to (Q2) was already known to be no. For (Q3) we give affirmative answers for arbitrary A in the strong case and for σ-unital (in particular, separable) A in the weak case. In the middle case the answer to (Q3) is no in general, but in Theorem 3.40 we give a positive result with strengthened hypotheses on h, k. Although the hypothesis of Theorem 3.40 is not as natural as one would like, it has so far been adequate for the applications which have occurred to us. We consider any technique for constructing multipliers to be potentially valuable, in part because of the use of multipliers in KK-theory, and urge the reader to look for improvements to or new proofs of Theorem 3.40.

A positive answer to (Q1) in the strong case is the same as the statement that A_{sa}^m, the smallest class of lower semicontinuous elements defined in **[5]**, is equal to $\overline{A_{sa}^m}$. Our intuitive feeling is that, regardless of the answer to (Q1), A_{sa}^m should not be regarded as giving a fourth concept of semi-

Received July 28, 1987. This research was done while the author was visiting the Mathematical Sciences Research Institute and was partially supported by M.S.R.I. and the National Science Foundation.

continuity, but rather should be regarded as an important sub-class of the class of strongly lsc elements. That is why we have chosen to speak of only three types of semicontinuity. The results of [**5**] on strong semicontinuity are quite powerful, and so far we know the problem considered in Section 4 (described below) was the first one that required an answer to (Q1). (Actually it is (Q2) that is needed for Section 4, and it is only in the separable case that (Q2) is the right question.) In any case the results on (Q1) are probably enough to convince the reader that our choice of terminology is justified.

The plan of the paper is as follows. Section 2 establishes the notation and proves a number of elementary or specialized results. Some of the results of Section 2 are used in later sections, and some are just facts that we consider interesting or potentially useful. In Theorem 2.36 we identify a sub-class of operator convex functions which is characterized by an operator inequality stronger than the usual one for operator convex functions, and which is also characterized in other ways (one of them related to semicontinuity). The function $x \mapsto 1/x$, $x > 0$, is in this sub-class. Section 3 includes the results on (Q1), (Q2), and (Q3) mentioned above, some applications, and also a number of results that can be considered noncommutative Tietze extension theorems. An example of the latter is Corollary 3.11: If L is a closed left ideal of a σ-unital C^*-algebra A, and $\theta:A \to A/L$ is a homomorphism of left A-modules, then θ can be lifted to a module homomorphism $\bar{\theta}:A \to A$ such that $\|\bar{\theta}\| = \|\theta\|$. An application of Theorem 3.40 is: If a σ-unital C^*-algebra $A = B + I$, where B is a hereditary C^*-subalgebra and I a closed two-sided ideal, and if h is a multiplier of A, then $h = h_1 + h_2$ where h_1 is a multiplier of A that is supported by B and h_2 a multiplier of A supported by I. Section 4 deals with the question: Given $0 \leqq h \in A^{**}$, when is $h = T^*T$ for T a right multiplier or quasi-multiplier? (The case T a left multiplier was dealt with in [**10**], and the case T a multiplier is trivial.) For A separable and stable the answer is that h must be strongly or weakly lsc, respectively. A related theorem is that if A is separable, then the norm closed complex vector space generated by the lsc elements of A^{**} is a C^*-algebra. If A is also stable, this C^*-algebra is the one generated by the quasi-multipliers. Section 4 also contains some density results, some of which are applications of the main results. For example, if A is separable and stable and $0 \leqq h \in (\tilde{A}^m_{sa})^-$, then

$$\{T \in QM(A):T^*T = h\}$$

is right strictly dense in $\{T \in QM(A):T^*T \leqq h\}$. Section 5 discusses several examples. None of these examples is exotic, though we do deal in some sense with arbitrary C^*-subalgebras of separable continuous trace algebras. We give criteria for the three types of semicontinuity, and for some of the examples we also discuss some of the questions raised in

Sections 3 and 4. Some of our results for the example $A = \mathcal{K}$, the algebra of compact operators on separable Hilbert space, may be new. In particular, $\{K \in \mathcal{K}: 0 \leqq K \leqq 1\}$ is directed upward.

In the field of "non-commutative topology" it is common for operator algebraists to gain intuition from analogies with the commutative case. For the subject of this paper this can lead to pitfalls, and a more complicated example should be used for analogies; namely, $A = C_0(X) \otimes \mathcal{K}$. For this example the elements of A ("strongly continuous" elements) are the norm continuous $\mathcal{K}$-valued functions on X vanishing at ∞, the elements of $M(A)$ ("middle continuous" elements) are the bounded double strongly continuous $B(H)$-valued functions on X, and the elements of $QM(A)$ ("weakly continuous" elements) are the bounded weakly continuous $B(H)$-valued functions on X. Pitfalls can also arise if one forgets that the elements of A vanish at ∞.

To some extent this paper is a sequel to **[10]**. However, for the most part no knowledge of **[10]** is assumed.

We gratefully acknowledge helpful conversations with C. Akemann, J. Anderson, E. Effros, J. Mingo, D. Olesen, and G. Pedersen.

2. Elementary or specialized results. The reader ought to be familiar with the basic results of **[28]** and **[5]**. Alternatively, Sections 3.11 and 3.12 of **[29]** should provide adequate background. It would not be appropriate to review all of the background material, and we merely explain the notation. For $M \subset A^{**}$, M_{sa} denotes $\{x \in M: x^* = x\}$ and

$$M_+ = \{x \in M: x \geqq 0\}.$$

For $M \subset A^{**}_{sa}$, M^m denotes the set of limits in A^{**} of monotone increasing nets of elements of M, M^σ the set of limits of increasing sequences, and M_m the set of limits of decreasing nets. $\tilde{A} = A + \mathbf{C} \cdot 1$, the result of adjoining a unit to A, and $^-$ means norm closure, unless some other topology is explicitly indicated.

$$M(A) = \{x \in A^{**}: xA, Ax \subset A\},$$

$$LM(A) = \{x \in A^{**}: xA \subset A\},$$

$$RM(A) = \{x \in A^{**}: Ax \subset A\}, \text{ and}$$

$$QM(A) = \{x \in A^{**}: AxA \subset A\}.$$

If $M \subset A$, her(M) denotes the smallest hereditary C^*-subalgebra of A containing M. If $q \in A^{**}$ is an open projection, her(q) denotes the corresponding hereditary C^*-subalgebra of A. If $q \in M(A)$, her(q) is called a corner of A. Ideals are closed and two-sided unless otherwise indicated.

$$\Delta(A) = \{\varphi \in A^*: \varphi \geqq 0 \text{ and } \|\varphi\| \leqq 1\},$$

$S(A)$ is the state space of A, and $P(A)$ is the set of pure states. A is called σ-unital if it has a strictly positive element, or equivalently a countable approximate identity. $\sigma(x)$ denotes the spectrum of x, and $E_S(h)$ is the spectral projection of h corresponding to the Borel set $S \subset \mathbf{R}$ $(h \in A_{sa}^{**})$. χ_S is the characteristic function of S, and co denotes convex hull.

Some of the results of this section may be known to experts, even if they have not appeared in print. In particular several were proved in Section 2.2 of [**15**] in the case of unital algebras. Also 2.D is based on things told to us by C. Akemann or G. Pedersen, and we disclaim originality for most of it.

Since not all of Section 2 is used in the rest of the paper, we offer some guidelines for the reader who wants to skip some on a first reading. Of the five subsections, only (parts of) A and D are used importantly in the main sections. Parts of B are also used. Theorem 2.36 (part of 2.C) is entirely independent of 2.B. There are relations between C, D, and E, but these have nothing to do with the later parts of the paper.

There are many examples in the paper, and we now establish notations for them which will be used throughout the paper. In dealing with $\mathscr{K}$, we will denote by $e_1, e_2, \dots$ a standard orthonormal basis for the Hilbert space H on which $\mathscr{K}$ operates. $v \times w$, $v, w \in H$, denotes the rank one operator $x \mapsto (x, w)v$. M_n is the C^*-algebra of $n \times n$ matrices, which we will consider embedded in $\mathscr{K}$; $a \in M_n$ is identified with $\sum a_{ij}e_i \times e_j$. $M_{k,l}$ is the space of $k \times l$ matrices. $E_1 = c \otimes \mathscr{K}$, the algebra of (norm) convergent sequences in $\mathscr{K}$. An element h of E_1^{**} is identified with a bounded collection, $\{h_n : 1 \leqq n \leqq \infty, h_n \in B(H)\}$.

$$E_2 = c \otimes M_2, \quad E_3 = \left\{x \in E_2 : x_\infty = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}\right\},$$

$$E_4 = \left\{x \in E_2 : x_\infty = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}, \quad \text{and} \quad E_5 = c_0 \otimes M_2.$$

The notation used in dealing with all these algebras is similar to that for E_1. $E_6 = \mathscr{K} + \mathbf{C}p$, where $p \in B(H)$ is a projection with infinite rank and co-rank. E_6 can also be described in an algebra of 2×2 operator matrices:

$$E_6 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in \widetilde{\mathscr{K}}, b, c, d \in \mathscr{K}\right\}.$$

In using these algebras for counterexamples, we will need criteria for deciding whether $h \in A_{sa}^{**}$ is semicontinuous. These criteria are proved in Section 5. (5.A and 5.C through 5.F.) The reader may want to glance ahead to read these criteria, but we suggest that it is not necessary to read the proofs before reading the rest of the paper. There is no circularity; our reason for presenting the material in this order is that we want to discuss some of the questions raised in Sections 3 and 4 for some of the examples.

Since the criteria for semicontinuity in E_1^{**} (5.C) are used in proving 2.36, we have arranged it so that the reader who wishes can read the proofs in 5.C before reading 2.C without undue difficulty. (2.36 is never used in the rest of the paper.)

2.A. *Basic facts.*

2.1. PROPOSITION. *Assume* $0 < \epsilon \leqq h \in A^{**}$. *Then*
(a) $h \in [(\widetilde{A}_{sa})_m]^- \Leftrightarrow h^{-1} \in \overline{A^m_{sa}}$.
(b) $h \in (\widetilde{A}_{sa})_m \Leftrightarrow \exists \delta > 0$ *such that* $h^{-1} - \delta \in \overline{A^m_{sa}}$.
(c) *It is impossible that* $h \in [(A_{sa})_m]^-$ *unless* $1 \in A$.

Proof. (a) is Proposition 3.5 of **[5]**. (Also (a) follows from (b).)
(b). If $h \in (\widetilde{A}_{sa})_m$, then $x_\alpha \searrow h$, where

$$x_\alpha \in \lambda_\alpha + A_{sa} \subset \widetilde{A}_{sa}.$$

Here (λ_α) is decreasing and positive (if $1 \in A$ we can choose λ_α as we please). Then

$$\lambda_\alpha^{-1} + A_{sa} \ni x_\alpha^{-1} \nearrow h^{-1}.$$

Choose $0 < \delta < \lambda_{\alpha_0}^{-1}$. Then

$$(x_\alpha^{-1} - \delta) \nearrow (h^{-1} - \delta) \quad \text{and} \quad x_\alpha^{-1} - \delta \in \lambda_\alpha^{-1} - \delta + A_{sa}.$$

Since $\lambda_\alpha^{-1} - \delta > 0$ (for α sufficiently large),

$$x_\alpha^{-1} - \delta \in \overline{A^m_{sa}}.$$

By **[5]** this implies

$$h^{-1} - \delta \in \overline{A^m_{sa}}.$$

If $\exists \delta > 0$ such that $h^{-1} - \delta \in \overline{A^m_{sa}}$, we may assume δ is small enough that $h^{-1} - \delta$ is still positive. Then by **[5]**

$$h^{-1} - \frac{\delta}{2} \in A^m_+.$$

If $a_\alpha \nearrow h^{-1} + \delta/2$, $a_\alpha \in A_+$, then

$$a_\alpha + \frac{\delta}{2} \nearrow h^{-1}.$$

Therefore $(a_\alpha + \delta/2)^{-1} \searrow h$.
(c). If $h \in [(A_{sa})_m]^-$, then by **[5]**

$$h - \frac{\epsilon}{2} \in (A_{sa})_m.$$

This implies $\exists a \in A$ such that

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$a \geqq h - \frac{\epsilon}{2} \geqq \frac{\epsilon}{2},$$

which implies $1 \in A$.

2.2. PROPOSITION. *Let A be a C^*-algebra, and consider the following conditions*: (i) $\forall 0 < \epsilon \leqq h \in \overline{A^m_{sa}}$, $\exists \delta > 0$ *such that*

$$h - \delta \in \overline{A^m_{sa}}.$$

(ii) $0 \leqq h \in \widetilde{A}^m_{sa} \Rightarrow h \in \overline{A^m_{sa}}$.
(iii) $\widetilde{A}^m_{sa} = (\widetilde{A}^m_{sa})^-$.
(iv) $QM(A) = M(A)$.
Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) *and* (i), (ii), (iii) $\Rightarrow$ (iv).

Remark. In Section 3.C it will be shown that if A is σ-unital, all the conditions are equivalent.

Proof. Assume $1 \notin A$, since otherwise (i)-(iv) are trivially true. Then $\forall h \in \widetilde{A}^m_{sa}$, $\exists$ a smallest $\lambda \in \mathbf{R}$ such that

$$h + \lambda \in \overline{A^m_{sa}}.$$

(By 2.1 (c) no negative invertible $h \in \overline{A^m_{sa}}$, and this implies $\{\lambda : h + \lambda \in \overline{A^m_{sa}}\}$ is bounded below.)

(i) $\Rightarrow$ (ii). Let λ be as above. If $\lambda > 0$, (i) would contradict the minimality of λ.

$$\lambda \leqq 0 \Rightarrow h = (h + \lambda) + (-\lambda) \in \overline{A^m_{sa}}.$$

(ii) $\Rightarrow$ (iii). Since $\widetilde{A}^m_{sa}$ is norm dense in $(\widetilde{A}^m_{sa})^-$ and both are invariant under translation by scalars, $(\widetilde{A}^m_{sa})_+$ is norm dense in $[(\widetilde{A}^m_{sa})^-]_+$. Thus by (ii)

$$0 \leqq h \in (\widetilde{A}^m_{sa})^- \Rightarrow h \in \overline{A^m_{sa}} \Rightarrow h \in \widetilde{A}^m_{sa}.$$

Now again using translation by scalars, we see that (iii) is true.

(iii) $\Rightarrow$ (i). $0 < \epsilon \leqq h \in \overline{A^m_{sa}} \Rightarrow h^{-1} \in \overline{[(\widetilde{A}_{sa})_m]^-}$ (by 2.1 (a)) $\Rightarrow h^{-1} \in (\widetilde{A}_{sa})_m$ (by (iii)) $\Rightarrow \exists \delta > 0$ such that $h - \delta \in \overline{A^m_{sa}}$ (by 2.1 (b)).

(iii) $\Rightarrow$ (iv) follows easily from **[28]** and **[5]**.

2.3. PROPOSITION.

$$M(A)_{sa} = \widetilde{A}^m_{sa} \cap [(\widetilde{A}_{sa})_m]^- = (\widetilde{A}^m_{sa})^- \cap (\widetilde{A}_{sa})_m.$$

Remark. In **[28]** and **[5]** it was shown that

$$M(A)_{sa} = \widetilde{A}^m_{sa} \cap (\widetilde{A}_{sa})_m \quad \text{and}$$

$$QM(A)_{sa} = (\widetilde{A}^m_{sa})^- \cap [(\widetilde{A}^m_{sa})_m]^-.$$

(Of course also $A_{sa} = \overline{A^m_{sa}} \cap [(A_{sa})_m]^-$.) The present proof is not different.

Proof. We prove the first. Let

$$x \in \widetilde{A}_{sa}^m \cap [(\widetilde{A}_{sa})_m]^-.$$

Then by [5] $x \in QM(A)$. Let $x_\alpha \nearrow x$ where $x_\alpha \in \widetilde{A}$, and let $a \in A$. Then

$$a^*x_\alpha a \nearrow a^*xa \quad \text{and} \quad a^*xa,\ a^*x_\alpha a \in A.$$

Since $a^*(x - x_\alpha)a \geqq 0$ and $a^*(x - x_\alpha)a \searrow 0$, Dini's theorem (for continuous functions on $\Delta(A)$) implies

$$\|(x - x_\alpha)^{1/2}a\|^2 = \|a^*(x - x_\alpha)a\| \to 0.$$

Therefore

$$\|(x - x_\alpha)a\| \to 0.$$

Since $x_\alpha a \in A$ and $x_\alpha a \to xa$, $xa \in A$. Since $x^* = x$, this implies $x \in M(A)$.

The next result also is just a refinement of a result of [5].

2.4. PROPOSITION. (a) *If* $h \in (\widetilde{A}_{sa}^m)^-$, *then*

$$a^*ha \in A_{sa}^m, \quad \forall a \in A.$$

(b) *If* $a^*ha \in (\widetilde{A}_{sa}^m)^-$, $\forall a \in A$, *then* $h \in (\widetilde{A}_{sa}^m)^-$. *If A is σ-unital, it is sufficient to verify this condition for a single strictly positive element a.*

Proof. (a). The map $x \mapsto a^*xa$ is positive, continuous with respect to all relevant topologies, and carries $\widetilde{A}$ into A. This shows that

$$h \in (\widetilde{A}_{sa}^m)^- \Rightarrow a^*ha \in \overline{A_{sa}^m}$$

and also that

$$h \in \widetilde{A}_{sa}^m \Rightarrow a^*ha \in A_{sa}^m.$$

Combining these, we see that

$$a_2^*a_1^*ha_1a_2 \in A_{sa}^m \quad (h \in (\widetilde{A}_{sa}^m)^-).$$

Since $A^2 = A$, (a) follows.

(b). Let $\varphi_\alpha \to \varphi$ in $S(A)$. $\forall \epsilon > 0$, $\exists a \in A$ such that $0 \leqq a \leqq 1$ and $\varphi(a) > 1 - \epsilon$. Then $\varphi_\alpha(a) > 1 - \epsilon$ for α sufficiently large. Hence

$$|\varphi_\alpha(a^*ha) - \varphi_\alpha(h)|,\ |\varphi(a^*ha) - \varphi(h)| < 2\sqrt{\epsilon}\|h\|$$

for α sufficiently large. Since

$$\varphi(a^*ha) \leqq \underline{\lim}\ \varphi_\alpha(a^*ha)$$

by hypothesis (and [5]),

$$\varphi(h) \leqq \underline{\lim}\ \varphi_\alpha(h) + 4\sqrt{\epsilon}\|h\|.$$

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

Since ϵ is arbitrary,

$$\varphi(h) \leqq \underline{\lim}\ \varphi_a(h)$$

and the result follows. In view of (a),

$$a^*ha \in (\widetilde{A}^m_{sa})^- \Rightarrow (aA)^*h(aA) \subset A^m_{sa} \Rightarrow (aA)^{-*}h(aA)^- \subset \overline{A^m_{sa}}.$$

If a is strictly positive, $(aA)^- = A$, and the last sentence follows.

Proposition 4.5 of [5] states that if $T \in QM(A)$ and $|T|, |T^*| \leqq a \in A$, then $T \in A$. Theorem 1.2 of [3] puts this into better perspective: The hypothesis $|T| \leqq a \in A$ is equivalent to

$$|T| \in \text{her}_{A^{**}}(A),$$

the hereditary C^*-subalgebra of A^{**} generated by A. The result of [5] becomes: If $T \in QM(A)$, then

$$T \in \text{her}_{A^{**}}(A) \Rightarrow T \in A.$$

Related results follow.

2.5. Proposition. *If* $h \in (\widetilde{A}^m_{sa})^-$ *and* $h \in \text{her}_{A^{**}}(A)$, *then* $h \in \overline{A^m_{sa}}$.

Proof. Let (e_α) be an approximate identity of A. Then $e_\alpha h e_\alpha \in \overline{A^m_{sa}}$ by 2.4, and $e_\alpha h e_\alpha \to h$ in norm.

2.6. Proposition. (a) *If* $T \in QM(A)$ *and* $T^*T \in \text{her}_{A^{**}}(A)$, *then* $T \in RM(A)$.

(b) *If* $T \in LM(A)$ *and* $T^*T \in \text{her}_{A^{**}}(A)$, *then* $T \in A$.

Remark. (b) applies in particular if $T \in QM(A)$ and $T^*T \in A$, since then Proposition 4.4 of [5] implies $T \in LM(A)$.

Proof. Let (e_α) be an approximate identity.

(a). $T \in QM(A) \Rightarrow Te_\alpha \in RM(A)$, and $T^*T \in \text{her}_{A^{**}}(A) \Rightarrow Te_\alpha \to T$ in norm.

(b). $T \in LM(A) \Rightarrow Te_\alpha \in A$, and again $Te_\alpha \to T$ in norm.

2.7. *Remark-Examples.* The hypothesis of (a) does not imply $T \in A$. $T \in QM(A) \Rightarrow T^*T$ weakly lsc (since $T^*e_\alpha T \nearrow T^*T$ and $T^*e_\alpha T \in QM(A)_{sa}$). The hypothesis T^*T strongly lsc would not imply special multiplier properties, but (still for $T \in QM(A)$) the hypothesis T^*T usc would have significance, in view of 4.1 and 4.4 of [5] and 2.3 above. In particular T^*T strongly usc would imply $T \in A$. (By 2.3 and the above $T^*T \in M(A)$. But every positive multiplier is strongly lsc, so that

$$T^*T \in \overline{A^m_{sa}} \cap [(A_{sa})_m]^- = A.$$

Then the earlier remark applies.) In (i) below the hypothesis of (a) is satisfied, TT^* and T^*T are strongly lsc and $TT^* \in M(A)$ but $T \notin A$ (and

$T \notin LM(A)$). In (ii) $T \in LM(A)$ and $T^*T = 1 \in M(A)$ (in particular T^*T is strongly lsc and middle usc, but $T \notin RM(A)$. If one takes the direct sum of the two examples, one obtains $S \in QM(A)$ such that SS^* and S^*S are strongly lsc but

$$S \notin LM(A) \cup RM(A).$$

(i) $A = E_1$. T is given by $T_n = e_n \times e_1$, $T_\infty = 0$.

(ii) $A = E_1$. T_∞ is the unilateral shift and (T_n) is a sequence of unitaries such that $T_n \to T_\infty$ strongly.

2.8. PROPOSITION.

(a) $\widetilde{A}^m_{sa} = M(A)^m_{sa}$

(b) $M(A)^m_+ \subset \overline{A^m_+}$

(c) $(\widetilde{A}^m_{sa})^m = (\widetilde{A}^m_{sa})^-$.

Proof. (a). One inclusion is obvious since $\widetilde{A} \subset M(A)$. For the other if $x_\alpha \in M(A)_{\text{sa}}$ and $x_\alpha \nearrow x$, choose $\lambda \in \mathbf{R}$ such that $\lambda + x_{\alpha_0} \geqq 0$. Then for α sufficiently large,

$$\lambda + x_\alpha \in M(A)_+ \Rightarrow \lambda + x_\alpha \in \overline{A^m_{sa}} \Rightarrow \lambda + x \in \overline{A^m_{sa}} \Rightarrow x \in \widetilde{A}^m_{sa}.$$

(b). This is a triviality, stated only for completeness. It is well known (and has already been used above) that $M(A)_+ \subset A^m_+$.

(c). It follows from [5] that

$$[(\widetilde{A}^m_{sa})^-]^m \subset (\widetilde{A}^m_{sa})^-,$$

and this gives one inclusion. For the other let

$$x \in (\widetilde{A}^m_{sa})^-.$$

For each n we can find $x_n \in \widetilde{A}^m_{sa}$ such that $\|x_n - x\| < 1/n$. Let

$$y_n = x_n - \frac{2}{n} \in \widetilde{A}^m_{sa}.$$

Then

$$x - \frac{3}{n} \leqq y_n \leqq x - \frac{1}{n}.$$

Then $y_{3^n} \nearrow x$.

2.9. *Remarks.* (i) By [5] $\overline{A^m_+} = (\overline{A^m_{sa}})_+$. The former notation is much more convenient.

(ii) (a) and (b) explain the remarks made about the middle cases of (Q1) and (Q2).

(iii) If $QM(A) \neq M(A)$, we see that $\widetilde{A}^m_{sa}$ is neither norm closed nor monotone (increasing) closed. It is obviously very unpleasant to work with a class of lsc elements with these failings. In the main parts of this paper we manage to avoid working directly with $\tilde{A}^m_{sa}$.

2.10. PROPOSITION. (a) *A^m_+ is boundedly quasi-strictly dense in* $[(\tilde{A}^m_{sa})^-)]_+$.

(b) *A^m_{sa} is boundedly quasi-strictly dense in $(\tilde{A}^m_{sa})^-$.*

Remarks. (i) This proof foreshadows some of the proofs of Section 3. It is not clear whether the result has any importance.

(ii) The quasi-strict topology is a sensible one to use, since by 2.4 $(\tilde{A}^m_{sa})^-$ is quasi-strictly closed.

Proof. (a). Let $x \in (\tilde{A}^m_{sa})^-$, $0 \leqq x \leqq 1$, and let

$$M = \{h \in A^m_+ : h \leqq 1\}.$$

We will show that x is in the quasi-strict closure of M. Thus we assume given $a_1, \dots, a_n, b_1, \dots, b_n \in A$ and $0 < \epsilon < 1$ and seek $h \in M$ such that

$$\|a_i h b_i - a_i x b_i\| < \epsilon, \ i = 1, \dots, n.$$

Let e be a strictly positive element for the (separable) C^*-algebra A_0 generated by $a_1, \dots, a_n, b_1, \dots, b_n$. Then there are $a'_1, \dots, a'_n, b'_1, \dots, b'_n \in A_0$ such that

$$\|a_i - a'_i e\| < \frac{\epsilon}{6(\|b_i\| + 1)}, \quad \|b_i - e b'_i\| < \frac{\epsilon}{6(\|a_i\| + 1)}.$$

This implies that $\exists \epsilon' > 0$ such that

$$\|ehe - exe\| < \epsilon' \Rightarrow \|a_i h b_i - a_i x b_i\| < \epsilon, \ i = 1, \dots, n.$$

Now let $\delta > 0$ and $y = exe + \delta \in A^m_+$ (by 2.4 and **[5]**). Since $y \leqq e^2 + \delta$, $\exists h \in A^{**}$, $0 \leqq h \leqq 1$, such that

$$y = (e^2 + \delta)^{1/2} h (e^2 + \delta)^{1/2}.$$

$$h = (e^2 + \delta)^{-1/2} y (e^2 + \delta)^{-1/2} \Rightarrow h \in A^m_+ \Rightarrow h \in M.$$

Since $\lambda^{1/2} \leqq (\lambda + \delta)^{1/2} \leqq \lambda^{1/2} + \delta^{1/2}$, $\forall \lambda \in \mathbf{R}_+$,

$$\|y - ehe\| \leqq \delta^{1/2}[\,\|e\| + \|e^2 + \delta\|^{1/2}].$$

Since also $\|y - exe\| \leqq \delta$,

$$\|exe - ehe\| < \epsilon'$$

if δ is sufficiently small.

(b). Given $x \in (\tilde{A}^m_{sa})^-$, choose $\lambda \in \mathbf{R}$ such that $x + \lambda \geqq 0$. By (a) there is a net (h_α) in A^m_+ such that $0 \leqq h \leqq \|x + \lambda\|$ and $h_\alpha \to x + \lambda$ quasi-strictly. Let (e_β) be an approximate identity for A then $h_\alpha - \lambda e_\beta \to x$ quasi-strictly.

2.B. *Subalgebras, etc.* If A_i is a C^*-algebra, $\forall i \in I$, then by the c_0-direct sum of the A_i's, we mean the C^*-algebra of functions f on I such that $f(i) \in A_i$ and $\|f(i)\| \to 0$ as $i \to \infty$. This is the appropriate concept of direct sum for C^*-algebras, as opposed to W^*-algebras, as is well known. If A is the c_0-direct sum, then A^{**} is the l_∞-direct sum of the A_i^{**}'s.

2.11. PROPOSITION. *Let A be the c_0-direct sum of C^*-algebras A_i, $i \in I$, and*

$$h = \bigoplus_{i \in I} h_i \in A_{sa}^{**}.$$

Then

(a) $h \in \overline{A_{sa}^m} \Leftrightarrow h_i \in \overline{(A_i)_{sa}^m}$, $\forall i \in I$, *and* $\forall \epsilon > 0$, $h_i \geqq -\epsilon$ *for all but finitely many* $i \in I$.

(b) $h \in (\tilde{A}_{sa}^m)^- \Leftrightarrow h_i \in [(\tilde{A}_i)_{sa}^m]^-$, $\forall i \in I$.

(c) $h \in \tilde{A}_{sa}^m \Leftrightarrow \exists \lambda$ *independent of i such that*

$$h_i + \lambda \in \overline{(A_i)_{sa}^m}, \forall i \in I.$$

Proof. (a). $\Rightarrow$: If $h \in \overline{A_{sa}^m}$, then $\forall \epsilon > 0$,

$$(h + \epsilon) \in A_{sa}^m.$$

This implies $\exists a \in A$ such that $a \leqq h + \epsilon$ ($a_i \leqq h_i + \epsilon$, $\forall i$), which implies $h_i + \epsilon \geqq -\epsilon$ for all but finitely many i. Also, examination of the map of A onto A_i makes it obvious that

$$h \in \overline{A_{sa}^m} \Rightarrow h_i \in \overline{(A_i)_{sa}^m}.$$

$\Leftarrow$: Choose $\epsilon > 0$. For each infinite set $F \subset I$, let

$$x_F = \bigoplus_{i \in F} (h_i + \epsilon).$$

Then the net (x_F) is eventually increasing. Since it is obvious that each x_F is in $\overline{A_{sa}^m}$ (even A_{sa}^m), it follows that $h + \epsilon$ is in $\overline{A_{sa}^m}$, $\forall \epsilon > 0$. Hence

$$h \in \overline{A_{sa}^m}.$$

(b) follows from (a) and 2.4.

(c). If $h \in \tilde{A}_{sa}^m$, $\exists \lambda$ such that $h + \lambda \in \overline{A_{sa}^m}$ and (a) implies

$$h_i + \lambda \in \overline{(A_i)_{sa}^m}, \quad \forall i \in I.$$

Conversely, if λ exists so that all

$$h_i + \lambda \in \overline{(A_i)_{sa}^m},$$

we may assume λ chosen large enough so that $h + \lambda \geqq 0$. Then $h_i + \lambda \geqq 0$, $\forall i \in I$, so that (a) implies

$$h + \lambda \in \overline{A_{sa}^m}.$$

2.12. *Example.* Let $A_0 = E_1$. Define $h(r) \in (A_0^{**})_{sa}$ by

$$h(r)_n = \left(\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_{n+1}\right) \times \left(\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_{n+1}\right),$$

$$h(r)_\infty = re_1 \times e_1.$$

If $r = 1/2$, then $h(r) \in QM(A_0)$ and is weakly lsc and usc, but not middle lsc or usc. If $r < 1/2$, then $h(r)$ is middle lsc, and $h(r) + \lambda$ is strongly lsc if and only if

$$\begin{pmatrix} r + \lambda & 0 \\ 0 & 0 \end{pmatrix} \leqq \begin{pmatrix} \frac{1}{2} + \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} + \lambda \end{pmatrix},$$

which is equivalent to

$$\lambda \geqq \frac{r}{1 - 2r}.$$

Thus by letting $r \to 1/2$, we can use 2.11 to construct $h \in A_{sa}^{**}$, such that h is "locally middle lsc" but not middle lsc. Here

$$A = A_0 \otimes c_0 \subset A_0 \otimes \mathscr{K} \cong E_1.$$

This example could also be done with $A_0 = E_3$,

$$h(r)_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in M_2,$$

and $h(r)_\infty = r \in \mathbf{C}$.

2.13. PROPOSITION. *If $q \in M(A)$ is a projection and $A_0 = \mathrm{her}(q) = qAq$, then the inclusion of A_0^{**} in A^{**} gives isomorphisms of $\overline{(A_0)_{sa}^m}$ with $\overline{A_{sa}^m} \cap A_0^{**}$, $(\widetilde{A}_0)_{sa}^m$ with $\widetilde{A}_{sa}^m \cap A_0^{**}$, and $[\,(\widetilde{A}_0)_{sa}^m]^-$ with $(\widetilde{A}_{sa}^m)^- \cap A_0^{**}$. Also the map $x \mapsto qxq$ gives surjections of $\overline{A_{sa}^m}$ onto $\overline{(A_0)_{sa}^m}$, $\widetilde{A}_{sa}^m$ onto $(\widetilde{A}_0)_{sa}^m$, and $(\widetilde{A}_{sa}^m)^-$ onto $[\,(\widetilde{A}_0)_{sa}^m]^-$.*

Proof. All that is required is to show that both maps preserve all three types of semicontinuity. For the map $x \mapsto qxq$ this is a complete triviality, since it carries A into A_0 and $\widetilde{A}$ into $\widetilde{A}_0$. For the inclusion of A_0^{**} in A^{**} it is necessary to observe that $-q \in \widetilde{A}_{sa}^m$, since $-q \in M(A)$, in order to see that $(\widetilde{A}_0)_{sa}^m$ maps into $\widetilde{A}_{sa}^m$. (Of course, q is the identity of A_0^{**}; in the present notation $\widetilde{A}_0 = A_0 + \mathbf{C} \cdot q$.)

2.14. PROPOSITION. *Let A_0 be a C^*-subalgebra of A, and let q be the identity of $A_0^{**} \subset A^{**}$. Then*

(a) $\overline{(A_0)_{sa}^m} = \overline{A_{sa}^m} \cap A_0^{**}$.

(b) $\widetilde{A}_{sa}^m \cap A_0^{**} \subset (\widetilde{A}_0)_{sa}^m$.

(c) $(\widetilde{A}_{sa}^m)^- \cap A_0^{**} \subset [(\widetilde{A}_0)_{sa}^m]^-$.

The reverse inclusions in (b), (c) *hold if and only if $q \in M(A)$, in particular if $q = 1$.*

Remark. q is an open projection, $\text{her}(q) = \text{her}(A_0)$.

Proof. (a). That $\overline{(A_0)_{sa}^m} \subset \overline{A_{sa}^m}$ is trivial. For the other inclusion note that $\Delta(A_0)$ is a topological quotient space of $\Delta(A)$, under $\varphi \mapsto \varphi_{|A_0}$, and that $h \in A^{**}$ is in A_0^{**} if and only if h (regarded as a function on $\Delta(A)$) factors through the quotient map. Clearly for $h \in A_0^{**}$, h is lsc as a function on $\Delta(A)$ if and only if h is lsc as a function on $\Delta(A_0)$.

(b). Assume $h \in \widetilde{A}_{sa}^m \cap A_0^{**}$ and $\lambda > 0$ is such that

$$h + \lambda \in \overline{A_{sa}^m}.$$

We claim that $h + \lambda q$ is lsc on $\Delta(A_0)$. Suppose $\varphi_\alpha \to \varphi$ in $\Delta(A_0)$. Extend φ_α to $\widetilde{\varphi}_\alpha$ in $\Delta(A)$ such that $\|\widetilde{\varphi}_\alpha\| = \|\varphi_\alpha\|$. By passing to a subnet (which is harmless in this context), we may assume $\widetilde{\varphi}_\alpha \to$ some $\widetilde{\varphi} \in \Delta(A)$. Clearly $\widetilde{\varphi}_{|A_0} = \varphi$, though possibly $\|\widetilde{\varphi}\| > \|\varphi\|$. Then by hypothesis

$$(h + \lambda)(\widetilde{\varphi}) = h(\widetilde{\varphi}) + \lambda\|\widetilde{\varphi}\| \leqq \underline{\lim}(h(\widetilde{\varphi}_\alpha) + \lambda\|\widetilde{\varphi}_\alpha\|).$$

Therefore

$$\begin{aligned}(h + \lambda q)(\varphi) &= h(\varphi) + \lambda\|\varphi\| \leqq h(\widetilde{\varphi}) + \lambda\|\widetilde{\varphi}\| \\ &\leqq \underline{\lim}(h(\widetilde{\varphi}_\alpha) + \lambda\|\widetilde{\varphi}_\alpha\|) \\ &= \underline{\lim}(h(\varphi_\alpha) + \lambda\|\varphi_\alpha\|) \\ &= \underline{\lim}(h + \lambda q)(\varphi_\alpha).\end{aligned}$$

(c). If $h \in (\widetilde{A}_{sa}^m)^- \cap A_0^{**}$ and $a \in A_0$, then by 2.4

$$a^*ha \in \overline{A_{sa}^m} \cap A_0^{**} \subset \overline{(A_0)_{sa}^m}.$$

Therefore 2.4 implies

$$h \in [(\widetilde{A}_0)_{sa}^m]^-.$$

If $q \in M(A)$, the reverse inclusions are proved just as in 2.13. If one of the reverse inclusions holds, then $-q \in (\widetilde{A}_{sa}^m)^- \Rightarrow 1 - q \in (\widetilde{A}_{sa}^m)^- \Rightarrow 1 - q$ is open (by **[5]**). q and $1 - q$ open $\Rightarrow q \in M(A)$.

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

2.15. COROLLARY. *If $p \in A^{**}$ is an open projection and $B = \text{her}(p)$, then*

$$p \in A_0^{**} \Rightarrow B = \text{her}(B \cap A_0).$$

Proof. By (a) p is open for A_0. Clearly

$$\text{her}_{A_0}(p) = B \cap A_0.$$

If (e_α) is an approximate identity for $B \cap A_0$, then $e_\alpha \nearrow p$; and this implies (e_α) is also an approximate identity for B (by Dini's theorem applied to $b^*(1 - e_\alpha)b$ as functions on $\Delta(B)$). This shows

$$B = \text{her}(B \cap A_0).$$

2.16. *Remarks.* $T \in A^{**}$ will be called *separable* if there is a separable C^*-subalgebra A_0 of A such that $T \in A_0^{**}$. This concept is most useful when A is σ-unital, since then it can be assumed that $\text{her}(A_0) = A$. Note that if $T \in A_0^{**}$, then

$$T \in QM(A) \Rightarrow T \in QM(A_0).$$

The same is true for $M(A)$, $LM(A)$, and $RM(A)$, and the converse ($QM(A_0)$, $LM(A_0)$, etc. $\subset QM(A)$, $LM(A)$, etc.) holds when $\text{her}(A_0) = A$. Also if A is σ-unital and $T \in QM(A)$, then T is separable (since $e_nTe_n \to T$ where (e_n) is a countable approximate identity); and hence any element of $QM(A)_{sa}^{\sigma}$, for example, is separable. An open projection p is separable if and only if $\text{her}(p)$ is σ-unital: One direction is trivial. For the other, apply 2.15, where A_0 separable $\Rightarrow B \cap A_0$ separable. These remarks will be used in Section 4. The point is to reduce the study of separable elements of A^{**} to the case when A itself is separable.

2.17. PROPOSITION. *If $q \in A^{**}$ is an open projection and $B = \text{her}(q)$, then*

$$q(\widetilde{A}_{sa}^m)^- q \subset (\widetilde{B}_{sa}^m)^-.$$

Proof. Let $\varphi_\alpha \to \varphi$ in $S(B)$. Let $\widetilde{\varphi}_\alpha$, $\widetilde{\varphi} \in S(A)$ be the unique norm-preserving extensions of φ_α, φ. Since each cluster point of $(\widetilde{\varphi}_\alpha)$ is an extension of φ of norm at most 1, $\widetilde{\varphi}_\alpha \to \widetilde{\varphi}$. Let

$$h \in (\widetilde{A}_{sa}^m)^-.$$

Then

$$(qhq)(\varphi) = h(\widetilde{\varphi}) \leqq \underline{\lim}\, h(\widetilde{\varphi}_\alpha) = \underline{\lim}(qhq)(\varphi_\alpha).$$

2.18. PROPOSITION. *Let I be an ideal of A with open central projection z. Then*

(a) $h \in \overline{A_+^m} \Rightarrow zh \in \overline{A_+^m}$ *and* $zh \in \overline{I_+^m}$.

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

(b) $0 \leqq h \in \widetilde{A}^m_{sa} \Rightarrow zh \in \widetilde{A}^m_{sa}$ *and* $zh \in \widetilde{I}^m_{sa}$.

(c) $0 \leqq h \in (\widetilde{A}^m_{sa})^- \Rightarrow zh \in (\widetilde{A}^m_{sa})^-$ *and* $zh \in (\widetilde{I}^m_{sa})^-$.

Remark. By 2.14 the two parts of (a) are equivalent and the first parts of (b) and (c) imply the last parts. 2.19 and 2.17 are strengthenings of the last parts of (b), (c).

Proof. (a). $zA_+ \subset M(I)_+ \subset I^m_+ \Rightarrow z\overline{A^m_+} \subset \overline{I^m_+}$.

(b). Assume $0 \leqq h \in \widetilde{A}^m_{sa}$, and take $\lambda > 0$ such that

$$h + \lambda \in \overline{A^m_{sa}}.$$

We claim that

$$zh + \lambda \in \overline{A^m_{sa}},$$

which implies the result. To see this, let $\varphi_\alpha \to \varphi$ in $\Delta(A)$. Passing to a subnet, we may assume

$$z\varphi_\alpha \to \theta \quad \text{and} \quad (1 - z)\varphi_\alpha \to \psi,$$

where $\theta + \psi = \varphi$. Since $(1 - z)\psi = \psi$,

$$\begin{aligned}\varphi(zh + \lambda) &= \theta(zh + \lambda) + \lambda\|\psi\| \\ &\leqq \theta(h + \lambda) + \lambda\|\psi\| \\ &\leqq \underline{\lim}(z\varphi_\alpha)(h + \lambda) + \lambda\, \underline{\lim}\|(1 - z)\varphi_\alpha\| \\ &\leqq \underline{\lim}[\varphi_\alpha(zh) + \lambda\|z\varphi_\alpha\| + \lambda\|(1 - z)\varphi_\alpha\|] \\ &= \underline{\lim}\ \varphi_\alpha(zh + \lambda).\end{aligned}$$

(c) follows from (b) (or from (a) via 2.4).

2.19. Corollary. $z(\widetilde{A}^m_{sa}) \subset \widetilde{I}^m_{sa}$.

Proof. If $h \in \widetilde{A}^m_{sa}$, choose $\lambda \in \mathbf{R}$ large enough that

$$h + \lambda \in \overline{A^m_+}.$$

Then

$$zh = z(h + \lambda) - \lambda z \in \overline{I^m_+} + \mathbf{R} \cdot z = \widetilde{I}^m_{sa}.$$

2.20. Corollary. *If B is a corner of an ideal of A, with open projection q, then*

$$q\overline{A^m_+}q \subset \overline{B^m_+} \quad \textit{and} \quad q\widetilde{A}^m_{sa}q \subset \widetilde{B}^m_{sa}.$$

Proof. Combine 2.13, 2.18, and 2.19.

2.21. *Remark*. The hypothesis that B be a corner of an ideal is weaker than the hypothesis that B be an ideal of a corner. In fact any ideal of a corner of an ideal is again a corner of an ideal.

2.22. COROLLARY (cf. [**5**, Proposition 3.7]). *In the notation of* 2.14 *let* $B = \text{her}(q)$.

(a) $\overline{B^m_{sa}} \subset q\overline{A^m_{sa}}q$, $\widetilde{B}^m_{sa} \subset q\widetilde{A}^m_{sa}q$, *and*

$(\widetilde{B}^m_{sa})^- \subset [q\widetilde{A}^m_{sa}q]^-$.

(b) $\overline{(A_0)^m_{sa}} \subset q\overline{A^m_{sa}}q$, $(\widetilde{A}_0)^m_{sa} \subset q\widetilde{A}^m_{sa}q$, *and*

$[(\widetilde{A}_0)^m_{sa}]^- \subset [q\widetilde{A}^m_{sa}q].^-$

(c) *If B is a corner of an ideal, then*

$\overline{B^m_+} = q\overline{A^m_+}q$, $\widetilde{B}^m_{sa} = q\widetilde{A}^m_{sa}q$, *and*

$(\widetilde{B}^m_{sa})^- = (q\widetilde{A}^m_{sa}q)^-$.

Proof. (a) is trivial, since

$$\overline{B^m_{sa}} \subset \overline{A^m_{sa}} \quad \text{and} \quad q = q \cdot 1 \cdot q \in q\widetilde{A}^m_{sa}q.$$

(b) follows from (a) and 2.14 (applied with A replaced by B).

(c) just combines (a) and 2.20.

2.23. *Remarks-Examples.* The point of 2.22 (b) is to have some kind of replacement for the missing reverse inclusions of 2.14.

(i) Unless B is a corner of an ideal, $\exists a \in A_+$ such that

$$qaq \notin \widetilde{B}^m_{sa}.$$

Proof. Since $qAq \subset QM(B)$ (by 2.17, for example), $qA_+q \subset \widetilde{B}^m_{sa}$ would imply $qA_+q \subset M(B)$ (by 2.3), which implies $qAq \subset M(B)$. Then

$$B \supset (qAq)B = qAB \Rightarrow qABA \subset BA \subset (ABA)^- \Rightarrow q \in M(I),$$

where $I = (ABA)^-$, the ideal generated by B.

(ii) LEMMA. *If A is a non-degenerate C^*-subalgebra of $B(H)$ and $0 \leqq P \in B(H)$, then $PAP \subset A \Rightarrow P \in M(A)$.*

Proof. Let $a \in A$. Then

$$(a^*Pa)^2 = a^*(Paa^*P)a \in A \Rightarrow a^*Pa \in A,$$

by uniqueness of positive square roots. By polarization, $P \in QM(A)$. Then $L = aP \in LM(A)$ and $L^*L \in A$. By the proof of 2.6 (b), $L \in A$. Hence $P \in RM(A)$, and since $P = P^*$, $P \in M(A)$.

Note. There can be pitfalls from using non-universal representations in connection with multipliers, for example in attempting to apply Proposition 4.4 of [**5**] when the hypothesis on T^*T is known only in $B(H)$ rather than in A^{**}. An example of this was shown to us by P. Fillmore and J. Mingo. We have avoided these pitfalls above.

(iii) Unless B is a corner, $\exists a \in A$ such that

$$qaq \notin \overline{B_{sa}^m}.$$

Proof. Since $A = -A$,

$$qAq \subset \overline{B_{sa}^m} \Rightarrow qAq \subset \overline{B_{sa}^m} \cap [(B_{sa})_m]^- = B \subset A.$$

By the lemma, $q \in M(A)$.

(iv) The first part of 2.18 (c) (also (a), (b) by (i)) fails if I is replaced by a hereditary subalgebra B; more precisely, there can be $a \in A_+$ such that

$$qaq \notin (\widetilde{A}_{sa}^m)^-.$$

If A is unital and B is not a corner of an ideal, this failure always occurs by (i) and 2.14. For example, take $A = E_2$ and define q by

$$q_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $a \in A_+$ be given by

$$a_n = a_\infty = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then $h = qaq$ is given by

$$h_n = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad h_\infty = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

h is not weakly lsc in A^{**}, though it is weakly lsc in B^{**}. ($B = E_3$.)

(v) It is not possible to replace $(q\widetilde{A}_{sa}^m q)^-$ by $q[(\widetilde{A}_{sa}^m)^-]q$ in 2.22, even if B is an ideal.

Consider, for example, the case where A is unital. If

$$(\widetilde{I}_{sa}^m)^- \subset z[(\widetilde{A}_{sa}^m)^-] = z\widetilde{A}_{sa}^m,$$

then

$$(\widetilde{I}_{sa}^m)^- = \widetilde{I}_{sa}^m,$$

by 2.19. All that is needed is an example of a C^*-algebra I such that $QM(I) \neq M(I)$, and then one can take $A = \widetilde{I}$. $A = E_4$, $I = E_3 \subset E_4$ would be a nice specific example. It will be shown below (2.28) that prim A cannot be T_2 in an example of this type.

2.24. PROPOSITION. *If (I_α) is an increasing net of ideals with open central projections z_α such that $A = (\cup I_\alpha)^-$ and $h \in A_{sa}^{**}$, then*

(a) $h \in \overline{A_+^m} \Leftrightarrow z_\alpha h \in \overline{(I_\alpha)_+^m}, \quad \forall \alpha.$

(b) $h \in \widetilde{A}_{sa}^m \Leftrightarrow \exists \lambda$ *independent of* α *such that*

$$z_\alpha(h + \lambda) \in \overline{(I_\alpha)_{sa}^m}, \quad \forall \alpha.$$

(c) $h \in (\widetilde{A}_{sa}^m)^- \Leftrightarrow z_\alpha h \in [(\widetilde{I}_\alpha)_{sa}^m]^-, \quad \forall \alpha.$

Proof. (a). One direction follows from 2.18 (a). For the other note that

$$z_\alpha h \nearrow h \quad \text{and} \quad z_\alpha h \in \overline{I_+^m} \subset \overline{A_+^m}.$$

(b) follows from (a) just as in the proof of 2.11 (c).

(c). One direction follows from 2.17. For the other we may assume $h \geqq 0$ (replace h by $h + \lambda$). Let $a \in (\cup I_\beta)$. Then $a \in I_\alpha$ for α sufficiently large, and hence

$$z_\alpha h \in [(\tilde{I}_\alpha)_{sa}^m]^- \Rightarrow z_\alpha a^* h a \in \overline{(I_\alpha)_+^m} \Rightarrow a^* h a \in \overline{A_+^m}.$$

Since $(\cup I_\beta)^- = A$, $a^*ha \in \overline{A_+^m}$, $\forall a \in A$, and 2.4 $\Rightarrow h \in (\tilde{A}_{sa}^m)^-$.

2.25. PROPOSITION. *If* prim A *is Hausdorff, I, J are ideals, with open central projections z, w, such that $A = I + J$, and $h \in A_{sa}^{**}$, then*

(a) $zh \in \overline{I_{sa}^m}$ *and* $wh \in \overline{J_{sa}^m} \Rightarrow h \in \overline{A_{sa}^m}$.

(b) $zh \in \tilde{I}_{sa}^m$ *and* $wh \in \tilde{J}_{sa}^m \Rightarrow h \in \tilde{A}_{sa}^m$.

(c) $zh \in (\tilde{I}_{sa}^m)^-$ *and* $wh \in (\tilde{J}_{sa}^m)^- \Rightarrow h \in (\tilde{A}_{sa}^m)^-$.

Proof. (a). It is enough to show

$$h + \delta \in A_{sa}^m, \forall \delta > 0.$$

Changing notation, we may assume

$$zh \in I_{sa}^m, wh \in J_{sa}^m.$$

Then $\exists i \in I_+$, $j \in J_+$ such that

$$i + zh, j + wh \geqq 0 \Rightarrow i + j + h \geqq 0.$$

Since $z(i + j) \in \overline{I_+^m}$ and $w(i + j) \in \overline{J_+^m}$, we may change notation again and assume $h \geqq 0$.

Assume $\varphi_\alpha \to \varphi$ in $\Delta(A)$ and let $\epsilon > 0$. There are open central projections z_0, w_0 and closed central projections z_1, w_1 such that

$$z_0 \leqq z_1 \leqq z, \quad w_0 \leqq w_1 \leqq w,$$

$$\|(z - z_0)\varphi\| < \epsilon, \quad \text{and}$$

$$\|(w - w_0)\varphi\| < \epsilon.$$

Write

$$\varphi_\alpha = \theta_\alpha + \psi_\alpha + \rho_\alpha,$$

where

$$\theta_\alpha = z_0\theta_\alpha, \psi_\alpha = w_0\psi_\alpha, \quad \text{and}$$

$$z_0\rho_\alpha = w_0\rho_\alpha = 0.$$

Passing to a subnet, we may assume $\theta_\alpha \to \theta$, $\psi_\alpha \to \psi$, and $\rho_\alpha \to \rho$. Then

$$\theta + \psi + \rho = \varphi, \quad \operatorname{supp} \theta \leqq z_1 \leqq z,$$
$$\operatorname{supp} \psi \leqq w_1 \leqq w, \quad \text{and}$$
$$z_0\rho = w_0\rho = 0 \Rightarrow ||z\rho||, ||w\rho|| < \epsilon \Rightarrow ||\rho|| < 2\epsilon.$$

Then

$$\begin{aligned} \varphi(h) &= \theta(h) + \psi(h) + \rho(h) \\ &\leqq \underline{\lim}\, \theta_\alpha(h) + \underline{\lim}\, \psi_\alpha(h) + 2\epsilon||h|| \\ &\leqq \underline{\lim}(\theta_\alpha(h) + \psi_\alpha(h) + \rho_\alpha(h)) + 2\epsilon||h|| \\ &= 2\epsilon||h|| + \underline{\lim}\, \varphi_\alpha(h). \end{aligned}$$

(b) follows from (a) by translation by scalars.

(c) The proof is the same as (actually easier than) (a) except that now φ_α, $\varphi \in S(A)$. We may reduce easily to the case $h \geqq 0$. It follows from $\varphi_\alpha, \varphi \in S(A)$ that

$$||\psi_\alpha|| \to ||\psi||, \ ||\theta_\alpha|| \to ||\theta||, \quad \text{and} \quad ||\rho_\alpha|| \to ||\rho||.$$

(Then $zh(\theta) \leqq \underline{\lim}\, zh(\theta_\alpha)$ follows from

$$zh\left(\frac{\theta}{||\theta||}\right) \leqq \underline{\lim}\, zh\left(\frac{\theta_\alpha}{||\theta_\alpha||}\right).)$$

2.26. Proposition. *If I and J are ideals, with open central projections z, w, $A = I + J$, and $T \in A^{**}$, then*

(a) *$zT \in M(I)$ and $wT \in M(J) \Leftrightarrow T \in M(A)$.*

(b) *$zT \in LM(I)$ and $wT \in LM(J) \Leftrightarrow T \in LM(A)$.*

(c) *$zT \in QM(I)$ and $wT \in QM(J) \Leftarrow T \in QM(A)$.*

(d) *If $T \in M(A)$, then $T \in A$ if and only if $T \in \operatorname{her}_{A^{**}}(A)$ by Proposition 4.5 of* **[5]**. *This is so if and only if zT, $wT \in \operatorname{her}_{A^{**}}(A)$, in particular if $zT \in I$, $wT \in J$.*

Proof. (a)$\Rightarrow$: If $a \in A$, write $a = i + j$, $i \in I$, $j \in J$. Then

$$Ta = Ti + Tj = (zT)i + (wT)j \in I + J \subset A.$$

Similarly $aT \in A$.

$\Leftarrow$: If $i \in I$,

$$(zT)i = Ti \in A \cap I^{**} = I.$$

Similarly, $i(zT) \in I$.

(b). Same as (a).

(c)$\Leftarrow$. Same as (a).

(d). Since $(zT)^*(zT) = zT^*T \leqq T^*T$ and $(zT)(zT)^* \leqq TT^*$,

$$T \in \operatorname{her}_{A^{**}}(A) \Rightarrow zT \in \operatorname{her}_{A^{**}}(A).$$

The converse follows from $T^*T \leqq zT^*T + wT^*T$.

2.27. *Remark-Examples.* (i) The converse to 2.26 (c) follows from 2.25 (c) and [**5**] if prim A is T_2, but is false in general. For example take $A = E_4$ and let T be given by

$$T_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n = 1, 2, \dots .$$

Here z, w are given by

$$z_n = w_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad w_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) For a counter-example to the rest of 2.25 when prim A is not T_2, take A, z, w as in (i). Let $h \in A_{sa}^{**}$ be given by

$$h_n = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}, \quad h_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $zh \in I_+^m$, since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leqq \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix},$$

and also $wh \in J_+^m$. But $h \notin \overline{A_{sa}^m} = (\widetilde{A}_{sa}^m)^-$, since

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nleqq \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}.$$

(iii) When prim A is T_2, 2.24 and 2.25 show that semi-continuity is local to the extent that this is reasonable: Assume I_α is an ideal, $\forall\alpha$, and $A = (\sum I_\alpha)^-$. We wish to decide whether h is lsc by looking at $z_\alpha h \in I_\alpha^{**}$, if possible. 3.22 below shows that a necessary condition for $h \in \overline{A_{sa}^m}$ is that $\exists a \in A$ such that $a \leqq h$. Since this necessary condition is clearly not local, one should assume it. Then

$$h \in \overline{A_{sa}^m} \Leftrightarrow h - a \in \overline{A_+^m},$$

and 2.24 (a), 2.25 (a) show that

$$h - a \in \overline{A_+^m} \Leftrightarrow z_\alpha(h - a) \in \overline{(I_\alpha)_+^m}, \forall\alpha.$$

There is also a hitch in locality for the middle case, illustrated by 2.12 above; but we still have from 2.24, 2.25 that $h \in \widetilde{A}_{sa}^m \Leftrightarrow \exists\lambda$ independent of α such that

$$z_\alpha(h + \lambda) \in \overline{(I_\alpha)_{sa}^m}, \forall\alpha.$$

For continuity, with the exception of weak continuity, one again has locality, even if prim A is not T_2. One should prove the analogue of 2.24 for left multipliers, but this is routine.

2.28. PROPOSITION. *If* prim A *is Hausdorff and* I *is an ideal with open central projection* z, *then*

(a) $0 \leqq h \in \tilde{I}^m_{sa} \Rightarrow h \in \tilde{A}^m_{sa}$.

(b) $0 \leqq h \in (\tilde{I}^m_{sa})^- \Rightarrow h \in (\tilde{A}^m_{sa})^-$.

Proof. (a). Let $\lambda > 0$ be such that

$$h + \lambda z \in \overline{I^m_{sa}}.$$

We claim that $h + \lambda \in \overline{A^m_{sa}}$. Thus let $\varphi_\alpha \to \varphi$ in $\Delta(A)$. Let $\epsilon > 0$. There is an open central projection z_0 and a closed central projection z_1 such that

$$z_0 \leqq z_1 \leqq z \quad \text{and} \quad \|(z - z_0)\varphi\| < \epsilon.$$

Passing to a subnet, we may assume

$$z_0\varphi_\alpha \to \theta \quad \text{and} \quad (1 - z_0)\varphi_\alpha \to \psi.$$

Then $\theta + \psi = \varphi$, supp $\theta \leqq z_1 \leqq z$, and

$$\psi = (1 - z_0)\psi \Rightarrow \|z\psi\| < \epsilon \Rightarrow \psi(h + \lambda) < \epsilon\|h\| + \lambda\|\psi\|.$$

Then

$$\begin{aligned} \varphi(h + \lambda) &= \theta(h + \lambda) + \psi(h + \lambda) \\ &\leqq \underline{\lim}(z_0\varphi_\alpha)(h + \lambda) + \epsilon\|h\| + \lambda\, \underline{\lim}\|(1 - z_0)\varphi_\alpha\| \\ &\leqq \epsilon\|h\| + \underline{\lim}[(z_0\varphi_\alpha)(h + \lambda) + (1 - z_0)\varphi_\alpha(h + \lambda)] \\ &= \epsilon\|h\| + \underline{\lim}\, \varphi_\alpha(h + \lambda). \end{aligned}$$

Since ϵ is arbitrary, the result follows.

(b) follows easily from (a).

2.29. *Examples-Remarks.* (i) 2.28 fails if prim A is not T_2. Take $A = E_4$ and define z by

$$z_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let h be given by

$$h_n = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad h_\infty = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $h \in (\tilde{I}^m_{sa})^-$ (h is even in $QM(I)$), but

$$h \notin (\tilde{A}^m_{sa})^- = \overline{A^m_{sa}},$$

since

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \nleqq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The fact that 2.28 (b) fails implies that 2.28 (a) also fails. To see this explicitly, take

$$h_n = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad h_\infty = \begin{pmatrix} 2-\epsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 < \epsilon < \frac{1}{2}.$$

If one replaces I in 2.28 with a hereditary C^*-subalgebra B, the result fails even if prim A is T_2. The same example just given can be used for $A = E_2$.

(ii) Consider the following facts from general topology. Let U be an open subset of the topological space X.

(1) If f is lsc on U and $\lambda \leqq f(x)$, $\forall x \in U$, then f' is lsc on X where

$$f'(x) = \begin{cases} f(x), & x \in U \\ \lambda, & x \notin U. \end{cases}$$

(2) If f is lsc on $X \backslash U$ and $\lambda \geqq f(x)$, $\forall x \notin U$, then f' is lsc on X where

$$f'(x) = \begin{cases} \lambda, & x \in U \\ f(x), & x \notin U. \end{cases}$$

We have been attempting to analyze the non-commutative analogue of (1). In both (1) and (2) we are dealing only with $h \in A^{**}$ such that $[h, p] = 0$, where p is the closed projection analogous to $X \backslash U$. Of course $[h, p] = 0$ always if p is central, and $[h, p] = 0$ in one of the Tietze extension theorems of Section 3. However, the result like (2) used in Section 3 does not seem worth formalizing. Below we discuss some effects of the hypothesis $[h, p] = 0$.

(iii) The reason 2.19 and 2.18 (a) are true is not that z is central but that $[h, z] = 0$. If in the notation of 2.17 one assumes $[h, q] = 0$, the proof given can easily be adapted to work for $h \in \overline{A_+^m}$ ("$qhq(\varphi) \leqq h(\tilde{\varphi})$" is the only real change), and then a result for $h \in \widetilde{A}_{sa}^m$ follows.

Note that $[h, q] = 0$ does not imply $h \in A_0^{**}$, where

$$A_0 = \{a \in A : [a, q] = 0\}.$$

(iv) In 2.22 it would be better to have

$$\overline{B_{sa}^m} \subset \{qx : x \in \overline{A_{sa}^m} \text{ and } [x, q] = 0\},$$

etc. This improvement is easily possible for the strong and middle cases. For the weak case one could state an unpleasant result,

$$(\widetilde{B}_{sa}^m)^- \subset \{qx : x \in \widetilde{A}_{sa}^m \text{ and } [x, q] = 0\}^-,$$

but 2.23 (v) rules out a nice result in general. Of course the only really satisfactory results of this type are the conclusions of 2.28, which are only sometimes available.

(v) Let $p \in A^{**}$ be a closed projection, $h \in A_{sa}^{**}$ such that $[h, p] = 0$, t the top point in $\sigma(h)$ and $h' = ph + t(1 - p)$. Then

(a) $h \in \overline{A_{sa}^m} \Rightarrow h' \in \overline{A_{sa}^m}$.

(b) $h \in \widetilde{A}_{sa}^m \Rightarrow h' \in \widetilde{A}_{sa}^m$.

(c) $h \in (\widetilde{A}_{sa}^m)^- \Rightarrow h' \in (\widetilde{A}_{sa}^m)^-$.

Proof. Let (e_α) be an approximate identity for her$(1 - p)$. Let

$$h_\alpha = (1 - e_\alpha)^{1/2} h (1 - e_\alpha)^{1/2} + te_\alpha.$$

Then $[h_\alpha, p] = 0$, $ph_\alpha = ph$, and

$$\begin{aligned}(1 - p)h_\alpha &\leqq (1 - p)[(1 - e_\alpha)^{1/2} t (1 - e_\alpha)^{1/2} + te_\alpha] \\ &= t(1 - p) = (1 - p)h'.\end{aligned}$$

Therefore $h_\alpha \leqq h'$. Also $h_\alpha \to h'$ strongly.

(a) It is easy to see that

$$h \in \overline{A_{sa}^m} \Rightarrow h_\alpha \in \overline{A_{sa}^m}.$$

Since h_α is lsc as a function on $\Delta(A)$, $h_\alpha \leqq h'$, and $h_\alpha \to h'$ pointwise on $\Delta(A)$, h' is lsc on $\Delta(A)$.

(b) follows from (a), since $(h + \lambda)' = h' + \lambda$.

(c) is proved in the same way as (a) with $\Delta(A)$ replaced by $S(A)$.

(vi) If p in (v) is central, t can be replaced by the top point in $\sigma(php)$ or $\sigma(php) \cup \{0\}$, computed relative to $pA^{**}p$ (thus giving a full analogue of (ii) (2)); but this is false in general.

Proof. In the central case there is an ideal I and ph is just the image of h in $(A/I)^{**} \cong pA^{**}$. Clearly ph is lsc in the same sense as h, and it is easy to prove directly that h' is lsc on $\Delta(A)$ or $S(A)$ (cases (a) or (c)). (If $\varphi_\alpha \to \varphi$, one can consider separately the cases "φ_α vanishes on I, $\forall\alpha$" and "supp $\varphi_\alpha \leqq (1 - p)$, $\forall\alpha$.") (b) still follows from (a).

Example. Take $A = E_2$, and define p by

$$p_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $h \in A_{sa}^m = (\widetilde{A}_{sa}^m)^-$ be given by

$$h_n = \begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad h_\infty = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

That h is lsc follows from

$$\begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix} \geqq \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$$

and the top point in $\sigma(php)$ is 6. Since

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} \not\geqq \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$$

$ph + 6(1 - p)$ is not lsc.

2.C. *Operator monotone and convex functions.* A real valued function f on an interval I is usually called operator monotone or convex if I is open and the map $h \mapsto f(h)$ is monotone increasing or convex for bounded self-adjoint operators h such that $\sigma(h) \subset I$. If f has finite limits at one or two (finite) endpoints of I, it is well known that, for the continuous extension of f to the enlarged interval, $h \mapsto f(h)$ will still be monotone or convex. Thus in this paper the interval I will not be required to be open.

2.30. PROPOSITION. *Let f be operator monotone on an interval I of the form $(-\infty, b)$, $(-\infty, b]$, or $(-\infty, \infty)$, and let $h \in A_{sa}^{**}$ such that $\sigma(h) \subset I$.*

(a) *If $1 \in A$ or $0 \in I$ and $f(0) \geqq 0$, then*

$$h \in \overline{A_{sa}^m} \Rightarrow f(h) \in \overline{A_{sa}^m}.$$

(b) $h \in \widetilde{A}_{sa}^m \Rightarrow f(h) \in \widetilde{A}_{sa}^m.$

(c) $h \in (\widetilde{A}_{sa}^m)^- \Rightarrow f(h) \in (\widetilde{A}_{sa}^m)^-.$

Remark. If $1 \notin A$ and $0 \notin I$, it is impossible that $h \in \overline{A_{sa}^m}$ by 2.1 (c).

Proof. (a). By [5] there is a net (x_α) in $\widetilde{A}_{sa}$ such that $x_\alpha \nearrow h$, and $x_\alpha \in \lambda_\alpha + A$ where $\lambda_\alpha \nearrow 0$. Then $f(x_\alpha) \nearrow f(h)$. If $1 \in A$, we are done. If $f(0) \geqq 0$, then

$$f(x_\alpha) \in f(\lambda_\alpha) + A \quad \text{and} \quad f(\lambda_\alpha) \nearrow f(0) \geqq 0.$$

By [5] this implies $f(h) \in \overline{A_{sa}^m}$ (if $f(0) > 0$, $f(x_\alpha) \in \overline{A_{sa}^m}$ for α sufficiently large).

(b). If $x_\alpha \nearrow h$, $x_\alpha \in \widetilde{A}_{sa}$, then $f(x_\alpha) \nearrow (h)$, $f(x_\alpha) \in \widetilde{A}_{sa}$.

(c) follows from (b), since $h \mapsto f(h)$ is norm continuous and we may choose $h_n \to h$ with $h_n \in \widetilde{A}_{sa}^m$ and $\sigma(h_n) \subset I$.

2.31. PROPOSITION. *Let f be operator monotone on an interval I of the form (a, ∞), $[a, \infty)$, or $(-\infty, \infty)$, and let $h \in \overline{A_{sa}^m}$ such that $\sigma(h) \subset I$.*

(a) *If $0 \in I$ and $f(0) \geqq 0$, then $f(h) \in \overline{A_{sa}^m}$.*

(b) *If $0 \in I$, then $f(h) \in \widetilde{A}_{sa}^m$.*

(c) *If $I = (0, \infty)$, then $f(h) \in (\widetilde{A}_{sa}^m)^-$.*

Proof. (a). First assume $h \geqq 0$. Let $\delta > 0$, and choose $a_\alpha \in A_+$ such that $a_\alpha \nearrow h + \delta$. Then

$$f(a_\alpha) \in f(0) + A \subset \overline{A_{sa}^m} \quad \text{and} \quad f(a_\alpha) \nearrow f(h + \delta).$$

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

Thus $f(h + \delta) \in \overline{A^m_{sa}}$, $\forall \delta > 0$; and letting $\delta \to 0$, we see that $f(h) \in \overline{A^m_{sa}}$.

If $h \ngeq 0$, let s be the least point in $\sigma(h)$, so that $s < 0$. Choose $x_\alpha \nearrow h$ such that $x_\alpha \in \lambda_\alpha + A$ and $\lambda_\alpha \nearrow 0$. If $\delta > 0$, then $\lambda_\alpha + \delta > 0$ for α sufficiently large, which implies $x_\alpha + \delta$ gives an lsc function on $\Delta(A)$. Since $x_\alpha + \delta \nearrow h + \delta$, which is $>s$ at each point of $\Delta(A)$, Dini's theorem implies $x_\alpha + \delta \geqq s$ for α sufficiently large. Thus

$$\sigma(x_\alpha + \delta) \subset I \quad \text{and} \quad f(x_\alpha + \delta) \nearrow f(h + \delta).$$

$f(x_\alpha + \delta) \in f(\lambda_\alpha + \delta) + A$, and $f(\lambda_\alpha + \delta) \geqq f(0) \geqq 0$ for α large. Hence

$$f(x_\alpha + \delta) \in \overline{A^m_{sa}} \Rightarrow f(h + \delta) \in \overline{A^m_{sa}}.$$

Again let $\delta \to 0$.

(b) follows from (a) applied to $f - f(0)$.

(c). If $\delta > 0$, then by **[5]** there are $a_\alpha \in A_+$ such that $a_\alpha \nearrow h + \delta$. Then

$$a_\alpha + \delta \nearrow h + 2\delta \Rightarrow f(a_\alpha + \delta) \nearrow f(h + 2\delta).$$

Hence $f(h + 2\delta) \in \widetilde{A}^m_{sa}$. As $\delta \to 0$, $f(h + 2\delta) \to f(h)$ in norm.

2.32. COROLLARY. *Let f be operator monotone on an interval I and $h \in \overline{A^m_{sa}}$ such that $\sigma(h) \subset I$.*

(a) *If $0 \in I$ and $f(0) \geqq 0$, then $f(h) \in \overline{A^m_{sa}}$.*

(b) *If $0 \in I$, then $f(h) \in \widetilde{A}^m_{sa}$.*

(c) *If 0 is the left endpoint of I, then $f(h) \in (\widetilde{A}^m_{sa})^-$.*

Proof. Let

$$I_- = \{x \in \mathbf{R}: x \leqq y \text{ for some } y \in I\} \quad \text{and}$$

$$I_+ = \{x \in \mathbf{R}: x \geqq y \text{ for some } y \in I\}.$$

Write $f = f_- + f_+$ where $f_\pm$ is operator monotone on $I_\pm$. If $f(0) \geqq 0$, we may assume $f_+(0), f_-(0) \geqq 0$. Apply 2.30 to f_- and 2.31 to f_+.

2.33. *Remarks.* (i) The sharpness of these results will be discussed in 2.41 below.

(ii) It is possible to translate the independent variable of f, replacing f by $f(\cdot - t)$ and I by $I + t$. If $1 \in A$, $h + t$ will be lsc if h is. Even if $1 \notin A$, $h + t$ may be lsc. In particular, in the context of 2.32 (c), if $\exists \delta > 0$ such that $h - \delta \in \overline{A^m_{sa}}$, then $f(h) \in \widetilde{A}^m_{sa}$.

2.34. PROPOSITION. *Let f be operator convex on an interval I and $h \in QM(A)_{sa}$ such that $\sigma(h) \subset I$. Then $f(h) \in (\widetilde{A}^m_{sa})^-$.*

Proof. It is well known that f has a representation

$$(1)\quad f(x) = ax^2 + bx + c + \int_{t<I}\left(\frac{1}{x-t} - \frac{1}{x_0 - t} + \frac{x - x_0}{(x_0 - t)^2}\right)d\mu_-(t) + \int_{t>I}\left(\frac{1}{t-x} - \frac{1}{t - x_0} - \frac{x - x_0}{(t - x_0)^2}\right)d\mu_+(t).$$

Here $a \geqq 0$, $t < I$ means $t < x$, $\forall x \in I$, $t > I$ means $t > x$, $\forall x \in I$, x_0 is any point in I°, and $\mu_\pm$ are positive measures such that

$$\int \frac{1}{(|t| + 1)^3} d\mu_\pm(t) < \infty.$$

Even if I contains an endpoint, (1) gives a norm convergent integral for $f(h)$. $ah^2 \in (\tilde{A}^m_{sa})^-$ by 2.7. By 2.1 (a) $(h - t)^{-1}$, $t < I$, and $(t - h)^{-1}$, $t > I$, are both in $\overline{A^m_{sa}}$, since $\pm h \in [(\tilde{A}_{sa})_m]^-$. This implies that the integrals are in $(\tilde{A}^m_{sa})^-$.

2.35. Proposition. *Let f be a continuous real-valued function on an interval I.*

(a) *If $h \in \overline{A^m_{sa}}$, $\sigma(h) \subset I \Rightarrow f(h) \in (\tilde{A}^m_{sa})^-$ for all C^*-algebras A (or for $A = E_1$), and if $\exists 0 \leqq t \in I$, then f is operator monotone.*

(a′) *If $h \in A^m_{sa}$, $\sigma(h) \subset I \Rightarrow f(h) \in A^m_{sa}$ for $A = c \otimes M_n$, $n = 1, 2, \ldots,$ then f is operator monotone.*

(b) *If $h \in QM(A)_{sa}$, $\sigma(h) \subset I \Rightarrow f(h) \in (\tilde{A}^m_{sa})^-$ for all C^*-algebras A (or for $A = E_1$), then f is operator convex.*

Remarks. (i) The hypothesis on I in (a) is necessary, since otherwise, it is impossible to have $h \in \overline{A^m_{sa}}$, $\sigma(h) \subset I$ when A is non-unital.

(ii) In (a′) the algebras are unital and hence there is only one kind of semicontinuity.

(iii) (b) is strictly a non-unital result.

Proof. (a). Choose $0 \leqq t \in I$. Let $h' \geqq h''$ in $M_k \subset \mathcal{K}$, where $\sigma(h')$, $\sigma(h'') \subset I$. Define $h \in \overline{A^m_{sa}}$ $(A = E_1)$ by $h_n = h' + tq$, $n = 1, 2, \ldots, h_\infty = h'' + tq$, where

$$q = \sum_{k+1}^{\infty} e_i \times e_i.$$

Then

$$f(h)_n = f(h') + f(t)q, \ \forall n, \text{ and } f(h)_\infty = f(h'') + f(t)q;$$

and clearly

$$f(h) \in (\tilde{A}^m_{sa})^- \Rightarrow f(h') \geqq f(h'').$$

(a′). This is left to the reader.

(b). Let $a \in (M_k)_{sa}$, $b \in M_{k,l}$, $c \in (M_l)_{sa}$ be such that

$$\sigma\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \subset I.$$

Fix $t \in I$ and consider $h \in QM(A)_{sa}$ such that

$$h_\infty = a + tq,$$

$$h_n = \sum_{1,1}^{k,k} a_{ij} e_i \times e_j$$

$$+ 2 \operatorname{Re} \sum_{1,1}^{k,l} b_{ij} e_i \times e_{n+j+k}$$

$$+ \sum_{1,1}^{l,l} c_{ij} e_{n+i+k} \times e_{n+j+k} + tq_n,$$

where

$$q = \sum_{k+1}^{\infty} e_i \times e_i,$$

$$q_n = \sum_{k+1}^{n+k} e_i \times e_i + \sum_{n+k+l+1}^{\infty} e_i \times e_i.$$

Then all the operators $f(h_n)$ are unitarily equivalent, though not equal as in (a), and all the operators

$$q' f(h_n) q' = q' f\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} q',$$

where $q' = 1 - q$. Thus $f(h) \in (\widetilde{A}_{sa}^m)^-$ implies

$$q' f\begin{pmatrix} a & 0 \\ 0 & t \end{pmatrix} q' \leqq q' f\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} q'.$$

This inequality for all choices of k, l, a, b, c implies f is operator convex. (See [**16**], for example; also cf. Remark 2.37 (b) below. The t on the left of the inequality drops out.)

2.36. THEOREM. *Let f be operator convex on an interval $I \ni 0$. The following are equivalent.*

(i) *For $A = E_1$, $h \in QM(A)_{sa}$ and $\sigma(h) \subset I \Rightarrow f(h) \in \overline{A_{sa}^m}$.*

(ii) *For $p, h \in B(H)_{sa}$ such that p is a projection and $\sigma(h) \subset I$,*

$$pf(php)p \leqq f(h).$$

(iii) *For $p, h \in B(H)_{sa}$ such that $0 \leqq p \leqq 1$ and $\sigma(h) \subset I$,*

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$f(php) \leqq f(h) + f(0)(1 - p).$$

(iv) *The condition in* (i) *holds for arbitrary* A.

(v) *Either* $f = 0$, *or* $f(t) > 0\ \forall t \in I$ *and* $-1/f$ *is operator convex.*

(vi) f *has a representation*

$$(2)\quad f(x) = \int_{-r<I} \frac{1}{r + x} d\mu_{-}(r) + \int_{r>I} \frac{1}{r - x} d\mu_{+}(r) + c,$$

where $\mu_{\pm}$ *are positive measures such that*

$$\int \frac{1}{r}\, d\mu_{\pm}(r) < \infty \quad \text{and} \quad c \geqq 0.$$

Also if f *satisfies the conditions and* f *can be continued to an operator convex function on some interval* $J \supset I$, *then* f *satisfies the conditions on* J. *In particular, unless* $f = 0$, f *cannot approach* 0 *at a (finite) endpoint of* I.

2.37. *Remarks.* (a) (ii) only nominally requires $0 \in I$, since neither side of the inequality would be affected if we extended $f_{|\mathrm{co}(\sigma(h))}$ to a continuous function on all of **R**. (i), (iv), (v), and (vi) do not depend on the hypothesis $0 \in I$ at all. Since the conditions other than (iii) are easily seen to be invariant under translation of the independent variable, (iii) must be invariant under translations that preserve the hypothesis $0 \in I$. Alternatively, let f be a function on an arbitrary interval I, $a \in I$, and consider:

(iii)$_a$: $\forall p, h \in B(H)_{sa}$ such that $0 \leqq p \leqq 1$ and $\sigma(h) \subset I$,

$$f(p(h - a)p + a) \leqq f(h) + f(a)(1 - p).$$

Then (iii)$_a$ is independent of a.

(b) It would appear that the sharp case of (iii) occurs when p is a projection. The only reason we considered (iii), instead of being satisfied with (ii), was to have something that would make sense in a C^*-algebra without enough projections. It is interesting to compare various operator inequalities. Operator convexity is characterized by

$$pf(php)p \leqq pf(h)p,$$

p a projection. (Davis; See **[16]**, where the history is also discussed.) A slight reworking of Davis' inequality will occur below (2.54):

$$f\left(\sum_1^n \lambda_i F_i\right) \leqq \sum_1^n f(\lambda_i) F_i,$$

$$\lambda_1, \dots, \lambda_n \in I,\ F_i \geqq 0,\ \sum_1^n F_i = 1.$$

(Here h is a finite matrix, $\sigma(h) = \{\lambda_1, \dots, \lambda_n\}$, and $F_i = pE_i p$, where the E_i's are the spectral projections. Naimark's dilation theorem shows all

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

choices of $F_1, \ldots, F_n$ can occur.) If $0 \in I$ and $f(0) \leqq 0$, operator convexity is also characterized by $f(pxp) \leqq pf(x)p$ (Davis) and by the stronger inequality $f(a^*xa) \leqq a^*f(x)a$, $\|a\| \leqq 1$ (Hansen and Pedersen [**22**]). The relation between (ii) and (iii) is somewhat analogous to the relation between Davis' and Hansen and Pedersen's inequalities, but note that in (ii) and (iii) $f(0) > 0$.

(c) That the function $x \mapsto 1/x$, $x > 0$, satisfies (ii) can easily be verified directly. To see this, use the formula

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b^*a^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & c - b^*a^{-1}b \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix},$$

which makes it easy to compare

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}^{-1} \text{ and } \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) Each of (i)-(vi) already implies f operator convex. The proof of this for (v) (the only non-obvious one) is contained in 2.38 below.

Proof of 2.36. (i) $\Rightarrow$ (ii): First note that by taking $h = t \cdot 1$, $t \in I$, we can conclude that $f \geqq 0$ on I; in particular $f(0) \geqq 0$. Now consider the same h used in the proof of 2.35 (b) with $t = 0$. Then

$$pf(a)p \leqq f(h_\infty) \quad \text{where } p = 1 - q.$$

By the criterion for strong semicontinuity (see Section 5.C), $\forall \epsilon > 0$, $\exists N$ such that

$$pf(a)p \leqq f(h_n) + \epsilon, \quad \forall n \geqq N.$$

The inequality

$$(1 - q_n)pf(a)p(1 - q_n) \leqq (1 - q_n)[f(h_n) + \epsilon](1 - q_n)$$

is equivalent to

$$\begin{pmatrix} f(a) & 0 \\ 0 & 0 \end{pmatrix} \leqq f\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} + \epsilon,$$

as an inequality in M_{k+l}. Letting $\epsilon \to 0$ we obtain the special case of (ii) where

$$h = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \; p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since finite rank operators are dense, this is adequate.

(ii) $\Rightarrow$ (i): Let $p \leqq q$ be projections and $h \in B(H)_{sa}$ such that $\sigma(h) \subset I$. By applying (ii) to qhq, we see that

$$pf(php)p \leqq qf(qhq)q.$$

Now let (p_m) be a sequence of finite rank projections such that $p_m \nearrow 1$, and $h \in QM(A)_{sa}$ such that $\sigma(h) \subset I$. Then

$$p_m f(p_m h_\infty p_m) p_m \nearrow f(h_\infty).$$

By the criterion for strong semicontinuity it is sufficient to prove: $\forall m$, $\forall \epsilon > 0$, $\exists N$ such that

$$n \geqq N \Rightarrow p_m f(p_m h_\infty p_m) p_m \leqq f(h_n) + \epsilon.$$

But

$$h_n \to h_\infty \text{ weakly} \Rightarrow p_m h_n p_m \to p_m h_\infty p_m \text{ in norm}$$

$$\Rightarrow \exists N \text{ such that } f(p_m h_\infty p_m) \leqq f(p_m h_n p_m) + \epsilon, \forall n \geqq N.$$

Therefore

$$p_m f(p_m h_\infty p_m) p_m \leqq p_m f(p_m h_n p_m) p_m + \epsilon \leqq f(h_n) + \epsilon,$$

where we have used (ii) again.

(i) and (ii) $\Rightarrow$ (iii): Let $0 \leqq p \leqq 1$ and $x \in \mathcal{K}_{sa}$ such that $\sigma(x) \subset I$. Choose a sequence (p_n) of projections such that $p_n \to p$ weakly. Then we can define $h \in QM(A)_{sa}$ by

$$h_n = p_n x p_n, \quad h_\infty = pxp.$$

Choose $K \in \mathcal{K}$ such that $K \leqq f(h_\infty)$ and $\epsilon > 0$. Then $f(h) \in \overline{A_{sa}^m} \Rightarrow \exists N$ such that $K \leqq f(h_n) + \epsilon$, $\forall n \geqq N$. By (ii)

$$p_n f(h_n) p_n \leqq f(x),$$

and this means

$$f(h_n) \leqq f(x) + f(0)(1 - p_n).$$

Therefore

$$K \leqq f(x) + f(0)(1 - p_n) + \epsilon, \forall n \geqq N$$

$$\Rightarrow K \leqq f(x) + f(0)(1 - p) + \epsilon.$$

Since K and ϵ are arbitrary (and since $f(h_\infty) \geqq 0 \Rightarrow f(h_\infty) \in \mathcal{K}_+^m$), we conclude

$$f(pxp) = f(h_\infty) \leqq f(x) + f(0)(1 - p).$$

This inequality for $x \in \mathcal{K}$ implies the general inequality since finite rank operators are dense. (Also, the inequality for finite matrices implies the inequality even in non-separable Hilbert spaces.)

(iii) $\Rightarrow$ (iv): Let $x \in QM(A)_{sa}$ such that $\sigma(x) \subset I$) and (e_α) an approximate identity for A. Then

$$y_\alpha = f(e_\alpha x e_\alpha) - f(0)(1 - e_\alpha) \leqq f(x).$$

Also

$$e_\alpha x e_\alpha \in A \Rightarrow f(e_\alpha x e_\alpha) \in f(0) + A \Rightarrow y_\alpha \in A.$$

Since $y_\alpha \rightarrow f(x)$ in the strong topology of A^{**} (and in particular as functions on $\Delta(A)$), this implies $f(x) \in \overline{A^m_{sa}}$.

(iv) $\Rightarrow$ (v): Let I_0 be a subinterval of I such that $f > 0$ on I_0. (If necessary translate the independent variable so that $0 \in I_0$.) Then for $h \in QM(A)_{sa}$ and $\sigma(h) \subset I_0$,

$$f(h) \in \overline{A^m_{sa}} \Rightarrow f(h)^{-1} \in [(\tilde{A}_{sa})_m]^- \text{ (2.1 (a))}$$
$$\Rightarrow -f(h)^{-1} \in (\tilde{A}^m_{sa})^-.$$

By 2.35 (b), $-1/f$ is operator convex on I_0. But a convex function can never approach $-\infty$ at a finite endpoint. This implies that we can take $I_0 = I$, unless $f = 0$.

(v) $\Rightarrow$ (vi): In the proof of 2.34 we saw the integral representation (1), for an arbitrary operator convex function, and now we want the stronger form (2). In comparing (1) and (2), take $x_0 = 0$ (we may assume $0 \in I^\circ$ for (v) $\Rightarrow$ (vi)) and $r = |t|$. Thus we have:

$$(1') \quad f(x) = ax^2 + bx + c + \int_{-r<I} \left(\frac{1}{r + x} - \frac{1}{r} + \frac{x}{r^2}\right) du_-(r)$$
$$+ \int_{r>I} \left(\frac{1}{r - x} - \frac{1}{r} - \frac{x}{r^2}\right) du_+(r).$$

If

$$\int \frac{1}{r}\, du_{\pm}(r) < \infty$$

the $1/r$ and x/r^2 terms can be dropped from the integrals and absorbed into $bx + c$. Assume $f \neq 0$, and write

$$g(x) = \frac{f(x) - f(0)}{x},$$

so that g is operator monotone. Then since

$$\frac{-\frac{1}{f(x)} - \left(-\frac{1}{f(0)}\right)}{x} = \frac{g(x)}{f(0)f(x)}$$

and $f(0) > 0$,

$$\frac{g(x)}{f(x)} = \frac{g(x)}{f(0) + xg(x)}$$

is operator monotone. The case g = constant cannot occur, and the case

$$\frac{g(x)}{f(0) + xg(x)} = \text{constant}$$

yields

$$f(x) = \frac{A}{B + CX},$$

which is a trivial case of (vi). Thus we assume neither of the two operator monotone functions is constant, and this implies both carry the upper half plane into itself. If $\text{Im } z = y > 0$, then

$$\text{Im}\frac{g(z)}{f(0) + zg(z)} = \frac{f(0) \text{ Im } g(z) - y|g(z)|^2}{\text{positive}}$$

$$\Rightarrow f(0) \text{ Im } g(z) > y|g(z)|^2 \Rightarrow \text{Im } g(z) < \frac{f(0)}{y}.$$

From (1′) we obtain

$$(3) \quad g(x) = \int_{r>I} \left(\frac{1}{r - x} - \frac{1}{r}\right)\frac{1}{r} du_+(r) - \int_{-r<I} \left(\frac{1}{r + x} - \frac{1}{r}\right)\frac{1}{r} du_-(r) + ax + b, \quad \text{and}$$

$$(4) \quad \text{Im } g(z) = \int_{r>I} \frac{y}{|r - z|^2} \frac{1}{r} du_+(r) + \int_{-r<I} \frac{y}{|r + z|^2} \frac{1}{r} du_-(r) + ay.$$

This implies

$$ay^2 + \int_{r>I} \frac{y^2}{|r - z|^2} \frac{1}{r} du_+(r) + \int_{-r<I} \frac{y^2}{|r + z|^2} \frac{1}{r} du(r) < f(0).$$

If $\text{Re } z = 0$, then $|r \pm z|^2 = r^2 + y^2$ and

$$\int \frac{y^2}{|r \pm z|^2} \frac{1}{r} du_\pm(r) \geqq \frac{1}{2} \int_{r \leqq y} \frac{1}{r} du_\pm(r).$$

Thus we conclude

$$a = 0 \quad \text{and} \quad \int \frac{1}{r} du_{\pm}(r) < \infty.$$

We now assume that the $1/r$ and x/r^2 terms are dropped from the integrals in (1′) and absorbed into $bx + c$. This does not change (4) but causes the "$-1/r$" terms to be dropped from (3). Then

$$f(0) \cdot \left(\int_{r>I} \frac{1}{r|r - z|^2} d\mu_+(r) + \int_{-r<I} \frac{1}{r|r + z|^2} d\mu_-(r) \right)$$
$$> \left| b + \int_{r>I} \frac{1}{r(r - z)} d\mu_+(r) - \int_{-r<I} \frac{1}{r(r + z)} d\mu_-(r) \right|^2.$$

If we let $z \to \infty$ so that Im z is bounded away from 0, the dominated convergence theorem applies and gives

$$f(0) \cdot 0 \geqq |b|^2 \Rightarrow b = 0.$$

Now we calculate

$$\lim_{\substack{y\to\infty \\ \mathrm{Re}\, z=0}} y \operatorname{Im} g(z) = \lim_{y\to\infty} \int \frac{y^2}{y^2 + r^2} \frac{1}{r} d\mu(r),$$

where $\mu = \mu_+ + \mu_-$. Fix $n > 0$. For $r \leqq (1/n)y$,

$$y^2 \leqq r^2 + y^2 \leqq \left(1 + \frac{1}{n^2}\right) y^2 \Rightarrow 1 \geqq \frac{y^2}{r^2 + y^2} \geqq \frac{n^2}{n^2 + 1}.$$

Thus

$$\int \frac{1}{r} d\mu(r) \geqq y \operatorname{Im} g(z) \geqq \frac{n^2}{n^2 + 1} \int_{r \leqq (1/n)y} \frac{1}{r} d\mu(r).$$

Therefore

$$\int \frac{1}{r} d\mu(r) \geqq \overline{\lim}\, y \operatorname{Im} g(z)$$
$$\geqq \underline{\lim}\, y \operatorname{Im} g(z) \geqq \frac{n^2}{n^2 + 1} \int \frac{1}{r} d\mu(r).$$

Since n is arbitrary,

$$\lim y \operatorname{Im} g(z) = \int \frac{1}{r} d\mu(r);$$

and

$$\operatorname{Im} g(z) < \frac{f(0)}{y} \Rightarrow \int \frac{1}{r} d\mu(r) \leqq f(0) = c + \int \frac{1}{r} d\mu(r).$$

Thus $c \geqq 0$.

(vi) $\Rightarrow$ (i) follows easily from 2.1.

The fact that f still satisfies the conditions on J when f can be continued to J follows from (vi) and the uniqueness of the integral representation.

2.38. COROLLARY. *If $k < 0$ is an operator convex function, then $f = -1/k$ is operator convex and has integral representation of the special form* (vi).

Proof. We need only show that f is operator convex, and then use (v) $\Rightarrow$ (vi). Since k is operator convex,

$$h \in QM(A)_{sa}, \quad \sigma(h) \subset I \Rightarrow -k(h) \in [(\tilde{A}_{sa})_m]^- \Rightarrow f(h) \in \overline{A^m_{sa}}$$

by 2.1. Thus 2.35 $\Rightarrow f$ operator convex.

2.39. PROPOSITION. *Let f be operator monotone on an interval I of the form $(-\infty, b)$, $(-\infty, b]$, or $(-\infty, \infty)$.*

(a) *If $f \geqq 0$ on I, then*

$$h \in (\tilde{A}^m_{sa})^- \quad \textit{and} \quad \sigma(h) \subset I \Rightarrow f(h) \in \overline{A^m_{sa}}.$$

(b) *If f is bounded below on I, then*

$$h \in (\tilde{A}^m_{sa})^- \quad \textit{and} \quad \sigma(h) \subset I \Rightarrow f(h) \in \tilde{A}^m_{sa}.$$

Proof. (a) f has a representation

$$(5) \quad f(x) = ax + b + \int_{r>I} \frac{1}{r-x} - \frac{1}{r-x_0} d\mu(r),$$

where $a, \mu \geqq 0$ and

$$\int \frac{1}{r^2+1} d\mu(r) < \infty.$$

$$\lim_{x \to -\infty} f(x) > -\infty \Rightarrow a = 0 \quad \text{and} \quad \int_{r>I} \frac{1}{r-x_0} d\mu(r) < \infty.$$

Therefore the "$-(1/r - x_0)$" terms can be dropped from the integral in (5) and absorbed into b. We obtain

$$(6) \quad f(x) = b + \int_{r>I} \frac{1}{r-x} d\mu(r).$$

Since

$$b = \lim_{x \to -\infty} f(x) \geqq 0,$$

(6) and 2.1 (a) imply the result.

(b) follows from (a) applied to

$$f - \lim_{x \to -\infty} f(x).$$

2.40. *Remark.* It should be clear that if f is as in (5), the conditions in 2.36 are equivalent to the hypothesis of 2.39 (a).

In the next three remarks we discuss the sharpness of the above results. Consider the following questions, each of which is nine-fold because of the three types of semicontinuity.

(I) Given a C^*-algebra A and an interval I, is it true that for all $h \in A_{sa}^{**}$ with $\sigma(h) \subset I$ and all operator monotone functions f on I (or, where appropriate, all f such that $f(0) \geqq 0$ or $f \geqq 0$ on I) h lsc $\Rightarrow f(h)$ lsc?

(II) Given a function f on I, is it true for all C^*-algebras A that h lsc $\Rightarrow$ $f(h)$ lsc?

In Remark 2.41 we argue that a yes answer to (I) that does not follow from 2.30-2.32 or 2.39 can occur only if A is very special. Namely, A must be unital or satisfy (i), (ii), (iii) of 2.2. Moreover, in these cases the yes answer to (I) follows easily from 2.30-2.32 or 2.39, the special condition on A, and 2.33 (ii); so that it is not worth being stated formally. In Remark 2.42 we argue that any yes answer to (II) follows from 2.30-2.32 or 2.39. Moreover, E_1 is a universal test algebra for (II). Of course (I) and (II) are not the only questions that could be asked on this subject.

2.41. *Remark.* First we dispose of the case A unital. In this case there is only one meaning of lsc and translations of independent variable cause no problems. Thus 2.32 (b) gives a positive answer to (I) always.

Now if A is not unital, it will be impossible to have $h \in \overline{A_{sa}^m}$, $\sigma(h) \subset I$ if $I \subset (-\infty, 0)$. Hence such I should not be considered when the hypothesis is h strongly lsc.

(a) strong $\rightarrow$ weak.

The yes answer to (I) follows from 2.32 unless I has a left endpoint $s > 0$. Consider $0 < \delta < s$ and let

$$f(x) = -\frac{1}{x - \delta}.$$

By 2.1 (a) if f takes strongly lsc to weakly lsc, it must be true that $h \in \overline{A_{sa}^m}$, $\sigma(h) \subset I \Rightarrow h - \delta \in \overline{A_{sa}^m}$. It is useful to state:

LEMMA. *Let I be a non-degenerate interval such that $I \subset (s, \infty)$ for some $s > 0$. If the conditions of 2.2 are not satisfied, then $\exists h \in \overline{A_{sa}^m}$ such that $\sigma(h) \subset I$ and $h - \delta \notin \overline{A_{sa}^m}$ for any $\delta > 0$.*

Proof. Choose $x \in (\widetilde{A}_{sa}^m)^- \setminus \widetilde{A}_{sa}^m$, and choose $x_n \in \underline{\widetilde{A}_{sa}^m}$ such that $x_n \rightarrow x$ in norm. Let λ_n be minimal such that $x_n + \lambda_n \in \overline{A_{sa}^m}$. Since $x \notin \widetilde{A}_{sa}^m$, $\lambda_n \rightarrow \infty$. This implies that for n large the ratio between the top and bottom points in $\sigma(x_n + \lambda_n)$ (both of which will be positive) is close to 1. Let

$$h = r_n(x_n + \lambda_n),$$

where n is large and r_n is chosen so that the bottom point in $\sigma(h)$ is slightly more than the left endpoint of I.

If the conditions of 2.2 are satisfied, by 2.2 (ii) we may translate the independent variable to replace I by $I - s$.

(b) strong $\rightarrow$ middle.

If $0 \in I$, the yes answer to (I) follows from 2.32; and if I has a left endpoint $s > 0$, the reasoning in (a) above is decisive (when 2.2 (iii) holds, case (a) and case (b) are the same). The remaining case is $I = (0, t)$ or $(0, t]$. In this case consider $f(x) = -1/x$. By 2.1 (b), if f takes strongly lsc to middle lsc, then 2.2 (i) holds. In this case the yes answer to (I) follows from 2.32 (c) and 2.2 (iii).

(c) strong $\rightarrow$ strong.

Since $\lambda \cdot 1 \in \overline{A^m_{sa}}$ if and only if $\lambda \geqq 0$, clearly we need $f \geqq 0$ on $I \cap [0, \infty)$. This means that the portion of (I) in parentheses is applicable. If $0 \in I$, 2.32 gives the yes answer. If 0 is the left endpoint of I, then $f \geqq 0$ implies f has a finite limit at 0; so that 2.32 (a) still applies. If I has a left endpoint $s > 0$, the reasoning in (a) above shows that the conditions of 2.2 are satisfied; and again we can replace I by $I - s$. To see this, one should note that in (a) we could have taken

$$f(x) = \frac{1}{s - \delta} - \frac{1}{x - \delta},$$

which is positive on I, with equally good effect.

(d) middle $\rightarrow$ weak or weak $\rightarrow$ weak.

These cases are the same, since $h \mapsto f(h)$ is norm continuous. The yes answer to (I) follows from 2.30 unless I has a left endpoint $s > -\infty$. In this case we may perform a translation to reduce to the case $s = 0$. Now consider

$$f(x) = -\frac{1}{x + \delta}, \quad \delta > 0.$$

A yes answer to (I) would imply (by 2.1 (a)) that

$$h + \delta \in \overline{A^m_{sa}}, \quad \forall \delta > 0,$$

which implies $h \in \overline{A^m_{sa}}$, for all $h \in \widetilde{A}^m_{sa}$ such that $\sigma(h) \subset I$. This gives 2.2 (ii), which means that 2.32 (c) applies.

(e) middle $\rightarrow$ strong or weak $\rightarrow$ strong.

Again these cases are the same, and the parenthetical part of (I) is applicable (here we need $f \geqq 0$ on I). If I has a finite left endpoint s, again we may assume $s = 0$, and the reasoning in (d) (take $f(x) = 1/\delta - 1/(x + \delta)$) shows that the conditions of 2.2 are satisfied, if there is a positive answer. Thus 2.32 (a) applies (see (c) if this is not clear).

If the left endpoint $= -\infty$, 2.39 (a) gives a yes answer to (I).

(f) middle $\rightarrow$ middle.

The yes answer to (I) follows from 2.30 unless I has a finite left endpoint s. Since $h \mapsto f(h)$ is norm continuous, a positive answer here would imply a positive answer in case (d), which implies the conditions of 2.2. Then 2.32 applies (see (b) if this is not clear).

(g) weak $\rightarrow$ middle.

Since the function $f(x) = x$ is allowed here (unlike case (e)), this case can have a positive answer only when the conditions of 2.2 are satisfied. Then this becomes the same as (d).

2.42. *Remark*. Since A is arbitrary in (II), it is in particular non-unital, and we again exclude the case $I \subset (-\infty, 0)$ when h is required to be strongly lsc.

(a) strong $\rightarrow$ weak.

A positive answer to (II) follows from 2.32 unless I has a left endpoint $s > 0$. Consider

$$g(x) = f\left(\frac{1}{1 - x}\right) \quad \text{for } x \text{ near } 1 - \frac{1}{x_0},$$

where $x_0 \in I^\circ \subset (s, \infty)$, and the answer to (II) is yes. By 2.1 (a) g takes weakly lsc to weakly lsc, and 2.35 (b) implies g is operator convex. Since clearly f operator monotone $\Rightarrow g$ operator monotone, g must continue to a function (still both operator monotone and convex) on $(-\infty, 1 - 1/x_0)$; and this implies that f continues to a function (still operator monotone) on $I \cup (0, x_0)$. Now 2.32 (c) applies.

(b) strong $\rightarrow$ strong.

f must be $\geqq 0$ on $I \cap [0, \infty)$, and then a positive answer follows from 2.32 unless I has a left endpoint $s > 0$. Assume I of this type and a positive answer to (II), and consider the g used in (a) above. Now g takes weakly lsc to strongly lsc and a fortiori $QM(A)_{sa}$ to strongly lsc (on some subinterval of its largest domain). Now by 2.36 (see also 2.37 (a)) g must be positive on all of $(-\infty, 1 - 1/x_0)$. Thus not only does f continue to $I \cup (0, x_0)$ (which we already know from (a)), but the continuation is still positive. Hence 2.32 applies.

(c) strong $\rightarrow$ middle.

A positive answer follows from 2.32 unless $I \subset (0, \infty)$. Using 2.11, we see that for any compact $I_0 \subset I$, there must be $\lambda > 0$ such that

$$h \in \overline{A^m_{sa}} \quad \text{and} \quad \sigma(h) \subset I_0 \Rightarrow f(h) + \lambda \in \overline{A^m_{sa}}.$$

Then by (b) this relation must hold on $\text{co}(I_0 \cup \{0\})$. Thus f has a finite limit at 0 (we already know by (a) that f continues to 0) and 2.32 (b) applies.

(d) middle $\rightarrow$ weak or weak $\rightarrow$ weak.

These cases are the same and 2.35 (b) implies f must be operator convex (as well as operator monotone) for a positive answer. Thus 2.30 applies.

(e) middle $\rightarrow$ strong or weak $\rightarrow$ strong.

Again f is operator convex and operator monotone for a positive answer. Thus f can be extended to an interval whose left endpoint is $-\infty$. By 2.36 the extended function will be $\geqq 0$ on its entire domain. Hence 2.39 (a) applies.

(f) middle $\rightarrow$ middle.

Since a positive answer here implies a positive answer in case (d), f must extend to an interval to which 2.30 applies.

(g) weak $\rightarrow$ middle.

Again f must extend to an interval with left endpoint $-\infty$. By 2.11 for any compact $I_0 \subset I$ there is $\lambda > 0$ such that $f + \lambda$ takes weakly lsc to strongly lsc for $\sigma(h) \subset I_0$. Then by 2.36 $f + \lambda$ must be positive on its whole domain. Hence 2.39 (b) applies.

In parts (c) and (g) we did not quite prove that a positive answer to (II) for $A = E_1$ implies a positive answer for all A. When we used 2.11, we were invoking $A = c_0 \otimes E_1$. But $c_0 \otimes E_1$ can be embedded in E_1 so that

$$\text{her}(c_0 \otimes E_1) = E_1,$$

and then 2.14 implies that a positive answer for E_1 implies a positive answer for $c_0 \otimes E_1$.

2.43. *Remark*. We now discuss the sharpness of 2.34, 2.36.

(a) If $h \in A_{sa}^{**}$ and $f(h)$ is weakly lsc for all operator convex f, then $\pm h$ are weakly lsc so that h must be in $QM(A)$.

(b) If in (a) we replace weakly lsc with middle or strongly lsc, we would obtain that $h \in M(A)$ or $h \in A$. Since A and $M(A)$ are C^*-algebras, there are no interesting results here.

(c) Given f, 2.34 and 2.35 completely solve the problem "When does f take $QM(A)$ to weakly lsc?", and 2.36 completely solves "When does f take $QM(A)$ to strongly lsc?" By reasoning similar to that in 2.42 (c) and (g), we can see that f takes $QM(A)$ to middle lsc if and only if $f + \lambda$ satisfies the conditions of 2.36 for some $\lambda \in \mathbf{R}$.

2.D. *Relations with compact and open projections*. The next result is not original with us, but we do not know precisely to whom the credit belongs.

2.44. PROPOSITION. *Let* $0 \leqq h \in A^{**}$, *and let q be the range projection of h.*

(a) *If* $h \in \overline{A_+^m}$, *then q is open.*

(b) *If* $h \in [(\tilde{A}_{sa})_m]^-$ *and* $\exists \epsilon > 0$ *such that*

$$\sigma(h) \cap (0, \epsilon) = \emptyset,$$

then $1 - q$ *is open.*

Proof. (a). Assume $h \leqq 1$. By 2.31 (a) $h^\alpha \in \overline{A_+^m}$ for $0 < \alpha < 1$. Since $h^\alpha \nearrow q$ as $\alpha \searrow 0$, this shows $q \in \overline{A_+^m}$; and **[5]** implies q is open.

(b). By 2.30 (c),

$$h^\alpha \in [\,(\widetilde{A}_{sa})_m]^- \quad \text{for } 0 < \alpha < 1.$$

The condition on $\sigma(h)$ implies $h^\alpha \to q$ in norm, so that

$$q \in [\,(\widetilde{A}_{sa})_m]^-.$$

Hence $1 - q \in (\widetilde{A}_{sa}^m)^-$, and **[5]** implies $1 - q$ is open.

2.45. Corollary. (a) *If $h \in [\,(\widetilde{A}_{sa})_m]^-$ and the bottom point, s, in $\sigma(h)$ is isolated, then $E_{\{s\}}(h)$ is open.*

(b) *If $h \in (\widetilde{A}_{sa}^m)^-$ and the top point, t, in $\sigma(h)$ is isolated, then $E_{\{t\}}(h)$ is open.*

(c) *If $h \in QM(A)_{sa}$ and either extreme point of $\sigma(h)$ is isolated, then the corresponding spectral projection is open.*

2.46. *Examples.* (i) 2.44 (a) fails if we assume only $h \in \widetilde{A}_{sa}^m$. Take $A = E_1$ and define h by

$$h_\infty = \frac{1}{4} e_1 \times e_1,$$

$$h_n = \left(\frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}\right) \times \left(\frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}\right) \quad \text{(cf. 2.12)}.$$

(ii) Even if A is unital, there can be lsc $h \in A^{**}$, $t \in \mathbf{R}$, and $\epsilon > 0$ such that

$$\sigma(h) \cap (t - \epsilon, t + \epsilon) = \emptyset$$

but $E_{(t,\infty)}(h)$ is not open: Take

$$A = E_2, \quad h_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = 1, 2, \dots,$$

$$h_\infty = \begin{pmatrix} \frac{2}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{-2}{5} \end{pmatrix}.$$

h is lsc since $h_n \geqq h_\infty$. $\sigma(h) = \{-1/2, 0, 1/2, 1\}$ and $E_{\{1/2,1\}}(h)$ is not open.

2.47. Definition-Lemma. *Let $p \in A^{**}$ be a closed projection. Then p is called compact* (**[4]**) *if it satisfies one of the following equivalent conditions.*

(i) $\exists a \in A$ *such that* $p \leqq a \leqq 1$ (*this implies* $[a, p] = 0$).
(ii) $\exists a \in A$ *such that* $p \leqq a$.
(ii′) $p \in \text{her}_{A^{**}}(A)$.
(iii) p *is closed in* $\tilde{A}^{**}$ (*under* $A^{**} \subset \tilde{A}^{**} \cong A^{**} \oplus \mathbf{C}$).
(iv) $p \in (A_{sa})_m^-$.
(v) p *is closed in* $M(A)^{**}$.

Proof. (i) is Akemann's original definition, and he proved the equivalence of (i), (ii), and (iii) (II.4 and II.5 of [**4**] and their proofs). (ii′) is just a restatement of (ii) in view of Theorem 1.2 of [**3**].

(i) ⇒ (iv): Let (e_α) be an approximate identity of $\text{her}(1 - p)$. Then $(1 - e_\alpha) \searrow p$. Hence

$$a^{1/2}(1 - e_\alpha)a^{1/2} \searrow a^{1/2}pa^{1/2} = p.$$

(iv) ⇒ (v): It is obvious (trivial portion of 2.14 (a)) that $p \in (M(A)_{sa})_m^-$. This implies $1 - p \in M(A)_{sa}^m$ so that $1 - p$ is open in $M(A)^{**}$ by [**5**].

(v) ⇒ (i): (This is really the same as (iii) ⇒ (i).) Let

$$B = \text{her}_{M(A)}(1 - p).$$

Then since $p \mapsto 0$ in $(M(A)/A)^{**}$, B maps onto $M(A)/A$. In particular, some $b \in B$ maps onto $1 \in M(A)/A$. We may assume $0 \leqq b \leqq 1$ (use Lemma 2.2 of [**6**]). Then $b = 1 - a$, $a \in A$, where $0 \leqq a \leqq 1$; and $1 - a \leqq 1 - p \Rightarrow a \geqq p$.

Remarks. (i) [**5**] showed that all meanings of lsc are the same for projections, but this is not the case for usc. For projections weakly usc ⇔ middle usc ⇔ closed.

(ii) The proof of 2.47 used the hypothesis that p is closed, but (iii), (iv), and (v) already imply p closed.

By applying 2.14 (a), we obtain:

2.48. COROLLARY. *If* A_0 *is a* C^**-subalgebra of* A, *and* $p \in A_0^{**} \subset A^{**}$, *then* p *compact as an element of* A^{**} *implies* p *compact as an element of* A_0^{**}.

2.49. *Definitions.* Let $h \in A_{sa}^{**}$.

(i) h is called q-lsc if $E_{(t,\infty)}(h)$ is open, $\forall t \in \mathbf{R}$ (equivalently $E_{(-\infty,t]}(h)$ is closed $\forall t$).

(i′) h is q-usc if $-h$ is q-lsc.

(ii) h is called *strongly* q-lsc if h is q-lsc and $E_{(-\infty,-\epsilon]}(h)$ is compact, $\forall \epsilon > 0$.

(ii′) h is called *strongly* q-usc if $-h$ is strongly q-lsc.
(iii) h is *q-continuous* if it is q-lsc and q-usc.
(iv) h is *strongly q-continuous* if it is strongly q-lsc and strongly q-usc.

Remarks. q-continuity was defined and strong q-continuity was introduced (but not named) by Akemann [**1**] and [**4**]. [**4**] showed that h is strongly q-continuous if and only if $h \in A_{sa}$ and that $h \in M(A)_{sa} \Rightarrow h$ q-continuous. Pedersen [**28**] and Akemann, Pedersen, and Tomiyama [**7**] completed Akemann's conjecture by showing that h q-continuous implies $h \in M(A)$. q-semicontinuity was used (but not named) by Pedersen [**28**] and Olesen, Pedersen [**25**].

2.50. PROPOSITION.

(a) h q-lsc $\Rightarrow h \in \widetilde{A}^m_{sa}$.
(b) h *strongly* q-lsc $\Rightarrow h \in \overline{A^m_{sa}}$.

Proof. (b). Assume $\sigma(h) \subset [s, t]$ where $s < 0$, $t > 0$. For $1 \leqq k \leqq n$ let

$$q_{k,n} = E_{(kt/n,\infty)}(h) \quad \text{and} \quad p_{k,n} = E_{(-\infty,ks/n]}(h).$$

Then $q_{k,n}$ is open, $p_{k,n}$ is compact and hence

$$h_n = \frac{s}{n}\sum_1^n p_{kn} + \frac{t}{n}\sum_1^n q_{k,n}$$

is in $\overline{A^m_{sa}}$.

$$||h_n - h|| \leqq \frac{1}{n}||h|| \Rightarrow h \in \overline{A^m_{sa}}.$$

(a). Choose $\lambda > 0$ such that $h + \lambda \geqq 0$. It is obvious that $h + \lambda$ is still q-lsc, and for positive operators q-lsc and strongly q-lsc are the same. By (b)

$$h + \lambda \in \overline{A^m_{sa}} \Rightarrow h \in \widetilde{A}^m_{sa}.$$

Since q-semicontinuity is strictly (by 2.46 (ii)) stronger than all three types of semicontinuity, it has probably occurred to the reader that maybe we should adopt q-semicontinuity as the basic notion. It seems clear that this is wrong, and that we have to regard the q-lsc elements as just a class of particularly regular lsc elements. Since every element of A_{sa} is q-continuous, $\{x:x \text{ is } q\text{-lsc}\}$ is not closed under increasing limits. Also it will be shown in Section 5 that for $A = E_1$ or E_2 every middle lsc element of A^{**}_{sa} is the sum of a multiplier and a q-lsc element; i.e., $\{x:x \text{ is } q\text{-lsc}\}$ is not closed under addition.

(a) and (c) of the following was told to us by G. Pedersen.

2.51. PROPOSITION. *Let* $h \in A^{**}_{sa}$.

(a) *If h is* q-lsc, $f\nearrow$, *and f is continuous from the left, then $f(h)$ is* q-lsc.

(b) *If h is strongly* q-lsc, $f\nearrow$, *f is continuous from the left, and* $f(0) \geqq 0$, *then $f(h)$ is strongly* q-lsc.

(c) *If* $f(h)$ *is weakly* lsc *for all continuous, monotone increasing* f, *then* h *is* q-lsc.

(d) *If* $f(h)$ *is strongly* lsc *for all continuous, monotone increasing* f *such that* $f(0) = 0$, *then* h *is strongly* q-lsc.

Proof. (a). $E_{(-\infty,t]}(f(h)) = E_{(-\infty,t']}(h)$, where

$$f^{-1}((-\infty, t']) = (-\infty, t'].$$

(b) follows from the same formula as (a) and the observation that $t < 0 \Rightarrow t' < 0$ ($f(0) \geqq 0$).

(c). $\forall t \in \mathbf{R}$, there is a sequence (f_n) of continuous increasing functions such that $f_n \nearrow \chi_{(t,\infty)}$, pointwise. This implies

$$f_n(h) \nearrow E_{(t,\infty)}(h).$$

Hence

$$f_n(h) \in (\widetilde{A}^m_{sa})^-, \forall n \Rightarrow E_{(t,\infty)}(h) \in (\widetilde{A}^m_{sa})^- \Rightarrow E_{(t,\infty)}(h) \text{ open.}$$

(d) If $t \geqq 0$, the f_n's used in (c) can be chosen so that $f_n(0) = 0$. If $t < 0$, the f_n's can be chosen so that $f_n(0) = 1$. Then if $g_n = f_n - 1$,

$$g_n(h) \nearrow (-E_{(\infty,t]}(h)).$$

Since $g_n(h) \in \overline{A^m_{sa}}$, this shows

$$E_{(-\infty,t]}(h) \in (A_{sa})^-_m \Rightarrow E_{(-\infty,t]}(h) \text{ compact.}$$

2.52. COROLLARY. (a) h *is* q-lsc $\Leftrightarrow$ $f(h)$ *is weakly* (*middle*) lsc, $\forall f$ *as in* 2.51 (c).

(b) h *is strongly* q-lsc $\Leftrightarrow$ $f(h)$ *is strongly* lsc $\forall f$ *as in* 2.51 (d).

(c) $\{h{:}h$ *is* q-lsc$\}$ *and* $\{h{:}h$ *is strongly* q-lsc$\}$ *are norm closed.*

Call $h \in QM(A)_{sa}$ *smooth* if $f(h)$ is weakly lsc for all continuous convex functions f. Then h is smooth if and only if $(h - \lambda)_+$ is weakly lsc, $\forall \lambda \in \mathbf{R}$ (given $h \in QM(A)_{sa}$). We have not been able to find any other description of smooth quasi-multipliers or to make good use of the concept, but in view of 2.52 and Section 2.C it seems a reasonable analogue of q-semicontinuity. Also it seems to be in the right spirit for use in improving some of the results of Section 3. (3.49 and 3.47 are not as good as 3.48 and 3.46.)

2.53. PROPOSITION. *If* $h \in QM(A)_{sa}$ *and* $\sigma(h)$ *has only four points, then* h *is smooth.*

Proof. Assume $\sigma(h) \subset \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Since

$$(h - t\lambda' - (1 - t)\lambda'')_+ = t(h - \lambda')_+ + (1 - t)(h - \lambda'')_+$$

when $0 \leqq t \leqq 1$ and $\sigma(h) \cap (\lambda', \lambda'') = \emptyset$, it is sufficient to check that $(h - \lambda_i)_+$ is weakly lsc.

$$(h - \lambda_1)_+ = h - \lambda_1, \quad (h - \lambda_4)_+ = 0,$$
$$(h - \lambda_2)_+ = h - \lambda_2 + (\lambda_2 - \lambda_1)E_{\{\lambda_1\}}(h), \quad \text{and}$$
$$(h - \lambda_3)_+ = (\lambda_4 - \lambda_3)E_{\{\lambda_4\}}(h).$$

Thus the result follows from 2.45 (c).

2.54. *Example.* $\exists h \in QM(A)_{sa}$ such that $\sigma(h)$ has only five points but h is not smooth: Take $A = E_1$. Let

$$F_1 = \frac{2}{3}\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}, \quad F_3 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{pmatrix}, \quad F_5 = \frac{2}{3}\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

Then $F_1, F_3, F_5 \geqq 0$ and $F_1 + F_3 + F_5 = 1$ (in M_2). By Naimark's dilation theorem we can find projections E_1, E_3, E_5 on $\mathbf{C}^N$ for some $N > 2$ such that

$$E_1 + E_3 + E_5 = 1 \text{ and } \mathrm{pr}_{\mathbf{C}^2}(E_i) = F_i,$$

where pr denotes compression. Let

$$\lambda_1 = -1, \lambda_2 = -\frac{\sqrt{3}}{3}, \quad \lambda_3 = 0, \quad \lambda_4 = \frac{\sqrt{3}}{3}, \quad \lambda_5 = 1.$$

As in the proof of 2.35 (b) we can construct $h \in QM(A)$ such that

$$h_\infty = \sum \lambda_i F_i, \; h_n = \sum \lambda_i E_i(n), \quad \text{and} \quad E_i(n) \to F_i \text{ weakly.}$$

Here $(E_1(n), E_3(n), E_5(n))$ is "unitarily equivalent" to (E_1, E_3, E_5) and

$$F_2 = F_4 = E_2(n) = E_4(n) = 0.$$

For any f,

$$f(h_n) = \sum f(\lambda_i)E_i(n) + f(0)(1 - E_1(n) - E_3(n) - E_5(n)),$$

and

$$f(h_n) \to \sum f(\lambda_i)F_i + f(0)q,$$

weakly, where

$$q = 1 - e_1 \times e_1 - e_2 \times e_2.$$

Thus $f(h)$ is weakly lsc if and only if

$$f(\sum \lambda_i F_i) \leqq \sum f(\lambda_i)F_i.$$

Now take $f(x) = x_+$. Computation shows that

$$\sigma(\sum \lambda_i F_i) = \{\lambda_2, \lambda_4\} \quad \text{and}$$

$$f(\Sigma \lambda_i F_i) = \frac{\sqrt{3}}{3}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Since

$$\Sigma f(\lambda_i)F_i = F_5 \quad \text{and} \quad \frac{\sqrt{3}}{3} \cdot \frac{1}{2} > \frac{2}{3} \cdot \frac{1}{4},$$

$f(h)$ is not weakly lsc.

We remark that for $A = E_1$ it is possible to find a strongly lsc h such that $\sigma(h)$ has only three points but h is not q-lsc. However, this is not possible in a unital algebra. In 2.46 (ii) A is unital and $\sigma(h)$ has four points.

2.E. *Miscellaneous Results*. In this subsection we add a bit to Pedersen's classification of the lsc elements in the center of A^{**}, discuss the relation to semicontinuity of the map $h \mapsto T^*hT$ for T some kind of multiplier, and discuss when a function of a quasi-multiplier can be a quasi-multiplier. (We show that Proposition 4.4 of [**5**] is really a convexity result.)

Pedersen [**28**] (or 4.4.6 of [**29**]) showed that the weakly and middle lsc elements of the center of A^{**} are the same and can be identified with the bounded lsc functions on prim A. Also these elements are all q-lsc.

2.55. PROPOSITION. *Let h be a central middle* lsc *element of A^{**}. The following are equivalent.*

(i) *h is strongly q-*lsc.

(ii) $h \in \overline{A^m_{sa}}$.

(iii) $\forall \epsilon > 0$, *the quotient algebra of A corresponding to the closed central projection $E_{(-\infty,-\epsilon]}(h)$ is unital.*

Proof. (i) $\Rightarrow$ (ii) follows from 2.50.

(ii) $\Rightarrow$ (iii). If I is the ideal being considered (the open central projection of I is $E_{(-\epsilon,\infty)}(h)$), then clearly $\bar{h}$, the image of h in $(A/I)^{**}$, lies in $[(A/I)^m_{sa}]^-$. Since $\bar{h} \leqq -\epsilon$, 2.1 (c) implies A/I is unital.

(iii) $\Rightarrow$ (i). We can find $a \in A_{sa}$ such that $0 \leqq a \leqq 1$ and a maps to 1 in A/I. This means that

$$E_{(-\infty,-\epsilon]}(h) \leqq a \leqq 1,$$

so that $E_{(-\infty,-\epsilon]}(h)$ is compact.

Remark. If A is commutative, say $A = C_0(X)$, the strongly lsc elements correspond to the bounded lsc functions f on X such that f_- vanishes at ∞. To interpret the above in an analogous way, we would have to consider the closed compact subsets of prim A to be the closed subsets corresponding to compact central projections, rather than just using the topology of prim A.

2.56. PROPOSITION. *For* $T \in A^{**}$ *consider the map*

$$\varphi_T: A^{**}_{sa} \to A^{**}_{sa}$$

defined by $\varphi_T(h) = T^*hT$.

(a) *If* $T \in QM(A)$, φ_T *sends* $\overline{A^m_{sa}}$ *into* $(\widetilde{A}^m_{sa})^-$.

(b) *If* $T \in LM(A)$, φ_T *sends* $(\widetilde{A}^m_{sa})^-$ *into itself.*

(c) *If* $T \in RM(A)$, φ_T *sends* $\overline{A^m_{sa}}$ *into itself.*

(d) *If* $T \in M(A)$, φ_T *sends* $\widetilde{A}^m_{sa}$ *into itself* (*and* (b), (c) *also apply*).

(e) *If* $T \in A$, φ_T *sends* $(\widetilde{A}^m_{sa})^-$ *into* $\overline{A^m_{sa}}$.

Proof. (a). For $a \in A_{sa}$,

$$T^*aT \in QM(A)_{sa} \subset (\widetilde{A}^m_{sa})^-.$$

Since φ_T is positive and continuous, (a) follows.

(b). For $x \in \widetilde{A}_{sa}$, $T^*xT \in QM(A)_{sa}$.

(c). For $a \in A_{sa}$, $T^*aT \in A_{sa}$.

(d). For $x \in \widetilde{A}_{sa}$, $T^*xT \in M(A)_{sa}$.

(e) follows from 2.4.

2.57. *Examples-Remarks.* The following are enough to show there are no obvious improvements of 2.56.

(i) $\exists T \in LM(A)$ such that φ_T does not send $\overline{A^m_{sa}}$ into $\widetilde{A}^m_{sa}$: In fact any T such that $T^*T \notin M(A)$ will be an example (and there are many such), since

$$1 \in M(A)_+ \subset \overline{A^m_{sa}}.$$

Since $T^*T \in QM(A)$, if T^*T were in $\widetilde{A}^m_{sa}$, 2.3 would imply $T^*T \in M(A)$.

We also give an example where $T^*AT \not\subset M(A)$. (As above $T^*AT \subset QM(A)$ and $T^*A_{sa}T \subset \widetilde{A}^m_{sa} \Leftrightarrow T^*AT \subset M(A)$.) Take $A = E_1$. Define $T \in LM(A)$ by

$$T_n = e_1 \times e_1 + e_1 \times e_n, \quad T_\infty = e_1 \times e_1,$$

and $a \in A$ by $a_n = a_\infty = e_1 \times e_1$. In general for $T \in LM(A)$, $T^*AT \subset M(A)$ if and only if T induces an element of $M(I)$ where I is the smallest ideal such that $T \in I^{**}$. This can occur for $T \in LM(A)\backslash M(A)$ and it has some relation with certain pathologies in $M(A)$. See 3.56.

(ii) $\exists T \in RM(A)$ such that φ_T does not send $\widetilde{A}^m_{sa}$ into $(\widetilde{A}^m_{sa})^-$: In fact any $T \in RM(A)\backslash M(A)$ will be such an example. Since $-1 \in \widetilde{A}^m_{sa}$, if φ_T sends $\widetilde{A}^m_{sa}$ into $(\widetilde{A}^m_{sa})^-$, then

$$T^*T \in |(\widetilde{A}_{sa})_m]^-.$$

Since $T^*T \in \overline{A^m_{sa}}$ (by (c)), 2.3 implies $T^*T \in M(A)$; and Proposition 4.4 of [**5**] implies $T \in LM(A)$.

We also give an example of $0 \leqq h \in \widetilde{A}^m_{sa}$ such that

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$T^*hT \notin (\widetilde{A}_{sa}^m)^-.$$

Take $A = E_1$; define T by

$$T_\infty = e_1 \times e_1,\ T_n = e_1 \times e_1 - e_{n+1} \times e_1;$$

and define h by

$$h_\infty = \frac{1}{4} e_1 \times e_1,$$

$$h_n = \left(\frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}\right) \times \left(\frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}\right) \quad \text{(cf. 2.12).}$$

(iii) If

$$T \in QM(A) \quad \text{and} \quad \varphi_T : \overline{A_{sa}^m} \to \overline{A_{sa}^m},$$

then $T \in RM(A)$.

In fact the hypothesis implies $T^*AT \subset A$. Then 2.6 (b) applies to aT, $a \in A$.

(iv) There are no general results where φ_T, $T \in M(A)$, sends some semicontinuity class into a smaller class, since there are invertible multipliers.

2.58. LEMMA. *Assume* $0 < \epsilon \leqq h \in A^{**}$.

(a) *If h is weakly* usc *and* $h^{-1} \in QM(A)$, *then* $h \in M(A)$. (*In particular this applies if* $h, h^{-1} \in QM(A)$.)

(b) *If* $\exists \delta > 0$ *such that* $h - \delta$ *is strongly* lsc *and* $h^{-1} \in QM(A)$, *then* $h \in M(A)$.

Proof. (a). By 2.1 (a) h^{-1} is strongly lsc. Hence 2.3 implies $h^{-1} \in M(A)$. Since $M(A)$ is a C^*-algebra, this implies $h \in M(A)$.

(b). By 2.1 (b) h^{-1} is middle usc. Hence 2.3 implies $h^{-1} \in M(A)$, whence $h \in M(A)$.

In the following we will use a simple fact of general topology:

(F) If f_1 and f_2 are lsc functions on a topological space and $f_1 + f_2$ is continuous, then f_1, f_2 are continuous.

2.59. PROPOSITION. (a) *If f is a non-linear operator convex function on an interval I, $h \in QM(A)_{sa}$, $\sigma(h) \subset I$, and $f(h) \in QM(A)$, then $h \in M(A)$.*

(b) *If f is operator monotone, operator convex, and non-constant on an interval I, if $h \in (\widetilde{A}_{sa}^m)^-$, $\sigma(h) \subset I$, and if $f(h) \in QM(A)$, then $h \in QM(A)$.* (*Hence, by* (a), $h \in M(A)$ *unless f is linear.*)

(c) *If f is operator monotone on an interval I such that either* $0 \in I$ *or* $I = (0, b)$, *if* $h \in \overline{A_{sa}^m}$, $\sigma(h) \subset I$, *and if* $f(h) \in QM(A)$, *then* $h \in M(A)$ *except when* $I = (0, b)$ *and*

$$f(x) = -\frac{C}{x} + B, \quad C \geqq 0.$$

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

Proof. (a). We refer to the integral representation (1) for f (proof of 2.34). If $f = f_1 + f_2$ with f_1, f_2 operator convex, then by (F) $f_1(h)$, $f_2(h) \in QM(A)$. Any Borel set of $\mathbf{R}\setminus I$ can be used to obtain such a decomposition from (1). Since the integrand in (1) is norm continuous (in t), this implies that $(h - t)^{-1} \in QM(A)$ for any $t \notin I$ which is in the closed support of $\mu_{\pm}$. If there is such a t, 2.58 (a) implies $h \in M(A)$. Also if $a \neq 0$, (F) implies $h^2 \in QM(A)$; and then Proposition 4.4 of [**5**] implies $h \in M(A)$.

(b) is proved in the same way as (a).

(c) is proved the same way, except that 2.58 (b) is used.

2.60. *Remark.* We now discuss Proposition 4.4 of [**5**]. The function $T \mapsto T^*T$ is (operator) convex on $B(H)$. Suppose f is a real-valued function on an interval $I \subset [0, \infty)$ such that $0 \in I$. Then we have an operator function

$$\psi_f: T \mapsto f(T^*T)$$

defined for all $T \in B(H)$ such that $\|T\|$ is sufficiently small (this is a convex set). It is natural to ask when ψ_f is convex, and it can be shown that the answer is: ψ_f is convex if and only if f is both operator monotone and operator convex. Of course this implies f can be continued to $I \cup (-\infty, 0)$. For such an f we can apply 2.59 (b) with $h = T^*T$.

If f is as in 2.59 (b), $T \in QM(A)$, and $f(T^*T) \in QM(A)$, then $T \in LM(A)$. Moreover, $T^*T \in M(A)$ unless f is linear.

Of course one could also use 2.59 (c) for $h = T^*T$, $T \in RM(A)$.

We will now apply the above to answer the following: When is it possible to find $T \in QM(A)\setminus LM(A)$ such that T^n or $|T|^n \in QM(A)$? We will consider three possibilities: $T \in QM(A)_+$, $T \in QM(A)_{sa}$, or $T \in QM(A)$. Of course if we find an example in one class, there is no need to consider larger classes.

2.61. *If* $h \in QM(A)_+$ *and* $h^\alpha \in QM(A)$ *for* $1 \neq \alpha > 0$, *then* $h \in M(A)$.

Proof. Use 2.59 (a) and the operator convex function

$$x \mapsto -x^\alpha \quad \text{or} \quad x \mapsto -x^{1/\alpha},$$

according as $\alpha < 1$ or $\alpha > 1$.

2.62. *If* $0 < \epsilon \leqq h \in QM(A)$ *and* $h^\alpha \in QM(A)$ *for* $\alpha < 0$, *then* $h \in M(A)$.

Proof. Use 2.59 (a) and the operator convex function

$$x \mapsto x^\alpha \quad \text{or} \quad x \mapsto x^{1/\alpha}$$

according as $|\alpha| \leqq 1$ or $|\alpha| \geqq 1$.

2.63. *If* $T \in QM(A)$ *and* $|T|^\alpha \in QM(A)$ *for some* $\alpha > 2$, *then* $|T| \in M(A)$. *This implies* $T \in LM(A)$, *and if* $T = T^*$ *it implies* $T \in M(A)$.

Remark. A similar result with a slightly weaker conclusion is true for $\alpha = 2$, but this would be exactly Proposition 4.4 of [**5**].

Proof. Use 2.59 (a) and the operator convex function

$$f(x) = -x^{2/\alpha},$$

applied to $h = |T|^\alpha$. Note that 2.34 implies $f(h)$ weakly lsc, and 2.56 (a), for example, implies $f(h)$ weakly usc.

2.64. LEMMA. *If* $h \in QM(A)_{sa}$ *and* h_+, h_- *are weakly* usc, *then* $h_+, h_-, |h|$ *are in* $QM(A)$.

Proof. Let $\varphi_\alpha \to \varphi$ in $S(A)$. We want to show

$$\varphi_\alpha(h_\pm) \to \varphi(h_\pm).$$

Let π, π_α be the GNS representations for φ, φ_α, extended to A^{**}. We may assume the Hilbert spaces for π, π_α have the same dimension. (If not, replace A by $A \otimes \mathcal{K}(H)$ for H a Hilbert space of sufficiently large dimension.) Then, passing to a subnet, we may realize π, π_α on the same Hilbert space H, with one unit vector ξ cyclic for π and all π_α's and inducing φ, φ_α, so that $\pi_\alpha(x) \to \pi(x)$ strongly, $\forall x \in A$ (cf. Section 3.5 of [**18**]). We claim that $\pi_\alpha(h) \to \pi(h)$ weakly and $\forall v \in H$,

$$\overline{\lim}(\pi_\alpha(h_\pm)v, v) \leqq (\pi(h_\pm)v, v).$$

Assume $||v|| = 1$. Then we can define $\psi, \psi_\alpha \in S(A)$ by

$$\psi(x) = (\pi(x)v, v), \quad \psi_\alpha(x) = (\pi_\alpha(x)v, v), \quad \forall x \in A.$$

(The fact that π, π_α are non-degenerate is important here.) Then $\psi_\alpha \to \psi$ weak* (since $\pi_\alpha(x) \to \pi(x)$), and this and the hypotheses on h, $h_\pm$ give the claim. Now passing to a subnet, we may assume $\pi_\alpha(h_\pm) \to k_\pm$ weakly, for some operators $k_\pm$. Then

$$0 \leqq k_\pm \leqq \pi(h_\pm), \quad \text{and}$$

$$k_+ - k_- = \lim \pi_\alpha(h_+ - h_-) = \pi(h) = \pi(h_+) - \pi(h_-).$$

Since $\pi(h_+) \cdot \pi(h_-) = 0$, this implies $k_\pm = \pi(h_\pm)$; i.e., $\pi_\alpha(h_\pm) \to \pi(h_\pm)$ weakly. Hence

$$\varphi_\alpha(h_\pm) = (\pi_\alpha(h_\pm)\xi, \xi) \to (\pi(h_\pm)\xi, \xi) = \varphi(h_\pm).$$

2.65. COROLLARY. *If* $h \in QM(A)_{sa}$ *and* $|h|^\alpha \in QM(A)$ *for some* $\alpha > 1$, *then* $h \in M(A)$.

Proof. Since $|h|^\alpha$ is weakly usc, 2.30 implies

$$|h| = (\,|h|^\alpha)^{1/\alpha}$$

is weakly usc. Since $|h| = h_+ + h_-$, $h = h_+ - h_-$, and $-h = h_- - h_+$ are all weakly usc, we have the hypotheses of 2.64. Hence $|h| \in QM(A)$ and 2.61 implies $|h| \in M(A)$. Thus $h^2 = |h|^2 \in M(A)$, and Proposition 4.4 of **[5]** completes the proof.

2.66. *Example.* We show a general method of constructing examples of $h \in QM(A)_{sa} \backslash M(A)$ such that $f(h) \in QM(A)$. Assume for simplicity that $1 \in$ domain f. Let $b^* = b \in M_k$ such that $b_{11} = 1$. Take $A = E_1$ and define h by $h_\infty = 1$,

$$h_n = \sum_{i=1}^{n} \sum_{p,q} b_{pq} e_{i+(p-1)n} \times e_{i+(q-1)n} + 1 - \sum_{1}^{kn} e_i \times e_i.$$

Then $h_n \to h_\infty$ weakly, so that $h \in QM(A)_{sa}$. To insure that $h_n \not\to h_\infty$ strongly (so that $h \notin M(A)$), we simply need $b_{p1} \neq 0$ for some $p > 1$. Now for any f, $f(h_n) \to f(b)_{11} \cdot 1$ weakly. Thus $f(h) \in QM(A)$ if and only if $f(1) = f(b)_{11}$. Write

$$B = U \operatorname{diag}(\lambda_1, \ldots, \lambda_k) U^*$$

for U unitary, $\lambda_1, \ldots, \lambda_k \in$ domain f, and let $t_p = |U_{1p}|^2$. Then

$$t_p \geqq 0, \quad \sum_{1}^{k} t_p = 1,$$

and any such t_p's can arise. The conditions $b_{11} = 1$, $f(b)_{11} = f(1)$ are equivalent to

$$\sum_{1}^{k} t_p(\lambda_p, f(\lambda_p)) = (1, f(1)).$$

The condition $b_{p1} \neq 0$ for some $p > 1$ is equivalent to: $t_p \neq 0$ for some p such that $\lambda_p \neq 1$. Thus we can find the desired example by this method if and only if $(1, f(1))$ is not an extreme point of the graph of f. Note that this construction does not illuminate the distinction between operator convex functions and arbitrary convex functions.

Conclusions. If $f(x) = |x|^\alpha$, $0 < \alpha < 1$; x^n, n odd and positive; x^{-n}, n any positive integer; or $|x|^\alpha$, $\alpha < 0$, then

$$\exists h \in QM(E_1)_{sa} \backslash M(E_1) \text{ such that } f(h) \in QM(E_1).$$

Of course for the last two cases, $0 \notin$ domain f, and the h we construct is invertible. The cases $f(x) = |x|^\alpha$, $\alpha = 0$ or 1, are trivial, of course; and x^n, n even, is the same as $|x|^n$. Thus the problem is completed for $h \in QM(A)_+$ and $h \in QM(A)_{sa}$.

2.67. *Example.* For $1 < \alpha < 2$, $\exists T \in QM(A) \backslash LM(A)$ such that $|T|^\alpha \in QM(A)$: Take $A = E_1$ and define T by

$$T_\infty = e_1 \times e_1 \quad \text{and}$$
$$T_n = e_1 \times e_1 + e_1 \times e_{n+1} + (2^{(2/\alpha)-1} - 1)^{1/2}(e_{n+1} \times e_1 + e_{n+1} \times e_{n+1}).$$

2.68. *Example.* $\exists h \in QM(A)_{sa} \backslash M(A)$ such that $h^n \in QM(A)$, for all odd positive n: Take $A = E_1$. Choose a sequence (p_n) of projections such that $p_n \to 1/2$ weakly. Define h by $h_n = 2p_n - 1$, $h_\infty = 0$. Then $h^3 = h$.

2.69. $\nexists h \in QM(A)_{sa} \backslash M(A)$ *such that* $h^n \in QM(A)$, *for all odd positive* n, *and* h^{-1} *exists* (*in* A^{**}).

Proof. By the Weierstrass approximation theorem, the hypothesis on h implies $f(h) \in QM(A)$ for every odd continuous f. In particular, since $0 \notin \sigma(h)$, $u = \text{sgn}(h) \in QM(A)$. (If $p_\pm$ are the range projections of $h_\pm$, $u = p_+ - p_-$.) Proposition 4.4 of [5] (or 2.45 (c)) show that $u, p_\pm \in M(A)$. The proof is completed, for example, by applying 2.61 to h_+, h_- separately.

2.70. *Example.* $\exists T \in QM(A) \backslash LM(A)$ such that $T^n \in QM(A)$ for all integers n: Take

$$T = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix},$$

where $A = B \otimes M_2$ and $S \in QM(B) \backslash LM(B)$.

2.71. *Example.* $\exists T \in QM(A)$ such that $T^2 = T$ but the range projection of T is not open: Take $A = E_1$ and define T by

$$T_\infty = e_1 \times e_1, \quad T_n = e_1 \times e_1 + e_{n+1} \times e_1.$$

(In this example $T \in RM(A)$. 2.44 (a) rules out this phenomenon for $T \in LM(A)$ ($TT^* \in \overline{A_+^m}$). By looking at $T \oplus T'$ where $T' \notin RM(A)$, we could make an example where $T \notin LM(A) \cup RM(A)$.)

3. Main results. Any σ-compact locally compact (Hausdorff) space is normal. Also any σ-compact open subset of an arbitrary locally compact space X is normal, for example $\{x: f(x) \neq 0\}$ for some $f \in C_0(X)$. Toward a non-commutative analogue of this, consider (N1) to (N5) below, each of which is either a basic property of normal topological spaces or a non-commutative analogue.

(N1) Urysohn's lemma.

(N2) (interpolation) If f is an lsc function and g a usc function on a normal topological space, and if $f \geqq g$, then there is a continuous function h such that $f \geqq h \geqq g$.

(N3) If $\theta: A \to B$ is a surjective homomorphism of σ-unital C^*-algebras and $h \in M(B)_{sa}$, then there is $\tilde{h} \in M(A)_{sa}$ such that

$$\theta^{**}(\tilde{h}) = h \quad \text{and} \quad \sigma(\tilde{h}) \subset \text{co}(\sigma(h)).$$

(N4) If $p \in A^{**}$ is a closed projection, where A is an arbitrary or σ-unital C^*-algebra, and $h \in pA_{sa}^{**}p$ is strongly q-continuous or q-continuous on p, then there is $\tilde{h} \in A_{sa}$ or $M(A)_{sa}$ such that $[\tilde{h}, p] = 0$, $p\tilde{h} = h$, and

$$\sigma(\tilde{h}) \subset \text{co}(\sigma(h) \cup \{0\}) \quad \text{or} \quad \sigma(\tilde{h}) \subset \text{co}(\sigma(h)).$$

(N5) If F is a closed face of $\Delta(A)$ containing 0 and h a continuous real affine functional on F such that $h(0) = 0$, then there is a continuous extension $\tilde{h}$ of h to $\Delta(A)$ such that $\tilde{h}(\Delta(A)) \subset h(F)$.

The non-commutative version of (N1) for the strong case was found by Akemann [**4**]: If p is a compact projection, q a closed projection, and $pq = 0$, then $\exists h \in A_{sa}$ such that $p \leqq h \leqq 1 - q$. The middle case of (N1) is Lemma 3.31 below. (N2) provides an efficient method of establishing the basic properties of normal spaces. Its proof is similar to that of Urysohn's Lemma, and only slightly harder, and the Tietze extension theorem (as well as Urysohn's Lemma) is an immediate corollary. The non-commutative cases of (N2) were discussed in Section 1. (N3) is the middle case of an analogue of the Tietze extension theorem, with closed sets being replaced by ideals. It was proved by Pedersen [**30**], generalizing a version by Akemann, Pedersen, and Tomiyama [**7**]. The strong case of (N3) is trivial, and the weak case, which involves $QM(A)$, and also a version for $LM(A)$ were proved in [**10**]. (N4) contains the strong and middle cases of an analogue of the Tietze extension theorem, with closed sets being replaced by closed projections. It specializes to (N3) when p is central and will be proved below (3.43) as an application of interpolation. We have no weak version of (N4), but the weak version of (N3) could also be deduced from interpolation. (N5) is an even more non-commutative analogue of the Tietze extension theorem (strong case). We have no middle or weak version but do have some one-sided versions (involving non-self-adjoint operators). (N5) has nothing to do with interpolation or semicontinuity so far as we know. If the condition $\tilde{h}(\Delta(A)) \subset h(F)$ is dropped, it becomes a known result (though we do not know whose result); our reason for investigating the more precise version was to find out if there was a true analogue of the Tietze theorem.

Before taking up (N5), we discuss the techniques of Section 3. Our proof of interpolation does not resemble the classical proof of (N2), though our proof of Theorem 3.40 (middle case) does use some ideas of classical topology. (N2) for paracompact spaces follows from the most basic of Michael's selection theorems [**24**], and thinking about how to use Michael's theorem for the example $A = C_0(X) \otimes \mathcal{K}$ was a great help to us.

3.A. $x \mapsto pxp$, $x \mapsto xp$, *and* $x \mapsto (px, xp)$ (*maximally non-commutative Tietze extension theorems*).

3.1. LEMMA. *Let* p, q *be projections in a* W^*-*algebra* M, $\epsilon > 0$, *and* $x \in M$.

(a) *If* $\|xq\| \leqq 1$, $\|x\| \leqq 1 + \epsilon$, *then* $\exists y \in M(1 - q)$ *such that*

$$\|y\| \leqq \sqrt{2\epsilon + \epsilon^2} \text{ and } \|x - y\| \leqq 1.$$

(b) *If* $\|pxq\| \leqq 1$, $\|x\| \leqq 1 + \epsilon$, *then* $\exists y \in (1 - p)M + M(1 - q)$ *such that*

$$\|y\| \leqq 2\sqrt{2\epsilon + \epsilon^2} \text{ and } \|x - y\| \leqq 1.$$

Proof. We will use matrix notation.

(a). Write $x = (a \quad b)$, $a = xq$, $b = x(1 - q)$.

$$aa^* + bb^* \leqq 1 + 2\epsilon + \epsilon^2 \Rightarrow bb^* \leqq 1 + 2\epsilon + \epsilon^2 - aa^*$$

$$\Rightarrow b = (1 + 2\epsilon + \epsilon^2 - aa^*)^{1/2}t \quad \text{with } \|t\| \leqq 1.$$

Write $b' = (1 - aa^*)^{1/2}t$. Then $\|b' - b\| \leqq \sqrt{2\epsilon + \epsilon^2}$. ($x - y = (a \quad b')$.)

(b). Write

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} b \\ d \end{pmatrix}, \quad a = pxq, \quad \text{etc.}$$

$\gamma_1^*\gamma_1 = a^*a + c^*c \leqq (1 + \epsilon)^2$. Symmetrically to the proof of (a), write

$$c = t(1 + 2\epsilon + \epsilon^2 - a^*a)^{1/2}, \quad \|t\| \leqq 1, \quad \text{and}$$

$$c' = t(1 - a^*a)^{1/2}.$$

Thus $\|c' - c\| \leqq \sqrt{2\epsilon + \epsilon^2}$. Also if $\gamma_1' = \begin{pmatrix} a \\ c' \end{pmatrix}$, then

$$\|\gamma_1'\| \leqq 1 \quad \text{and} \quad \gamma_1' = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \left(\dfrac{1 - a^*a}{1 + 2\epsilon + \epsilon^2 - a^*a}\right)^{1/2} \end{pmatrix}.$$

This implies

$$\gamma_1'\gamma_1'^* + \gamma_2\gamma_2^* \leqq \gamma_1\gamma_1^* + \gamma_2\gamma_2^* \leqq (1 + \epsilon)^2.$$

Now if the argument of (a) is applied to $x' = (\gamma_1' \ \gamma_2)$, we find γ_2' such that

$$\|\gamma_2' - \gamma_2\| \leqq \sqrt{2\epsilon + \epsilon^2} \quad \text{and} \quad \|(\gamma_1' \ \gamma_2')\| \leqq 1.$$

Take $x - y = (\gamma_1' \ \gamma_2')$.

Note. In (a) the estimate $\sqrt{2\epsilon + \epsilon^2}$ is sharp, and in (b) the order of magnitude is sharp. Consider

$$x = \begin{pmatrix} 1 & \sqrt{2\epsilon + \epsilon^2} \\ 0 & 0 \end{pmatrix}.$$

3.2. LEMMA. *Let $p, q \in A^{**}$ be closed projections, and let R be the (norm) closed right ideal of A corresponding to p and L the closed left ideal corresponding to q.*

(a) *Let $x \in A$ such that $\|xq\| \leqq 1$ and $\|x\| \leqq 1 + \epsilon$, $\epsilon > 0$. Let $\delta > 0$. Then $\exists y \in L$ such that*

$$\|y\| \leqq \sqrt{2\epsilon + \epsilon^2} \quad \text{and} \quad \|x - y\| \leqq 1 + \delta.$$

(b) *Let $x \in A$ such that $\|pxq\| \leqq 1$ and $\|x\| \leqq 1 + \epsilon$, $\epsilon > 0$. Then $\exists y \in L + R$ such that*

$$\|y\| \leqq 2\sqrt{2\epsilon + \epsilon^2} \quad \text{and} \quad \|x - y\| \leqq 1 + \delta.$$

Proof. (a) Assume not. Let

$$B = \{z \in A : \|z - x\| < 1 + \delta\},$$

$$B_1 = \{z \in A : \|z - x\| \leqq 1\}, \quad \text{and}$$

$$C = \{y \in L : \|y\| \leqq \sqrt{2\epsilon + \epsilon^2}\}.$$

Then

$$0 \notin B - C \Rightarrow \operatorname{dist}(0, B_1 - C) \geqq \delta.$$

Therefore $\exists f \in A^*$ such that

$$\inf \operatorname{Re} f_{|B_1} > \sup \operatorname{Re} f_{|C}.$$

This implies $\overline{B_1}^{w^*} \cap \overline{C}^{w^*} = \emptyset$ in A^{**}. But

$$\overline{B_1}^{w^*} = \{z \in A^{**} : \|z - x\| \leqq 1\} \quad \text{and}$$

$$\overline{C}^{w^*} = \{y \in \overline{L}^{w^*} : \|y\| \leqq \sqrt{2\epsilon + \epsilon^2}\}.$$

(For any closed subspace X of A, $\overline{X}^{w^*}$ is the bidual of X, and the unit ball of X is dense in the unit ball of its bidual.) But $\overline{L}^{w^*} = A^{**}(1 - q)$, so that 3.1 (a) is contradicted.

(b) is proved in the same manner as (a). A result of Combes **[14]** states that $L + R$ is closed.

$$(L + R)^{-w^*} = A^{**}(1 - q) + (1 - p)A^{**}.$$

3.3. THEOREM. *Let $p, q \in A^{**}$ be closed projections and let R and L be the closed right and left ideals of A corresponding to p and q.*

(a) *Let $x \in A$ be such that $\|xq\| \leqq 1$ and $\|x\| \leqq 1 + \epsilon$, $\epsilon > 0$. Then $\forall \epsilon' > \sqrt{2\epsilon + \epsilon^2}$, $\exists y \in L$ such that $\|y\| \leqq \epsilon'$ and $\|x - y\| \leqq 1$.*

(b) *Let $x \in A$ be such that $\|pxq\| \leqq 1$ and $\|x\| \leqq 1 + \epsilon$, $\epsilon > 0$. Then $\forall \epsilon' > 2\sqrt{2\epsilon + \epsilon^2}$, $\exists y \in L + R$ such that $\|y\| \leqq \epsilon'$ and $\|x - y\| \leqq 1$. In particular* if $\epsilon < 2$, *we may take $\epsilon' = 4\epsilon^{1/2}$.*

Proof. (a). Choose $0 < \epsilon_n \searrow 0$ such that

$$\epsilon_1 = \epsilon \quad \text{and} \quad \sum \epsilon_n^{1/2} < \infty.$$

Choose y_1 as in 3.2 (a) with $\delta = \epsilon_2$. Then choose y_2 as in 3.2 (a) with x replaced by $x - y_1$, ϵ replaced by ϵ_2, and $\delta = \epsilon_3$. Continue. Then

$$\|y_n\| \leqq \sqrt{2\epsilon_n + \epsilon_n^2} \leqq 2\epsilon_n^{1/2}$$

for n sufficiently large, and

$$\|x - y_1 - \ldots - y_n\| \leqq 1 + \epsilon_{n+1}.$$

Therefore $y = \sum y_n$ exists, $\|x - y\| \leqq 1$, and

$$\|y\| \leqq \sum \sqrt{2\epsilon_n + \epsilon_n^2} = \sqrt{2\epsilon + \epsilon^2} + \sum_2^\infty \sqrt{2\epsilon_n + \epsilon_n^2}.$$

By choosing $\epsilon_2, \epsilon_3, \ldots$ appropriately, we can achieve $\|x - y\| \leqq \epsilon'$.

(b) is proved in exactly the same way, using 3.2 (b).

3.4. COROLLARY. *Let $p \in A^{**}$ be a closed projection and $h \in pA_{sa}p$ such that $\sigma(h)$ (computed in $pA^{**}p$) $\subset [s, t]$. Then if either* $0 \in [s, t]$ *or* $1 \in A$, *$\exists \tilde{h} \in A_{sa}$ such that $p\tilde{h}p = h$ and $\sigma(\tilde{h}) \subset [s, t]$.*

Remark. It was proved by Akemann, Pedersen, and Tomiyama (Proposition 4.4 of [7]) that the map $x \mapsto pxp$ is an isometry of $A/L + L^*$ onto pAp (which is therefore closed). 3.3 (b), applied with $p = q$, gives the additional information that each $x \in pAp$ can be written $p\tilde{x}p$ with $\|\tilde{x}\| = \|x\|$, rather than $\|\tilde{x}\| < \|x\| + \delta$. 3.4 simply gives the self-adjoint version.

Proof. First assume $1 \in A$. If $s = -t$, the conclusion is immediate. The general case can be reduced to this by translation: Replace h by $h - ((s + t)/2)p$.

Now if $1 \notin A$, consider

$$A^{**} \subset \tilde{A}^{**} \cong A^{**} \oplus \mathbf{C}.$$

Let $p_\infty = 0 \oplus 1 \in \tilde{A}^{**}$. Then $p' = p + p_\infty$ is closed in $\tilde{A}^{**}$, $p'\tilde{A}^{**}p' = pA^{**}p \oplus \mathbf{C}$, and $\sigma(h)$, computed in $p'\tilde{A}^{**}p'$, $\subset [s, t]$ (since $0 \in [s, t]$). Hence $\exists \tilde{h} \in \tilde{A}_{sa}$ such that $\sigma(\tilde{h}) \subset [s, t]$ and $p'\tilde{h}p' = h$. Since $p_\infty h = 0$, $p_\infty \tilde{h}$ must $= 0$; and $\tilde{h} \in A_{sa}$.

3.5. COROLLARY (restatement of 3.4). *If F is a closed face of $\Delta(A)$ containing 0, f is a continuous real affine functional on F vanishing at 0, and $f_{|F\cap S(A)}$ takes values in $[s, t]$, then there is a continuous real affine functional $\tilde{f}$ on $\Delta(A)$ such that $\tilde{f}_{|F} = f$ and $\tilde{f}_{|S(A)}$ takes values in $[s, t]$, provided either $1 \in A$ or $0 \in [s, t]$.*

Remark. Our contribution is only that it is not necessary to use $[s - \delta, t + \delta]$ in the conclusion.

Proof. Let p be the closed projection corresponding to F. The elements of $pA^{**}p$ may be regarded as affine functionals on F, vanishing at 0; and for $h \in pA^{**}_{sa}p$, $\mathrm{co}(\sigma(h))$, computed in $pA^{**}p$, is the same as the range of $h_{|F\cap S(A)}$. We need to show that $pA_{sa}p \subset pA^{**}_{sa}p$ consists precisely of the continuous functionals. An elementary theorem in Choquet theory states that any vector space of continuous real affine functionals on a compact convex set F which separates points and contains the constants is norm dense in the space of all continuous real affine functionals. Now $pA_{sa}p$ clearly separates the points of F, and it follows routinely from the above theorem that $pA_{sa}p$ is norm dense in the space of continuous affine functionals vanishing at 0. Since $pA_{sa}p$ is closed (by **[7]**), the result follows.

We give a short proof (following **[17]**) of a theorem of Ch. Davis and S. Parrott (**[27]**).

3.6. THEOREM. *Let p, q be projections in a W^*-algebra M and $a \in pMq$, $b \in pM(1 - q)$, $c \in (1 - p)Mq$. If $\|a + b\|, \|a + c\| \leqq 1$, then $\exists d \in (1 - p)M(1 - q)$ such that*

$$\|a + b + c + d\| \leqq 1.$$

Proof. We use matrix notation. Thus we are given $\|(a \quad b)\|, \left\|\binom{a}{c}\right\| \leqq 1$, and we wish to find d such that

$$\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| \leqq 1.$$

$a^*a + c^*c \leqq 1 \Rightarrow c^*c \leqq 1 - a^*a \Rightarrow \exists t$ such that

$$\|t\| \leqq 1 \quad \text{and} \quad c = t(1 - a^*a)^{1/2}.$$

Similarly $aa^* + bb^* \leqq 1 \Rightarrow \exists u$ such that

$$\|u\| \leqq 1 \quad \text{and} \quad b = (1 - aa^*)^{1/2}u.$$

Take $d = -ta^*u$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} a & (1 - aa^*)^{1/2} \\ (1 - a^*a)^{1/2} & -a^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

The reader may need a little thought to see that the factorization makes sense. Of course the middle factor is unitary.

3.7. Corollary. *Let R, L be norm closed right, left ideals of a C^*-algebra A. Let $\pi_1:A \to A/R$, $\pi_2:A \to A/L$, and $\pi:A \to A/R \cap L$ be the quotient maps. Then*

$$\|\pi(x)\| = \max(\|\pi_1(x)\|, \|\pi_2(x)\|), \quad \forall x \in A.$$

Proof. Let p, q be the closed projections corresponding to R, L. Then

$$R^{**} = (1 - p)A^{**}, \quad L^{**} = A^{**}(1 - q), \quad \text{and}$$

$$(R \cap L)^{**} = (1 - p)A^{**}(1 - q).$$

Since $A/R \to A^{**}/R^{**}$, etc. are isometries, the result follows.

3.8. Corollary. *With the same notations, if R^0, L^0 are the annihilators in A^*, then $R^0 + L^0$ is isometrically isomorphic to the natural quotient of $R^0 \oplus L^0$, where the direct sum is given the 1-norm.*

Proof. $R^0 \oplus L^0 \to R^0 + L^0$ is the adjoint of $A/R \cap L \to A/R \oplus A/L$ (where the latter direct sum is given the ∞-norm). (Combes [**14**] showed $R^0 + L^0$ weak* closed.)

3.9. Theorem. *Let A be a σ-unital C^*-algebra, $p \in A^{**}$ a closed projection, and $T \in A^{**}p$ such that $\|T\| = 1$ and $AT \subset Ap$. Then $\exists R \in RM(A)$ such that $\|R\| = 1$ and $T = Rp$.*

Remark. For p central this specializes to (N3) for right multipliers (4.13 of **10**]).

Proof. Let (e_n) be a sequential approximate identity of A such that $e_{n+1}e_n = e_n$, $\forall n$. We will construct a sequence of $a_n \in A$ such that:

(i) $\|a_n\| \leqq 1$.

(ii) $a_np = e_nT$.

(iii) $\exists \delta_n \in A$ such that $\|\delta_n\| \leqq 2^{1-(n/2)}$ and

$$a_n - a_{n-1} \in \delta_n + [(1 - e_{n-1})A]^-.$$

a_1 can be chosen arbitrarily such that $a_1p = e_1T$ and $\|a_1\| = \|e_1T\| \leqq 1$ (3.3 (a)). (Note: $a_0 = e_0 = 0$.) Suppose $a_1, \ldots, a_n$ are constructed. Choose b such that $\|b\| \leqq 1$ and $bp = e_{n+1}T$ (3.3 (a)). Let

$$l = e_nb - a_n \in L = \{x \in A:xp = 0\}.$$

Let $R = [(1 - e_n)A]^-$ and $\pi_1:A \to A/R$, $\pi_2:A \to A/L$ the quotient maps. Then

$$\|\pi_2(b - l)\| = \|\pi_2(b)\| \leqq 1.$$

Also

$$\|\pi_1(b - l)\| \leqq \|\pi_1(b - e_nb)\| + \|\pi_1(e_nb - l)\| = \|\pi_1(a_n)\| \leqq 1.$$

Thus by 3.7 $\exists d \in R \cap L$ such that

$$\|b - l - d\| \leqq 1 + 2^{-(n+1)}.$$

Since $\|(b - l - d)p\| = \|bp\| \leqq 1$, $\exists \delta \in L$ such that

$$\|\delta\| \leqq 2 \cdot 2^{-((n+1)/2)} \quad \text{and} \quad \|b - l - d - \delta\| \leqq 1 \quad (3.3\text{ (a)}).$$

Take $a_{n+1} = b - l - d - \delta$. (i) and (ii) are clear. Also

$$\pi_1(a_{n+1}) = \pi_1(b - e_n b) + \pi_1(e_n b - l) + 0 - \pi_1(\delta)$$
$$= \pi_1(a_n) - \pi_1(\delta).$$

Take $\delta_{n+1} = -\delta$.

Now since $e_k(1 - e_{n-1}) = 0$, $\forall k < n - 1$, (iii) $\Rightarrow$

$$\|e_k a_n - e_k a_{n-1}\| \leqq \|e_k \delta_n\| \leqq 2^{1-(n/2)} \text{ for } n \geqq k + 2.$$

Therefore $(e_k a_n)$ converges in norm as $n \to \infty$, $\forall k$; and in view of (i) (a_n) converges right strictly to some $R \in RM(A)$ with $\|R\| \leqq 1$. Also

$$Rp = \lim(a_n p) = \lim(e_n T) = T.$$

For p a closed projection and L the corresponding left ideal of A, let

$$\widetilde{L} = RM(A) \cap A^{**}(1 - p)$$
$$= \{S \in RM(A) : AS \subset L\} = \{S \in RM(A) : Sp = 0\}.$$

3.10. Corollary. *If A is σ-unital and p a closed projection, then $RM(A)p$ is norm closed and equal to*

$$\{T \in A^{**}p : AT \subset Ap\}.$$

$\forall R \in RM(A)$, $\exists y \in \widetilde{L}$ such that $\|R - y\| = \|Rp\|$.

3.11. Corollary. *If A is σ-unital, L is a closed left ideal, and $\theta : A \to A/L$ is a homomorphism of left A-modules, then $\exists \widetilde{\theta} : A \to A$, a homomorphism of left A-modules, such that $\widetilde{\theta}$ lifts θ and $\|\widetilde{\theta}\| = \|\theta\|$.*

Proof. θ is automatically bounded, by the same proof as for right centralizers (see 3.12.2 of [**29**], for example). Since $A/L \cong Ap$, we may regard θ as a map from A to $Ap \subset A^{**}p$. If (e_n) is an approximate identity of A, $(\theta(e_n))$ has a weak cluster point $T \in A^{**}p$, with $\|T\| \leqq \|\theta\|$. Then $\forall a \in A$, aT is a weak cluster point of $(a\theta(e_n)) = (\theta(ae_n))$. But $\theta(ae_n) \to \theta(a)$ in norm. Hence $\theta(a) = aT$, and by 3.9, $\exists R \in RM(A)$ such that $\|R\| = \|T\| \leqq \|\theta\|$ and $T = Rp$. Let $\widetilde{\theta}(a) = aR$.

3.12. *Remark-Example.* Since $A/L \cap R$ embeds isometrically in $A^{**}/(1 - p)A^{**}(1 - q)$, we can replace the map

$$\pi : A \to A/L \cap R$$

with

$$\pi': a \mapsto (pa, aq),$$

where the map takes values in

$$\{(x, y) \in pA^{**} \oplus A^{**}q : xq = py\}$$

(notation as in 3.7). Although π' gives an isometry of $A/L \cap R$ onto its range, which is therefore norm closed, it is not in general true that $z \in \pi'(A)$ can be written $\pi'(\tilde{z})$ with $\|\tilde{z}\| = \|z\|$: Take $A = E_2$ and let $p = q$ be given by

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = 1, 2 \ldots, \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$B = \text{her}(1 - p) = L \cap L^* = L \cap R$$
$$= \left\{ x : x_n = \begin{pmatrix} 0 & 0 \\ 0 & d_n \end{pmatrix} \text{ with } d_n \to 0, x_\infty = 0 \right\}.$$

Take a sequence (ϵ_n) such that $0 < \epsilon_n < 1$ and $\epsilon_n \searrow 0$, and let $a \in A$ be given by

$$a_n = \begin{pmatrix} 1 - \epsilon_n & \sqrt{\epsilon_n(1 - \epsilon_n)} \\ \sqrt{\epsilon_n(1 - \epsilon_n)} & \dfrac{1 + 3\epsilon_n}{2} \end{pmatrix}, \quad a_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{2} \end{pmatrix}.$$

Then $a^* = a$, $\|a_n\| = 1 + \epsilon_n$, $\|a_\infty\| = 1$. If

$$a'_n = \begin{pmatrix} 1 - \epsilon_n & \sqrt{\epsilon_n(1 - \epsilon_n)} \\ \sqrt{\epsilon_n(1 - \epsilon_n)} & \epsilon_n \end{pmatrix},$$

then $\|a'_n\| = 1$. If $a^{(N)}$ is given by

$$a_n^{(N)} = \begin{cases} a'_n, & n < N, \\ a_n, & n \geqq N \end{cases}$$

then $a^{(N)} - a \in B$ and $\|a^{(N)}\| = 1 + \epsilon_N$. This and $\|a_\infty\| = 1$ imply $\|\pi(a)\| = 1$. But $\nexists b \in B$ such that $\|a - b\| \leqq 1$. If b existed, then it could be taken self-adjoint. Then

$$(a - b)_n = \begin{pmatrix} 1 - \epsilon_n & \sqrt{\epsilon_n(1 - \epsilon_n)} \\ \sqrt{\epsilon_n(1 - \epsilon_n)} & y_n \end{pmatrix},$$

and $(a - b)_n \leqq 1 \Rightarrow y_n \leqq \epsilon_n$. This implies

$$|b_n| \geqq \frac{1 + 3\epsilon_n}{2} - \epsilon_n \to \frac{1}{2},$$

a contradiction.

If p and q are finite rank projections (i.e., L and R are finite intersections of maximal one-sided ideals), then E. Effros pointed out to us that the Kadison density theorem can be regarded as giving information about any of the maps $a \mapsto paq$, $a \mapsto pa$, $a \mapsto aq$, or $\pi':a \mapsto (pa, aq)$. Although the formally strongest version of the theorem deals with π', the things that are true about any one of these maps for arbitrary closed projections, specialized to finite rank projections, are adequate to imply the theorem (provided we know the Kaplansky density theorem and that finite rank projections are closed). Whether our results will have any real applications remains to be seen.

3.13. *Example*. We have discussed the maps $x \mapsto pxp$, $x \mapsto xp$, and $\pi':x \mapsto (px, xp)$ for $x \in A$, and the second of these maps for $x \in RM(A)$. We show that equally good results do not hold for the other obvious variants. Specifically, for $A = E_1$, $\exists$ a closed projection $p \in A^{**}$ such that:

(i) $\exists x \in [pM(A)p]^-$ such that $x \notin pQM(A)p$.
(ii) $\exists x \in [M(A)p]^-$ such that $x \notin LM(A)p$.
(iii) $\exists x \in [\pi'(M(A))]^-$ such that $x \notin \pi'(QM(A))$.
(iv) $\exists x \in [LM(A)p]^-$ such that $x \notin QM(A)p$.

This phenomenon is related to the fact, which will be discussed in Section 3.F, that closed projections for A need not be regular relative to $M(A)$ (i.e., as elements of $M(A)^{**} \supset A^{**}$).

For $A = E_1$, define $p \in A^{**}$ by

$$p_\infty = 1, \quad p_n = \sum_{k=1}^{n} v_{k,n} \times v_{k,n},$$

where

$$v_{k,n} = \sqrt{1 - \frac{1}{k}}\, e_k + \sqrt{\frac{1}{k}}\, e_{k+n}.$$

The fact that $p_\infty = 1$ implies p is closed.

(i) Let $x_k \in pA^{**}p$ be given by

$$(x_k)_\infty = 0 \quad \text{and} \quad (x_k)_n = \begin{cases} 0, & k > n \\ v_{k,n} \times v_{k,n}, & k \leqq n. \end{cases}$$

Then $x_k = p\tilde{x}_k p$, where $\tilde{x}_k \in M(A)$ is given by

$$(\tilde{x}_k)_\infty = 0 \quad \text{and} \quad (\tilde{x}_k)_n = \begin{cases} 0, & k > n \\ k(e_{k+n} \times e_{k+n}), & k \leqq n. \end{cases}$$

Let $t_k \geqq 0$ be such that

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$t_k \to 0 \text{ as } k \to \infty \text{ and } t_k \neq O\left(\frac{1}{k^{1/2}}\right).$$

Then

$$x = \sum_1^\infty t_k x_k \in [pM(A)p]^- .$$

Suppose $x = p\widetilde{x}p$ for some $\widetilde{x} \in QM(A)$. Choose $M > \|\widetilde{x}\|$ and choose k_0 such that

$$t_{k_0} > \frac{3M}{k_0^{1/2}}.$$

For $n \geqq k_0$,

$$t_{k_0} = (\widetilde{x}_n v_{k_0,n}, v_{k_0,n}) = \left(1 - \frac{1}{k_0}\right)(\widetilde{x}_n e_{k_0}, e_{k_0})$$

$$+ \sqrt{\frac{1}{k_0}\left(1 - \frac{1}{k_0}\right)}[(\widetilde{x}_n e_{k_0}, e_{k_0+n}) + (\widetilde{x}_n e_{k_0+n}, e_{k_0})]$$

$$+ \frac{1}{k_0}(\widetilde{x}_n e_{k_0+n}, e_{k_0+n}).$$

The first term approaches 0 as $n \to \infty$, since $\widetilde{x}_n \to \widetilde{x}_\infty = 0$, weakly. The sum of the last three terms is majorized by

$$\|\widetilde{x}\|\left[2\sqrt{\frac{1}{k_0}\left(1 - \frac{1}{k_0}\right)} + \frac{1}{k_0}\right] \leqq \frac{3\|\widetilde{x}\|}{k_0^{1/2}}.$$

By choosing n sufficiently large, we obtain

$$t_{k_0} < \frac{3M}{k_0^{1/2}},$$

a contradiction.

(ii) Let $x_k \in A^{**}p$ be given by

$$(x_k)_\infty = 0 \quad \text{and} \quad (x_k)_n = \begin{cases} 0, & k > n \\ e_{k+n} \times v_{k,n}, & k \leqq n. \end{cases}$$

Then $x_k = \widetilde{x}_k p$ where $\widetilde{x}_k \in M(A)_{sa}$ is given by

$$(\widetilde{x}_k)_\infty = 0 \quad \text{and} \quad (\widetilde{x}_k)_n = \begin{cases} 0, & k > n \\ k^{1/2}(e_{k+n} \times e_{k+n}), & k \leqq n. \end{cases}$$

Choose t_k as in (i) and

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$x = \sum_1^\infty t_k x_k \in [M(A)p]^-.$$

If $x = \tilde{x}p$, $\tilde{x} \in LM(A)$, choose $M > \|\tilde{x}\|$ and k_0 such that

$$t_{k_0} > \frac{M}{k_0^{1/2}}.$$

For $n \geqq k_0$,

$$t_{k_0} e_{k_0+n} = \tilde{x}_n v_{k_0,n} = \left(1 - \frac{1}{k_0}\right)^{1/2} \tilde{x}_n e_{k_0} + \frac{1}{k_0^{1/2}} \tilde{x}_n e_{k_0+n}.$$

$\|\tilde{x}_n e_{k_0}\| \to 0$ as $n \to \infty$ since $\tilde{x}_n \to \tilde{x}_\infty = 0$, strongly, and hence we see

$$\|t_{k_0} e_{k_0+n}\| < \frac{M}{k_0^{1/2}}$$

for n large, a contradiction.

(iii) is almost the same as (ii), since the x of (ii) is actually in $[M(A)_{sa}p]^-$. It follows that

$$(x^*, x) \in [\pi'(M(A)_{sa})]^-.$$

If $(x^*, x) = \pi'(\tilde{x})$, $\tilde{x} \in QM(A)$, we may assume $\tilde{x} = \tilde{x}^*$. Then for $n \geqq k_0$,

$$t_{k_0} e_{k_0+n} = \left(1 - \frac{1}{k_0}\right)^{1/2} \tilde{x}_n e_{k_0} + \frac{1}{k_0^{1/2}} \tilde{x}_n e_{k_0+n}.$$

Therefore

$$\left(1 - \frac{1}{k_0}\right)^{1/2} (\tilde{x}_n e_{k_0}, e_{k_0}) = -\frac{1}{k_0^{1/2}} (\tilde{x}_n e_{k_0+n}, e_{k_0})$$

$$= -\frac{1}{k_0^{1/2}} (\tilde{x}_n e_{k_0}, e_{k_0+n})^-$$

(complex conjugate). Since $(\tilde{x}_n e_{k_0}, e_{k_0}) \to 0$ as $n \to \infty$ ($\tilde{x}_n \to 0$ weakly), it follows that

$$(\tilde{x}_n e_{k_0}, e_{k_0+n}) \to 0 \quad \text{as } n \to \infty.$$

Then we proceed as above from

$$t_{k_0} = \left(1 - \frac{1}{k_0}\right)^{1/2} (\tilde{x}_n e_{k_0}, e_{k_0+n}) + \frac{1}{k_0^{1/2}} (\tilde{x}_n e_{k_0+n}, e_{k_0+n}).$$

(iv) Define $x_k \in A^{**}p$ by

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$(x_k)_\infty = 0, \quad (x_k)_n = \begin{cases} 0, & k > n \\ e_k \times v_{k,n}, & k \leqq n. \end{cases}$$

Then $x_k = \widetilde{x}_k p$ where $\widetilde{x}_k \in LM(A)$ is given by

$$(\widetilde{x}_k)_\infty = 0 \quad \text{and} \quad (\widetilde{x}_k)_n = \begin{cases} 0, & k > n \\ k^{1/2}(e_k \times e_{k+n}), & k \leqq n. \end{cases}$$

Choose t_k as in (i) and

$$x = \sum_1^\infty t_k x_k \in [LM(A)p]^-.$$

If $x = \widetilde{x}p$, $\widetilde{x} \in QM(A)$, then choose M and k_0 as above. For $n \geqq k_0$,

$$\begin{aligned} t_{k_0} &= (\widetilde{x}_n v_{k_0,n}, e_{k_0}) \\ &= \left(1 - \frac{1}{k_0}\right)^{1/2} (\widetilde{x}_n e_{k_0}, e_{k_0}) + \frac{1}{k_0^{1/2}} (\widetilde{x}_n e_{k_0+n}, e_{k_0}). \end{aligned}$$

Proceed as above.

(v) For later use we point out that the x of (i) is in $(pC)^-$, where

$$C = \{y \in QM(A): yp = py\}.$$

To see this, define $y_k \in C$ by $(y_k)_\infty = 0$,

$$(y_k)_n = \begin{cases} 0, & k > n \\ v_{k,n} \times v_{k,n} - k\left(1 - \frac{1}{k}\right)(w_{k,n} \times w_{k,n}), & k \leqq n, \end{cases}$$

where

$$w_{k,n} = -\sqrt{\frac{1}{k}} e_k + \sqrt{1 - \frac{1}{k}} e_{k+n}.$$

Then $py_k = x_k$.

3.B. *Strong interpolation.*

3.14. LEMMA. *Let A be a C^*-algebra. Assume $k \leqq h$, $\|h - k\| \leqq 2/3$, $k \in (A_{sa})_m^-$, $h \in \overline{A_{sa}^m}$, $0 < \epsilon < 1/6$, $k - \epsilon \leqq x \leqq h + \epsilon$, $x \in A$, and $\delta > 0$. Then $\exists x' \in A$ such that $k - \delta \leqq x' \leqq h + \delta$ and $\|x' - x\| \leqq 4\epsilon^{1/2}$.*

Proof. Let $0 < \eta < \epsilon$. By **[5]** there are nets (a_α), (b_β) in A_{sa} such that $a_\alpha \nearrow h + \eta$, $b_\beta \searrow k - \eta$. Since

$$a_\alpha - b_\beta \nearrow h - k + 2\eta \geqq 0,$$

Dini's theorem (for functions on $\Delta(A)$) implies $a_\alpha - b_\beta \geqq -\eta$ for α, β sufficiently large. Also since

$$a_\alpha + \epsilon - \eta - x \nearrow h + \epsilon - x \geqq 0$$

(and since $a_\alpha + (\epsilon - \eta) - x$ is lsc on $\Delta(A)$), Dini's theorem implies

$$a_\alpha + \epsilon - \eta - x \geqq -\eta$$

for α sufficiently large. Similarly

$$b_\beta - (\epsilon - \eta) - x \leqq \eta$$

for β sufficiently large. Thus we can choose $a, b \in A$ such that $a \leqq h + \eta$, $b \geqq k - \eta$, $a - b + \eta \geqq 0$, and $b - \epsilon \leqq x \leqq a + \epsilon$. Thus $0 \leqq x + \epsilon - b \leqq a - b + 2\epsilon$. Since $2\epsilon > \eta$, $a - b + 2\epsilon$ is invertible, and

$$x + \epsilon - b = (a - b + 2\epsilon)^{1/2} t (a - b + 2\epsilon)^{1/2},$$

where $0 \leqq t \leqq 1$, $t \in \widetilde{A}$, and $t \equiv 1/2 \pmod{A}$. Thus

$$x = b - \epsilon + (a - b + 2\epsilon)^{1/2} t (a - b + 2\epsilon)^{1/2}.$$

Let

$$x' = b - \frac{\eta}{2} + (a - b + \eta)^{1/2} t (a - b + \eta)^{1/2}.$$

Then $x' \in A$,

$$x' \leqq \left(b - \frac{\eta}{2}\right) + (a - b + \eta) = a + \frac{\eta}{2} \leqq h + \frac{3}{2}\eta,$$

and $x' \geqq b - \eta/2 \geqq k - (3/2)\eta$.

$$\begin{aligned} \|x' - x\| &\leqq \left(\epsilon - \frac{\eta}{2}\right) + (2\epsilon - \eta)^{1/2} \|a - b + 2\epsilon\|^{1/2} \\ &\quad + (2\epsilon - \eta)^{1/2} \|a - b + \eta\|^{1/2} \\ &\leqq \epsilon - \frac{\eta}{2} + (2\epsilon - \eta)^{1/2} [\, \|h - k + 2\eta + 2\epsilon\|^{1/2} \\ &\qquad + \|h - k + 3\eta\|^{1/2}] \\ &\leqq \epsilon + 2 \cdot (2\epsilon)^{1/2} \leqq 4\epsilon^{1/2} \end{aligned}$$

if η is sufficiently small. Choose $\eta \leqq (2/3)\delta$ and small enough for the above to be true.

3.15. THEOREM. *Let A be a C^*-algebra. Assume $k \leqq h$, $\|h - k\| \leqq 2/3$, $k \in (A_{sa})_m^-$, $h \in \overline{A_{sa}^m}$, $0 < \epsilon < 1/6$, and $k - \epsilon \leqq x \leqq h + \epsilon$, $x \in A$. Then $\exists x' \in A$ such that $k \leqq x' \leqq h$ and $\|x' - x\| \leqq 5\epsilon^{1/2}$.*

Proof. Choose $\epsilon_n > 0$, $n = 1, 2, \dots$, such that $\epsilon_1 = \epsilon$, $\epsilon_n \searrow$, and $\sum_1^\infty \epsilon_n^{1/2} < \infty$. Let $x_1 = x$ and apply 3.14 with $\delta = \epsilon_2$, to obtain $x_2 \in A$ such that $k - \epsilon_2 \leqq x_2 \leqq h + \epsilon_2$ and $\|x_2 - x_1\| \leqq 4\epsilon_1^{1/2}$. Continuing, we obtain $x_n \in A$ such that

$$k - \epsilon_n \leqq x_n \leqq h + \epsilon_n \quad \text{and} \quad \|x_n - x_{n-1}\| \leqq 4\epsilon_{n-1}^{1/2}.$$

Then if $x' = \lim x_n$, we see $k \leqq x' \leqq h$ and

$$\|x' - x\| \leqq 4 \sum_1^{\infty} \epsilon_n^{1/2} = 4\epsilon^{1/2} + 4 \sum_2^{\infty} \epsilon_n^{1/2} \leqq 5\epsilon^{1/2}$$

if the ϵ_n's are chosen suitably.

3.16. COROLLARY. *If $k \leqq h$, $k \in (A_{sa})_m^-$, $h \in \overline{A_{sa}^m}$, then $\exists a \in A$ such that $k \leqq a \leqq h$.*

Proof. We may assume $\|h\|, \|k\| \leqq 1/12 \leqq 1/3$. Then the hypotheses of 3.15 are satisfied with $\epsilon = 1/12$, $x = 0$.

The following indicates the order of magnitude of the best estimate obtainable with our method.

3.17. COROLLARY. *There are universal constants C_1, C_2 such that for any C^*-algebra A, if $k \leqq h$, $k - \epsilon \leqq x \leqq h + \epsilon$, where $k \in (A_{sa})_m^-$, $h \in \overline{A_{sa}^m}$, $x \in A$, and $\epsilon > 0$, then $\exists x' \in A$ such that*

$$k \leqq x' \leqq h \quad \text{and} \quad \|x' - x\| \leqq \max(C_1\epsilon, C_2\|h - k\|^{1/2}\epsilon^{1/2}).$$

Proof. Choose $t > 0$ such that

$$t\|h - k\| \leqq \frac{2}{3} \quad \text{and} \quad t\epsilon < \frac{1}{6}.$$

By 3.15, $\exists x'' \in A$ such that

$$tk \leqq x'' \leqq th \quad \text{and} \quad \|x'' - tx\| \leqq 5t^{1/2}\epsilon^{1/2}.$$

With $x' = t^{-1}x''$,

$$\|x' - x\| \leqq 5t^{-1/2}\epsilon^{1/2}.$$

If for example $t = \min((2/3)\|h - k\|^{-1}, (1/7)\epsilon^{-1})$, one obtains

$$C_1 \leqq 5\sqrt{7} \quad \text{and} \quad C_2 \leqq \left(\frac{75}{2}\right)^{1/2}.$$

Remark. Anyone who cares what C_1 and C_2 are should use 3.41 below.

3.18. COROLLARY. *If $x \leqq h + k$, $x \in A$, $h, k \in \overline{A_{sa}^m}$, then $\exists a, b \in A$ such that $a \leqq h$, $b \leqq k$, and $x \leqq a + b$.*

Proof. First apply 3.16 to solve the interpolation problem: $x - k \leqq a \leqq h$. Then solve $x - a \leqq b \leqq k$.

3.19. COROLLARY. *If $x \leqq h + k$, $x \in A$, $h, k \in \overline{A_+^m}$, then $\forall \epsilon > 0$, $\exists a$, $b \in A_+$ such that $a \leqq h$, $b \leqq k$, and $x \leqq a + b + \epsilon$.*

Proof. Let $\delta > 0$, and choose nets (a_α), (b_β) in A_+ such that $a_\alpha \nearrow h + \delta$, $b_\beta \nearrow k + \delta$ ([**5**]). By Dini's theorem

$$x \leqq a_{\alpha_0} + b_{\beta_0} + \delta$$

for suitable α_0, β_0. Since $0 \leqq a_{\alpha_0} \leqq h + \delta$, 3.15 (or 3.17) implies that $\exists a \in A_+$ such that $a \leqq h$ and $\|a - a_{\alpha_0}\| \leqq f(\delta)$, where $f(\delta) \to 0$ as $\delta \to 0$. Similarly $\exists b \in A_+$ such that

$$b \leqq k \quad \text{and} \quad \|b_{\beta_0} - b\| \leqq f(\delta).$$

Then $x \leqq a + b + 2f(\delta) + \delta$, and we need only choose δ sufficiently small.

3.20. Corollary. *If $x \leqq h + \epsilon$, $x \in A$, $h \in \overline{A_+^m}$, $\epsilon \geqq 0$, then $\forall \epsilon' > \epsilon$, $\exists a \in A$ such that $0 \leqq a \leqq h$ and $x \leqq a + \epsilon'$.*

Proof. Apply 3.19 with $k = \epsilon$.

3.21. Corollary. *If $a \in A$, $h \in \overline{A_+^m}$, $\epsilon > 0$, and $a^*a \leqq h + \epsilon$, then $\forall \epsilon' > \epsilon$, $\exists b \in A$ such that $b^*b \leqq h$ and $\|a - b\| \leqq (\epsilon')^{1/2}$.*

Proof. By 3.20, $\exists c \in A$ such that $0 \leqq c \leqq h$ and $a^*a \leqq c + \epsilon'$. Therefore $a = t(c + \epsilon')^{1/2}$, where $t \in A$ (since $c + \epsilon'$ is invertible) and $\|t\| \leqq 1$. Let $b = tc^{1/2}$.

3.22. Corollary. *If $h \in \overline{A_{sa}^m}$, then $\exists a \in A$ such that $a \leqq h$.*

Proof. Choose $\lambda \geqq \|h\|$, and apply 3.16 with $k = -\lambda$.

3.23. *Remark-Examples.* Consider the following properties for a given C^*-algebra A.

(D1) $\forall h \in \overline{A_{sa}^m}$, $\{x \in A_{sa} : x \leqq h\}$ is directed upward.

(D2) If $x \leqq h + k$, $x \in A$, $h, k \in \overline{A_+^m}$, then $\exists a, b \in A_+$ such that $a \leqq h$, $b \leqq k$, and $x \leqq a + b$.

(D2′) Same as (D2) except that $x \geqq 0$.

(D3) If $x \leqq h + \epsilon$, $x \in A$, $h \in \overline{A_+^m}$, $\epsilon > 0$, then $\exists a \in A$ such that $0 \leqq a \leqq h$ and $x \leqq a + \epsilon$.

(D3′) Same as (D3) except that $x \geqq 0$.

(D4) $\{x \in A : x \leqq 1\}$ is directed upward.

It is not hard to see that (D1) $\Leftrightarrow$ (D2) $\Leftrightarrow$ (D2′), (D1) $\Rightarrow$ (D3) $\Leftrightarrow$ (D3′), (D3′) $\Rightarrow$ (D1) if A is unital, and (D1) $\Rightarrow$ (D4).

Unlike the Riesz interpolation and decomposition properties, (D1) and (D2′) are satisfied if A is finite dimensional. (D4) will remind the reader of Dixmier's result that $\{x \in A_+ : \|x\| < 1\}$ is always directed upward. (In (D4) it is irrelevant whether we require $x \in A_+$ or $x \in A_{sa}$.) Of course (D2) is just 3.19 without the ϵ and (D3) is 3.20 with $\epsilon' = \epsilon$. It will be shown in Section 5 that for $A = \mathscr{K}$ (D3) and (D4) are true but not (D1).

(i) For $A = E_2$, (D4) is trivially true, since A is unital, but (D1) is false: Let h be given by

$$h_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n = 1, 2, \dots .$$

Let $p, q \in A$ be given by

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = \infty, 1, 2, \dots, \quad q_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$

$$q_n = \begin{pmatrix} \cos^2 \theta_n & \cos \theta_n \sin \theta_n \\ \cos \theta_n \sin \theta_n & \sin^2 \theta_n \end{pmatrix},$$

where $0 < \theta_n < \pi/2$ and $\theta_n \to 0$ as $n \to \infty$. Then $p, q \leqq h$, but $\nexists x \in A$ such that $p, q \leqq x \leqq h$.

(ii) For $A = E_5$, (D4) is still (almost) trivially true, and (D1) is still false: Let $h \in \overline{A_+^m}$ be given by

$$h_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let p, q be as in (i) (except that now $n = \infty$ does not occur), and take

$$x = h^{1/2} p h^{1/2}, \quad y = h^{1/2} q h^{1/2}.$$

Then $x, y \in A$; $x, y \leqq h$; but $\nexists a \in A$ such that $x, y \leqq a \leqq h$.

(iii) For $A = E_3$, (D4) is false: Example (i) actually shows this also.

(iv) For $A = E_1$, (D4) is false: Let x be given by $x_n = e_1 \times e_1$, $n = \infty, 1, 2, \dots$. Let y be given by

$$y_\infty = e_1 \times e_1,$$
$$y_n = \cos^2 \theta_n (e_1 \times e_1) + \cos \theta_n \sin \theta_n [e_1 \times e_{n+1} + e_{n+1} \times e_1]$$
$$+ \sin^2 \theta_n e_{n+1} \times e_{n+1},$$

θ_n as in (i). If $x, y \leqq z \leqq 1$, then $e_{n+1} \times e_{n+1} \leqq z_n$, $\forall n$. For n sufficiently large

$$\|z_n - z_\infty\| < \frac{1}{2} (z \in A) \Rightarrow (z_\infty e_{n+1}, e_{n+1}) > \frac{1}{2}.$$

This contradicts $z_\infty \in \mathcal{K}$.

3.C. *Monotone limits, weak interpolation.*

3.24. THEOREM. *Let A be a C^*-algebra and $h \in \overline{A_+^m}$.*

(a) *If A is separable, $h \in A_+^\sigma$.*

(b) *For arbitrary A, $\exists$ a net (b_α) in A such that $0 \leqq b_\alpha \leqq h$, $b_\alpha \to h$ strongly, and $\forall \eta > 0$, $\forall c \in A$ such that $c \leqq h$, $c \leqq b_\alpha + \eta$ for α sufficiently large.*

Proof. By **[5]** there is a net $(x_\alpha)_{\alpha \in D}$ in $\widetilde{A}$ such that $x_\alpha = \lambda_\alpha + a_\alpha$, $a_\alpha \in A$, $\lambda_\alpha \nearrow 0$, and $x_\alpha \nearrow h$. If $\delta > 0$, then $\lambda_\alpha > -\delta$ eventually and hence $x_\alpha + \delta$ is lsc on $\Delta(A)$ eventually. Since $x_\alpha + \delta \nearrow h + \delta \geqq 0$, Dini's theorem implies $x_\alpha + \delta \geqq -\delta$ for α sufficiently large. Thus for α sufficiently large,

$$a_\alpha \geqq x_\alpha \geqq -2\delta, \quad \text{and} \quad a_\alpha \leqq h - \lambda_\alpha \leqq h + 2\delta.$$

The basic idea is to apply 3.15 (or 3.17) with $k = 0$ and $\epsilon = 2\delta$.

(a) Since A is separable, $\Delta(A)$ is second countable. Therefore we may assume (x_α) is a sequence, and we denote it by (x_n). (This follows from a standard result in topology: If $(f_\alpha)_{\alpha \in D}$ is a family of lsc functions on a second countable space X, then there is a countable $D_0 \subset D$ such that

$$\sup\{f_\alpha(x):\alpha \in D_0\} = \sup\{f_\alpha(x):\alpha \in D\}, \forall x \in X.$$

It is enough to apply this to $x_{\alpha|S(A)}$, for example.) We construct recursively $0 = b_0 \leqq b_1 \leqq \ldots \leqq h$ such that

$$b_m \in A \quad \text{and} \quad b_m \geqq x_m - \frac{1}{m}, \quad \forall m \geqq 1.$$

Assume $b_0, \ldots, b_{m-1}$ have already been constructed. Then the above reasoning applies to $(x_n - b_{m-1}) \nearrow (h - b_{m-1})$. Choose $n \geqq m$ such that $\exists c \in A$ with

$$0 \leqq c \leqq h - b_{m-1} \quad \text{and} \quad \|c - (a_n - b_{m-1})\| \leqq \frac{1}{m}.$$

This is possible by 3.15 if the δ used above is sufficiently small. Then let $b_m = b_{m-1} + c$. Note that

$$b_m \geqq b_{m-1} + (a_n - b_{m-1}) - \frac{1}{m} = a_n - \frac{1}{m} \geqq x_m - \frac{1}{m}.$$

Now clearly $\lim b_m$ exists and $\lim b_m \leqq h$. Also

$$b_m \geqq x_m - \frac{1}{m} \Rightarrow \lim b_m \geqq \lim x_m = h.$$

(b) Let $D' = D \times (0, \infty)$, with $(0, \infty)$ ordered downwards, and let

$$D'' = \{ (\alpha, \epsilon) \in D' : \exists b \in A \text{ with } 0 \leqq b \leqq h \text{ and } \|b - a_\alpha\| < \epsilon\}.$$

By 3.15 and the above D'' is cofinal in D'. For $(\alpha, \epsilon) \in D''$ choose $b_{\alpha,\epsilon} \in A$ such that

$$0 \leqq b_{\alpha,\epsilon} \leqq h \quad \text{and} \quad \|b_{\alpha,\epsilon} - a_\alpha\| < \epsilon.$$

Since $x_\alpha \to h$ strongly and $\lambda_\alpha \to 0$, $a_\alpha \to h$ strongly. Therefore $b_{\alpha,\epsilon} \to h$ strongly. If $A \ni c \leqq h$, then by the above reasoning, applied to $x_\alpha - c \nearrow$

$h - c$, $a_\alpha \geqq c - 2\delta$ for α sufficiently large. Thus it is clear that $b_{\alpha,\epsilon} \geqq c - \eta$ for (α, ϵ) sufficiently "large".

Remark. Just the fact that $b_\alpha \leqq h$ and $b_\alpha \to h$ weakly is enough to imply h lsc on $\Delta(A)$. The last part of (b) is intended to compensate for the fact that Dini's theorem is available only for monotone nets. In fact it follows from (b) that $\forall \eta > 0$, $\forall \alpha_1, \dots, \alpha_k$,

$$b_\alpha + \eta \geqq b_{\alpha_1}, \dots, b_{\alpha_k}$$

for α sufficiently large. This last is an adequate hypothesis for Dini's theorem.

3.25. COROLLARY. (a) *If A is a separable C^*-algebra and $h \in \overline{A^m_{sa}}$, then $h \in A^\sigma_{sa}$.*

(b) *If A is any C^*-algebra and $h \in \overline{A^m_{sa}}$, then $\exists$ a bounded net (b_α) in A such that $b_\alpha \leqq h$, $b_\alpha \to h$ strongly, and $\forall c \in A$ such that $c \leqq h$, $c \leqq b_\alpha + \eta$ for α sufficiently large.*

Proof. Combine 3.24 and 3.22.

3.26. THEOREM. *Let A be a σ-unital C^*-algebra.*

(a) *If A is separable, then*

$$[(\tilde{A}^m_{sa})^-]_+ = QM(A)^\sigma_+ \quad \text{and} \quad (\tilde{A}^m_{sa})^- = QM(A)^\sigma_{sa}.$$

(b) *In any case if $h \in (\tilde{A}^m_{sa})^-$, then there is a bounded net (x_α) in $QM(A)_{sa}$ such that $x_\alpha \leqq h$ and $x_\alpha \to h$ strongly. If $h \geqq 0$, then x_α can be taken positive.*

(c) *If $k \leqq h$, $k \in [(\tilde{A}_{sa})_m]^-$, $h \in (\tilde{A}^m_{sa})^-$, then $\exists x \in QM(A)$ such that $k \leqq x \leqq h$.*

Remark. 3.26 (a) and (b) are the weak counter-parts of 3.24-3.25 (a) and (b). 3.26 (c) is the weak counter-part of 3.16.

Proof. The basic method of deducing these results from their strong counter-parts is the same in all cases. Let e be a strictly positive element of A.

(a). If $0 \leqq h \in (\tilde{A}^m_{sa})^-$, then by 2.4 $ehe \in \overline{A^m_+}$. By 3.24 there are $a_n \in A_+$ such that $a_n \nearrow ehe$. Since $0 \leqq a_n \leqq \|h\|e^2$, $\exists! t_n \in A^{**}$ such that $a_n = et_ne$ and $0 \leqq t_n \leqq \|h\|$.

$$et_ne \in A \Rightarrow (Ae)t_n(eA) \subset A \Rightarrow At_nA \subset A$$

(since $(eA)^- = A$)

$$\Rightarrow t_n \in QM(A).$$

Clearly $a_n \nearrow \Rightarrow t_n \nearrow$, and $et_ne \to ehe$ weakly $\Rightarrow t_n \to h$ weakly (since $\|t_n\|$ is bounded and e has a dense range when regarded as an operator on the universal Hilbert space of A). The case where h is not positive follows by translation by scalars.

(b) is proved in the same way. Since the convergence here is not monotone, one should note that $x_\alpha \leqq h$, $x_\alpha \to h$ weakly, and $\|x_\alpha\|$ bounded imply $x_\alpha \to h$ strongly.

(c). Apply 3.16 and 2.4 to obtain $a \in A$ such that

$$eke \leqq a \leqq ehe.$$

If $\lambda \geqq \max(\|h\|, \|k\|)$, then

$$-\lambda e^2 \leqq a \leqq \lambda e^2 \Rightarrow 0 \leqq a + \lambda e^2 \leqq 2\lambda e^2$$

$$\Rightarrow \exists t \in A^{**} \text{ such that } a + \lambda e^2 = ete.$$

Let $x = t - \lambda$, so that $a = exe$. Then as in (a), $x \in QM(A)$, and $eke \leqq exe \leqq ehe \Rightarrow k \leqq x \leqq h$.

Remarks. (i) σ-unitality cannot be dropped from the hypothesis of (c), as is seen already from the commutative case. If $A = C_0(X)$, then the weakly lsc and usc elements of A^{**} are just the bounded lsc and usc functions on X ([**28**]). Thus (c) is true if and only if X is normal. Of course there are normal locally compact spaces which are not σ-compact, but not every locally compact space is normal.

(ii) The answer to the middle case of (Q3) (see Section 1) is "no" whenever $QM(A) \neq M(A)$: Let

$$T \in QM(A)_{sa} \backslash M(A).$$

Since $T \in (\widetilde{A}^m_{sa})^-$, $\exists h_n \in \widetilde{A}^m_{sa}$ such that

$$T \leqq h_n \leqq T + \frac{1}{n}.$$

Similarly $\exists k_n \in (\widetilde{A}_{sa})_m$ such that

$$T - \frac{1}{n} \leqq k_n \leqq T.$$

If the answer to (Q3) were yes, there would be $x_n \in M(A)$ such that

$$T - \frac{1}{n} \leqq k_n \leqq x_n \leqq h_n \leqq T + \frac{1}{n}.$$

Then $x_n \to T$ in norm and $T \in M(A)$.

3.27. THEOREM. *If A is a σ-unital C^*-algebra, then the following are equivalent*:

(i) *If* $0 < \epsilon \leqq h \in \overline{A^m_{sa}}$, *then* $\exists \delta > 0$ *such that* $h - \delta \in \overline{A^m_{sa}}$.

(ii) $0 \leqq h \in \widetilde{A}^m_{sa} \Rightarrow h \in \overline{A^m_{sa}}$.

(iii) $\widetilde{A}^m_{sa} = (\widetilde{A}^m_{sa})^-$.

(iv) $QM(A) = M(A)$.

Proof. In view of 2.2 it is enough to prove (iv) $\Rightarrow$ (ii). If $0 \leqq h \in \tilde{A}^m_{sa}$, we can apply 3.26 (b) to h. Thus there is a net (x_α) in $QM(A)_+ = M(A)_+$ such that $x_\alpha \leqq h$ and $x_\alpha \to h$ strongly. Since $M(A)_+ \subset A^m_+$, each x_α is lsc on $\Delta(A)$. Therefore h is lsc on $\Delta(A)$.

3.D. *Middle interpolation.* If B and C are hereditary C^*-subalgebras of A with open projections p and q, we say that B and C *q-commute* if $[p, q] = 0$. In this case it follows from a result of Akemann [**1**] that pq is the open projection for $B \cap C$.

3.28. THEOREM. *Let B and C be q-commuting hereditary C^*-subalgebras of A. Then there is an (increasing) approximate identity (e_α) of $B \cap C$ such that*

$$||b(1 - e_\alpha)c|| \to 0, \quad \forall b \in B, c \in C.$$

Moreover, if B, C, and $B \cap C$ are σ-unital, then (e_α) can be taken as a sequence.

Proof. (cf. proof of 3.12.14 of [**29**]). Let p and q be the open projections for B and C, $r = pq$, and $(r_\beta)_{\beta \in D}$ an approximate identity of $B \cap C$. Note that $\forall b \in B$, $c \in C$,

$$bc = (bp)(qc) = brc \Rightarrow b(1 - r)c = 0.$$

Let $b_1, \ldots, b_n \in B$, $c_1, \ldots, c_n \in C$, and consider

$$d_\beta = \langle b_i(1 - r_\beta)c_i\rangle_{i=1}^n \in \underbrace{A \oplus \ldots \oplus A}_{n \text{ times}}.$$

Since $r_\beta \to r$ strongly in A^{**},

$$b_i(1 - r_\beta)c_i \to 0$$

in the weak* topology of A^{**}, $\forall i$; and therefore $d_\beta \to 0$ in the weak Banach space topology of $A \oplus \ldots \oplus A$. It follows that $\forall \beta_0 \in D$, 0 is in the norm closed convex hull of $\{d_\beta : \beta \geqq \beta_0\}$.

Now let $\mathcal{F}$ be the collection of all finite subsets of $B \times C$ and $D' = D \times \mathcal{F} \times (0, \infty)$. For each $\alpha = (\beta_0, F, \epsilon) \in D'$ let e_α be one element of $\text{co}(\{r_\beta : \beta \geqq \beta_0\})$ such that

$$||b(1 - e_\alpha)c|| < \epsilon, \forall (b, c) \in F.$$

Order D' by

$$\alpha_1 = (\beta_1, F_1, \epsilon_1) \geqq (\beta_0, F_0, \epsilon_0) = \alpha_0$$

if and only if

$$\beta_1 \geqq \beta_0, F_1 \supset F_0, \epsilon_1 \leqq \epsilon_0, \text{ and } e_{\alpha_1} \geqq e_{\alpha_0}.$$

Then D' is directed and $(e_\alpha)_{\alpha \in D'}$ has the required properties.

For the last sentence let b, c, and x be strictly positive elements of B, C, and $B \cap C$, respectively. If (e_α) is as above, we can choose $\alpha_1 \leqq \alpha_2 \leqq \dots$ such that

$$\|b(1 - e_{\alpha_n})c\|, \|(1 - e_{\alpha_n})x\| < \frac{1}{n}.$$

Let $u_n = e_{\alpha_n}$. Then

$$b(1 - u_n)c \to 0 \Rightarrow (Bb)(1 - u_n)(cC) \to 0$$
$$\Rightarrow b'(1 - u_n)c' \to 0, \forall b' \in B, c' \in C,$$

since $(Bb)^- = B$ and $(cC)^- = C$. Similarly, $(1 - u_n)x' \to 0$, $\forall x' \in B \cap C$ (and also $x'(1 - u_n) = [(1 - u_n)x'^*]^*$).

3.29. LEMMA. *If p, q, r are projections such that $r \leqq p$, $r \leqq q$, and $p(1 - r)q = 0$, then $[p, q] = 0$ and $r = pq$.*

(Proof left to reader.)

3.30. LEMMA. *Let B and C be q-commuting hereditary C^*-subalgebras of A, $b \in B_+$, $c \in C_+$, and $r_0 \in (B \cap C)_+$. Then there are q-commuting hereditary C^*-subalgebras B', C' such that $b \in B' \subset B$, $c \in C' \subset C$, $r_0 \in B' \cap C'$ and B', C', $B' \cap C'$ are all σ-unital.*

Remark. Actually the facts that B' and C' are σ-unital and their open projections have a positive angle imply $B' \cap C'$ σ-unital.

Proof. Let (e_α) be as in the conclusion of 3.28. By choosing appropriate elements of (e_α), we can recursively construct $r_n \in B \cap C$ such that

$$0 \leqq r_n \leqq r_{n+1} \leqq 1 \quad (\text{for } n \geqq 1),$$

$$\|b(1 - r_n)c\| < \frac{1}{n}, \quad \text{and}$$

$$\|(1 - r_n)r_k\| < \frac{1}{n}, \quad k = 0, 1, \dots n - 1.$$

Let $B' = \text{her}(b, r_0, r_1, \dots)$, $C' = \text{her}(c, r_0, r_1, \dots)$, p the open projection for B', q the open projection for C', and $r = \lim r_n$.

$$\|(1 - r_n)r_k\| < \frac{1}{n}, \quad k < n \Rightarrow (1 - r)r_k = 0\ \forall k \Rightarrow (1 - r)r = 0.$$

Also

$$\|b(1 - r_n)c\| < \frac{1}{n} \Rightarrow b(1 - r)c = 0.$$

Thus r is a projection and $x(1 - r)y = 0$ whenever x, y are in the $*$-algebras generated by $\{b, r_0, r_1, \dots\}$, $\{c, r_0, r_1, \dots\}$ respectively. It follows that $x(1 - r)y = 0\ \forall x \in B'$, $y \in C'$ and hence $p(1 - r)q = 0$. Therefore 3.29 implies that B' and C' q-commute and (with help of **[1]**) r is the open projection for $B' \cap C'$. The fact that $r_n \nearrow r$ implies that (r_n) is an approximate identity for $B' \cap C'$ (Dini's theorem or **[6]**), and the proof is complete.

3.31. LEMMA. *Let A be a σ-unital C^*-algebra and $p_1, p_2 \in A^{**}$ closed projections such that $p_1p_2 = 0$. Then $\exists h \in M(A)$ such that $p_1 \leqq h \leqq 1 - p_2$.*

Proof. Let $B_i = \text{her}(1 - p_i)$. Then B_1 and B_2 q-commute and $\text{her}(B_1 \cup B_2) = A$. It follows that $\exists b_i \in (B_i)_+$ such that $b_1 + b_2$ is a strictly positive element of A. By 3.30 there are q-commuting hereditary C^*-subalgebras B_1', B_2' such that $b_i \in B_i' \subset B_i$, and B_1', B_2', $B_1' \cap B_2'$ are σ-unital. Then

$$\text{her}(B_1' \cup B_2') = A.$$

Hence if p_1', p_2' are the closed projections corresponding to B_1', B_2', then $p_1'p_2' = 0$ and $p_i' \geqq p_i$. It is sufficient to construct h such that $p_1' \leqq h \leqq 1 - p_2'$. Changing notation, we may assume B_1, B_2, and $B_1 \cap B_2$ are σ-unital.

Now let b_i be a strictly positive element of B_i, c a strictly positive element of $C = B_1 \cap B_2$, and choose $\epsilon_n > 0$ such that

$$\epsilon_n \searrow 0 \quad \text{and} \quad \sum_1^\infty \epsilon_n^{1/2} < \infty.$$

By 3.28 there is $s_1 = x_1 \in C$ such that

$$0 \leqq s_1 \leqq 1,\ \|(1 - s_1)c\| < 2^{-1}, \quad \text{and}$$
$$\|b_1(1 - s_1)b_2\| < \epsilon_1.$$

Next apply 3.28 to $b_1(1 - s_1)^{1/2} \in B_1$ and $(1 - s_1)^{1/2}b_2 \in B_2$ to obtain $x' \in C$ such that

$$0 \leqq x' \leqq 1,\ \|(1 - x')(1 - s_1)^{1/2}c\| < 2^{-2}, \text{ and}$$
$$\|b_1(1 - s_1)^{1/2}(1 - x')(1 - s_1)^{1/2}b_2\| < \epsilon_2.$$

Then

$$\|b_1(1 - s_1)^{1/2}x'(1 - s_1)^{1/2}b_2\| \leqq \|b_1(1 - s_1)^{1/2}(1 - s_1)^{1/2}b_2\|$$
$$+ \|b_1(1 - s_1)^{1/2}(1 - x')(1 - s_1)^{1/2}b_2\| < \epsilon_1 + \epsilon_2 \leqq 2\epsilon_1.$$

Let $e_i = |b_i(1 - s_1)^{1/2}x'^{1/2}|$, so that $e_1, e_2 \in C$ and

$$\|e_1e_2\| = \|b_1(1 - s_1)^{1/2}x'(1 - s_1)^{1/2}b_2\| < 2\epsilon_1.$$

Let $f:[0, \infty) \rightarrow [0, 1]$ be a continuous function such that $f = 1$ on $[2\epsilon_1^{1/2}, \infty)$ and $f = 0$ on $[0, \epsilon_1^{1/2}]$. Let $y = f(e_2) \in C$. Then since $f(t)/t \leqq \epsilon_1^{-1/2}$,

$$||e_1 y|| \leqq ||e_1 e_2|| \cdot \epsilon_1^{-1/2} < 2\epsilon_1^{1/2}.$$

Also $||(1 - y)e_2|| \leqq 2\epsilon_1^{1/2}$. Let

$$x_2 = (1 - s_1)^{1/2} x'^{1/2} y x'^{1/2} (1 - s_1)^{1/2},$$

$$x_2' = (1 - s_1)^{1/2} x'^{1/2} (1 - y) x'^{1/2} (1 - s_1)^{1/2}, \text{ and}$$

$$s_2 = s_1 + x_2 + x_2'.$$

Then

$$||b_1 x_2|| \leqq ||b_1(1 - s_1)^{1/2} x'^{1/2} y|| \cdot ||x'^{1/2}(1 - s_1)^{1/2}|| \leqq 2\epsilon_1^{1/2}, \text{ and}$$

$$\begin{aligned} ||x_2' b_2|| &\leqq ||(1 - s_1)^{1/2} x'^{1/2}|| \cdot ||(1 - y) x'^{1/2} (1 - s_1)^{1/2} b_2|| \\ &\leqq ||(1 - y)e_2|| \leqq 2\epsilon_1^{1/2}. \end{aligned}$$

Also

$$\begin{aligned} 1 - s_2 &= 1 - s_1 - (1 - s_1)^{1/2} x' (1 - s_1)^{1/2} \\ &= (1 - s_1)^{1/2} (1 - x')(1 - s_1)^{1/2} \geqq 0, \end{aligned}$$

$$||b_1(1 - s_2)b_2|| < \epsilon_2, \text{ and } ||(1 - s_2)c|| < 2^{-2}.$$

If we repeat this process recursively, we obtain $x_n, x_n' \in C_+$, $n = 1, 2, \ldots$ $(x_1' = 0)$ such that

$$s_n = \sum_1^n (x_k + x_k') \leqq 1,$$

$$||b_1 x_n||, ||x_n' b_2|| \leqq 2\epsilon_{n-1}^{1/2} \quad (n > 1),$$

$$||b_1(1 - s_n)b_2|| < \epsilon_n, \text{ and}$$

$$||(1 - s_n)c|| \leqq 2^{-n}.$$

It follows that (s_n) is an approximate identity of C. Hence

$$\lim s_n = \sum_1^\infty (x_k + x_k') = r,$$

the open projection for C. Let

$$h = p_1 + \sum_1^\infty x_k \in A^{**}.$$

Then

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$(1 - h) = p_2 + r - \sum_1^\infty x_k = p_2 + \sum_1^\infty x_k'.$$

By construction $b_1h \in B_1 \subset A(b_1p_1 = 0)$, and $(1 - h)b_2 \in B_2 \subset A$. Therefore

$$B_1h = (B_1b_1)^- h \subset A \Rightarrow hB_1 \subset A, \text{ and}$$

$$(1 - h)B_2 = (1 - h)(b_2B_2)^- \subset A \Rightarrow hB_2 \subset A.$$

Since $\text{her}(B_1 \cup B_2) = A$, this implies $hA \subset A$, and since $h = h^*$, $h \in M(A)$.

A direct proof of the following would make it possible to adapt Urysohn's proof of Urysohn's lemma to the non-commutative case.

3.32. COROLLARY. *With the hypotheses of* 3.31 *there exist open projections* $q_1, q_2 \in A^{**}$ *such that* $q_i \geqq p_i$ *and* $q_1q_2 = 0$.

Proof. Let

$$q_2 = E_{[0,(1/3))}(h), \quad q_1 = E_{((2/3),1]}(h).$$

For a projection $p \in A^{**}$ we denote by $\bar{p}^M$ its closure in $M(A)^{**}$, relative to $M(A)$ (under $A^{**} \subset M(A)^{**} \cong A^{**} \oplus (M(A)/(A)^{**})$.

3.33. COROLLARY. *If* A *is* σ*-unital and* $p_1, p_2 \in A^{**}$ *are closed projections such that* $p_1p_2 = 0$, *then* $\bar{p}_1^M\bar{p}_2^M = 0$.

Proof. If we consider the spectral projections in $M(A)^{**}$ of $h \in M(A)$, then, for the h of 3.31,

$$\bar{p}_1^M \leqq E_{\{1\}}(h) \quad \text{and} \quad \bar{p}_2^M \leqq E_{\{0\}}(h).$$

3.34. COROLLARY. *Let* A *be a* σ*-unital* C^**-algebra and* $q \in A^{**}$ *an open projection. The following are equivalent.*

(i) $\text{her}(q)$ *is* σ*-unital.*
(ii) $q = \bigvee_{i=1}^\infty p_i$, p_i *a compact projection.*
(iii) $q = \bigvee_{i=1}^\infty p_i$, p_i *a closed projection.*
(iv) $\exists h \in M(A)_+$ *such that* $q = E_{(0,\infty)}(h)$.

Proof. (i) $\Rightarrow$ (ii). Let e be a strictly positive element of $\text{her}(q)$ and

$$p_i = E_{[(1/i),\infty)}(e).$$

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Apply 3.31 to p_i and $1 - q$, obtaining $h_i \in M(A)$ such that $p_i \leqq h_i \leqq q$. Let

$$h = \sum_1^\infty 2^{-i}h_i.$$

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

(iv) $\Rightarrow$ (i). Let e be a strictly positive element of A. Then $h^{1/2}eh^{1/2}$ is a strictly positive element of her(q).

3.35. COROLLARY. *With the hypotheses of* 3.31, *if* her$(1 - p_1 - p_2)$ *is* σ-*unital, then* h *can be chosen so that* $p_1 = E_{\{1\}}(h)$ *and* $p_2 = E_{\{0\}}(h)$.

Proof. Let r_i be closed projections such that

$$1 - p_1 - p_2 = \bigvee_{i=1}^{\infty} r_i.$$

By [**1**] $p_1 + r_i$ and $p_2 + r_i$ are closed $\forall i$. Choose $h_i', h_i'' \in M(A)_{sa}$ such that

$$p_1 \leqq h_i' \leqq (1 - p_2 - r_i) \text{ and } p_1 + r_i \leqq h_i'' \leqq 1 - p_2.$$

Let

$$h = \sum_1^{\infty} 2^{-i-1}(h_i' + h_i'').$$

3.36. COROLLARY. *Let* A *be a* σ-*unital* C^*-*algebra and* B, C q-*commuting hereditary* C^*-*subalgebras such that* $A =$ her$(B \cup C)$. *Then there are* $b_i \in B_+$, $c_i \in C_+$ *such that* $(\sum_1^n (b_i + c_i))$ *is an approximate identity of* A. *In particular if* $1 \in A$, *then* $1 \in B_+ + C_+$.

Proof. The hypothesis (and conclusion) of 3.36 is equivalent to that of 3.31 (where p_1, p_2 are the closed projections corresponding to B, C). Let h be as in 3.31 and let $a_i \in A_+$ be such that $(\sum_1^n a_i)$ is an approximate identity of A. Take

$$b_i = (1 - h)^{1/2}a_i(1 - h)^{1/2} \text{ and } c_i = h^{1/2}a_ih^{1/2}.$$

Of course the last sentence follows from Akemann's Urysohn lemma ([**1**] or [**2**]).

Remarks. 3.34 and 3.35 benefitted from conversations with J. Anderson. There are other things along these lines that one would like to do, but non-commutativity seems to interfere. 3.36 applies in particular if $A = B + I$, B hereditary, I an ideal. G. Pedersen asked whether in this case $A_+ = B_+ + I_+$. Although the answer to Pedersen's question is no (3.53), the question was helpful.

For B a hereditary C^*-subalgebra of A, let

$$M(A, B) = M(A) \cap B^{**} \subset A^{**} \text{ and}$$

$$QM(A, B) = QM(A) \cap B^{**}.$$

$M(A, B) = \{x \in M(A):Ax \subset L \text{ and } xA \subset R\}$, where L and R are the closed left and right ideals of A corresponding to B (since $L = A \cap L^{**}$, $R = A \cap R^{**}$). If B is an ideal, this notation agrees with that of [**30**] and $M(A, B)$ is also described (by Pedersen) as the kernel of $M(A) \to M(A/B)$. In general $M(A, B)$ is a hereditary C^*-subalgebra of $M(A)$.

3.37. COROLLARY. *Let X be a $B_1 - B_2$ Hilbert bimodule, where B_1 and B_2 are σ-unital, and U a partial isometry in $QM(X)$. Then $U \in LM(X) + RM(X)$ in the following way: Let $C_1 = \mathrm{her}(UU^*)$, $C_2 = \mathrm{her}(U^*U)$, and $\theta:C_2 \to C_1$ the isomorphism $c \mapsto UcU^*$. Then there is $h_0 \in M(B_2, C_2)$ such that*

$$0 \leqq h_0 \leqq U^*U \text{ and } UU^* - \theta^{**}(h_0) \in M(B_1, C_1);$$

and

$$U = Uh_0 + (1 - \theta^{**}(h_0))U \in LM(X) + RM(X).$$

Proof. 3.31 deals with a special case of the present situation: Let $X = (B_1AB_2)^-$ and $U = r$ in the notation of 3.31. In this special case $C_1 = C_2 = C$, θ is the identity on C,

$$h_0 = \sum x'_k = 1 - h - p_2 \quad \text{and}$$
$$UU^* - \theta^{**}(h_0) = \sum x_k = h - p_1.$$

$((1 - \theta^{**}(h_0))U = (UU^* - \theta^{**}(h_0))U.)$ The construction of N. T. Shen [**31**] reduces the general case to this special case. (Our hypothesis that B_1, B_2 be σ-unital is too strong of course. We only need that the A produced by Shen's construction be σ-unital.) It may be helpful for the reader to consult Section 2 of [**10**] and 2.3 in particular.

For the reader who does not understand the above, it is possible to make the proof of 3.31 work in the present context. One should note that U^*U and UU^* are indeed open projections by 2.45 (b). Also UC_2, $C_1U \subset X$ by Proposition 4.4 of [**5**] (cf 2.6 (b)). (These results should be applied in a suitable linking algebra; or the reader may assume X is a C^*-algebra and $X = B_1 = B_2$.)

3.38. *Remark.* Consider the following situation: B_1 and B_2 are C^*-algebras with hereditary C^*-subalgebras C_1 and C_2. $\theta:C_2 \to C_1$ is an isomorphism. We ask when can B_1 and B_2 be "patched" along θ. In other words: When does there exist a C^*-algebra A containing B_1 and B_2 as q-commuting hereditary C^*-subalgebras such that $B_1 \cap B_2 = C_1 = C_2$ (and $c_1 = \theta(c_2)$ for $c_2 \in C_2$)? [**31**] produces an answer to this question, but it does not seem easy to apply; namely, to get an A it is necessary and sufficient to have a suitable partial isometry $U \in QM(X)$. (Note that for q_1, q_2 projections, $[q_1, q_2] = 0 \Leftrightarrow q_1q_2$ is a partial isometry.)

Now in general for the problem of [**31**] X should be given, but here one can construct X. Let $L_1 = (B_1C_1)^-$, $R_2 = (C_2B_2)^-$, and regard L_1 as a $B_1 - C_2$ bimodule (via θ) and R_2 as a $C_2 - B_2$ bimodule. Then

$$X_0 = L_1 \underset{C_2}{\otimes} R_2$$

is a $B_1 - B_2$ Hilbert bimodule. Moreover, there is a well-defined partial isometry U in X_0^{**}:

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$U = \lim(\theta(e_n^{1/2}) \otimes e_n^{1/2}),$$

where (e_n) is an approximate identity of C_2. Then B_1 and B_2 can be patched if and only if $U \in QM(X_0)$, but how does one check whether $U \in QM(X_0)$? 3.37 gives an answer: B_1 and B_2 can be patched if and only if $\exists h_0 \in M(B_2, C_2)$ such that

$$0 \leqq h_0 \leqq r_2 \quad \text{and} \quad r_1 - \theta^{**}(h_0) \in M(B_1, C_1)$$

($r_i \in B_i^{**}$ is the open projection for C_i). (The bimodule X of [**31**] need not be X_0: X_0 is the cutdown of X to the ideals generated by C_1 and C_2. The question whether B_1 and B_2 can be patched depends only on X_0, though the result of patching depends on all of X.)

It is interesting to consider the special case of the problem where C_i is an ideal in B_i and B_1, B_2 are required to be ideals of A. (In this case $X = X_0 \cong C_i$.) This problem (or rather a more elaborate but similar one) came up in connection with work done four years ago ([**11**]) and was solved independently of [**31**]: B_1 and B_2 can be patched if and only if $B_1B_2 \subset C_1 \subset M(C_1)$, where B_1 maps to $M(C_1)$ in the usual way and

$$B_2 \to M(C_2) \overset{\theta^{**}}{\to} M(C_1).$$

(In other words certain products in $M(C_1)/C_1$ must $= 0$.) How does one show that this answer agrees with that based on 3.37 (under the σ-unitality hypothesis of 3.37)? The bridge is provided by the following: If C is a σ-unital C^*-algebra, $x, y \in M(C)$, and $xy \in C \subset M(C)$, then $\exists h_0 \in M(C)$ such that $0 \leqq h_0 \leqq 1$, $h_0 y \in C$, and $x(1 - h_0) \in C$. This last result is Theorem 13 of Pedersen [**30**] and follows, in a simplified proof due to J. Cuntz, from (N3). Since 3.31 is the main lemma needed to prove (N4), we have now come full circle.

3.39. DEFINITION-LEMMA. *For $h, k \in A_{sa}^{**}$, write $h \overset{q}{\geqq} k$ if and only if $\forall s < t \in \mathbf{R}$,*

$$E_{(-\infty,s]}(h) \cdot E_{[t,\infty)}(k) = 0.$$

(Note that if $[h, k] = 0$, $h \overset{q}{\geqq} k \Leftrightarrow h \geqq k$.) Then

$$h \overset{q}{\geqq} k \Rightarrow h \geqq k.$$

Proof. Assume $\sigma(h) \cup \sigma(k) \subset [s, t]$. Let

$$p_i = E_{(s+(i/N)(t-s),\infty)}(h) \quad \text{and}$$

$$q_i = E_{(s+(i/N)(t-s),\infty)}(k), \quad i = 1, \dots N - 1.$$

Then $p_i \geqq q_i$, since

$$E_{(-\infty,s+(i/N)(t-s)]}(h)$$

is orthogonal to

$$E_{[s+(1/N)(t-s)+\epsilon,\infty)}(k),$$

$\forall \epsilon > 0$. Also

$$\left\| s + \frac{t-s}{N} \sum_{1}^{N-1} p_i - h \right\|, \left\| s + \frac{t-s}{N} \sum_{1}^{N-1} q_i - k \right\| \leqq \frac{t-s}{N}.$$

3.40. THEOREM. *If A is a σ-unital C^*-algebra, $h, k \in A_{sa}^{**}$, h is q-*lsc, *k is q-*usc, *and $h \overset{q}{\geqq} k$, then $\exists x \in M(A)_{sa}$ such that $k \leqq x \leqq h$ and $h - x$, $x - k \in \overline{A_+^m}$.*

Proof. Let

$$p_t = E_{(-\infty,t]}(h) \quad \text{and} \quad q_s = E_{[s,\infty)}(k).$$

Then p_t, q_s are closed $p_{t_1} \leqq p_{t_2}$ for $t_1 \leqq t_2$, $q_{s_1} \geqq q_{s_2}$ for $s_1 \leqq s_2$, and $p_t q_s = 0$ for $t < s$. Let

$$\widetilde{p}_t = \overline{p}_t^M \text{ and } \widetilde{q}_s = \overline{q}_s^M.$$

Then $\widetilde{p}_t$, $\widetilde{q}_s$, which are elements of $M(A)^{**}$, have the same properties as p_t, q_s, by 3.33. There is a standard way to construct h', $k' \in M(A)_{sa}^{**}$ such that

$$E_{(-\infty,t]}(h') = \bigwedge_{t'>t} \widetilde{p}_{t'} \text{ and } E_{[s,\infty)}(k') = \bigwedge_{s'<s} \widetilde{q}_{s'}.$$

To do this, choose a countable dense set D in some sufficiently large interval (s_0, t_0). Represent the Boolean σ-algebra of projections generated by the $\widetilde{p}_t$'s as a σ-field of subsets of some set S modulo a σ-ideal (Loomis-Sikorski theorem). One can represent the projections $\widetilde{p}_t$, $t \in D$, by subsets P_t of S such that $t_1 < t_2 \Rightarrow P_{t_1} \subset P_{t_2}$. Also let $P_{t_0} = S$ and define a measureable function f on S by

$$f(y) = \inf\{t : y \in P_t\}, \; y \in S.$$

Then

$$f^{-1}((-\infty, t]) = \bigcap_{\substack{t'>t \\ t' \in D}} P_{t'},$$

and if h' is the operator corresponding to f, h' has the required properties. The construction of k' is similar.

Now h' and k' have the same properties, relative to $M(A)$, as h, k have relative to A. The q-semicontinuity follows from the fact that the infimum of any family of closed projections is closed. That $h' \overset{q}{\geqq} k'$ follows since if $t < s$, $\exists t < t' < s' < s$. Then

$$E_{(-\infty,t]}(h') \leqq \widetilde{p}_{t'} \text{ and } E_{[s,\infty)}(k') \leqq \widetilde{q}_{s'}.$$

Since $M(A)$ is unital 3.16 and 2.50 imply $\exists x \in M(A)_{sa}$ with $k' \leqq x \leqq h'$. Now if z is the open central projection in $M(A)^{**}$ corresponding to the ideal A of $M(A)$, then $zk' = k$, $zh' = h$. Thus $k \leqq zx \leqq h$ in $A^{**} \subset M(A)^{**}$, and this simply means $k \leqq x \leqq h$ in A^{**} in the notation of the theorem. The fact that $h - x, x - k \in \overline{A_+^m}$ follows from 2.18 (a); i.e.,

$$h' - x, x - k' \in (M(A)_+^m)^- \Rightarrow z(h' - x), z(x - k') \in \overline{A_+^m}.$$

Remark. It is not true that $h \in \widetilde{A}_{sa}^m$, $k \in (\widetilde{A}_{sa})_m$, and $h - k \in \overline{A_+^m} \Rightarrow \exists x \in M(A)$ such that $k \leqq x \leqq h$. This fails, for example, for $A = E_6$, as will be shown in Section 5.E.

3.E. *Applications of interpolation*. The following result concerns the closure in A^{**} of certain bounded convex subsets in the σ-weak (or equivalently σ-strong) topology. In view of the proof of 3.2, this seems to be of interest.

3.41. THEOREM. *Assume $h \geqq k$ in A^{**}.*

(a) *If $h \in \overline{A_{sa}^m}$ and $k \in (A_{sa})_m^-$, then*

$$\mathscr{S} = \{a \in A : k \leqq a \leqq h\}$$

is strongly dense in

$$\mathscr{T} = \{a \in A^{**} : k \leqq a \leqq h\}.$$

(b) *If $h \in \overline{A_+^m}$, then $\mathscr{S} = \{a \in A : a^*a \leqq h\}$ is double-strongly dense in $\mathscr{T} = \{a \in A^{**} : a^*a \leqq h\}$.*

(c) *If A is σ-unital, h is q-lsc, k is q-usc, and $h \overset{q}{\geqq} k$, then $\mathscr{S} = \{y \in M(A) : h \leqq y \leqq k\}$ is strongly dense in $\mathscr{T} = \{a \in A^{**} : k \leqq a \leqq h\}$.*

(d) *If A is σ-unital, $h \in (\widetilde{A}_{sa}^m)^-$, and $k \in [(\widetilde{A}_{sa})_m]^-$, then $\mathscr{S} = \{y \in QM(A) : h \leqq y \leqq k\}$ is weakly dense in $\mathscr{T} = \{a \in A^{**} : k \leqq a \leqq h\}$.*

(e) *If A is σ-unital and $h \in [(\widetilde{A}_{sa}^m)^-]_+$, then $\mathscr{S} = \{T \in LM(A) : T^*T \leqq h\}$ is double-strongly dense in $\mathscr{T} = \{x \in A^{**} : x^*x \leqq h\}$.*

Proof. (a). By 3.16 $\exists x \in \mathscr{S}$. Since $h - x, x - k \in \overline{A_+^m}$, by 3.24 (b) there are nets (b_α), (c_β) in A such that $0 \leqq b_\alpha \leqq h - x$, $0 \leqq c_\beta \leqq x - k$ and $b_\alpha \to h - x$, $c_\beta \to x - k$ strongly. Now let $a \in \mathscr{T}$. Then $0 \leqq a - k \leqq h - k \Rightarrow \exists t \in A^{**}$ such that

$$0 \leqq t \leqq 1 \quad \text{and} \quad a - k = (h - k)^{1/2} t (h - k)^{1/2}.$$

Thus

$$a = k + (h - k)^{1/2} t (h - k)^{1/2}.$$

By the Kaplansky density theorem there is a net (t_γ) in A such that $0 \leqq t_\gamma \leqq 1$ and $t_\gamma \to t$ strongly. Then

$$z_{\alpha\beta\gamma} = x - c_\beta + (b_\alpha + c_\beta)^{1/2} t_\gamma (b_\alpha + c_\beta)^{1/2}$$

is in $\mathscr{S}$ and $z_{\alpha\beta\gamma} \to a$ strongly.

(b). Choose a net (b_α) in A such that $0 \leqq b_\alpha \leqq h$ and $b_\alpha \to h$ strongly (3.24 (b)). If $a \in \mathcal{T}$, then $a = th^{1/2}$ for some $t \in A^{**}$ such that $\|t\| \leqq 1$. Choose a net (c_β) in A such that $\|c_\beta\| \leqq 1$ and $c_\beta \to t$ double-strongly (Kaplansky). Then

$$c_\beta b_\alpha^{1/2} \in \mathcal{S} \quad \text{and} \quad c_\beta b_\alpha^{1/2} \to a.$$

(c) is proved in the same way as (a). The only differences are that x is now in $M(A)$ and 3.40 is used.

(d). Let $a \in \mathcal{T}$ and e a strictly positive element of A. Then $eke \leqq eae \leqq ehe$. By (a) and 2.4 there is a net (b_α) in A such that $eke \leqq b_\alpha \leqq ehe$ and $b_\alpha \to eae$ strongly. As in the proof of 3.26 (c) there are $y_\alpha \in QM(A)$ such that $b_\alpha = ey_\alpha e$. Then $y_\alpha \in \mathcal{S}$, and $ey_\alpha e \to eae$ strongly $\Rightarrow y_\alpha \to a$ weakly.

(e). Choose a net (S_α) in $QM(A)$ such that $0 \leqq S_\alpha \leqq h$ and $S_\alpha \to h$ strongly (3.26 (b)). If $a \in \mathcal{T}$, then $a = th^{1/2}$ for some $t \in A^{**}$ such that $\|t\| \leqq 1$. It is enough to show that $tS_\alpha^{1/2}$ is in the double-strong closure of $\mathcal{S}$ for each α. Choose $T_\alpha \in LM(A)$ such that $T_\alpha^* T_\alpha = S_\alpha$ ([**10**], 4.9). Then $tS_\alpha^{1/2} = rT_\alpha$ for some $r \in A^{**}$ with $\|r\| \leqq 1$. Choose a net $(c_\beta) \in A$ such that $\|c_\beta\| \leqq 1$ and $c_\beta \to r$ double-strongly (Kaplansky). Then $c_\beta T_\alpha \in \mathcal{S}$ and $c_\beta T_\alpha \to rT_\alpha$.

3.42. *Remark*. Both Akemann's Urysohn lemma [**2**]and a well known result of Størmer [**32**] follow easily from 3.16.

(a) If $pq = 0$, p compact, q closed, then the interpolation problem $p \leqq x \leqq 1 - q$ satisfies the hypotheses of 3.16.

(b) $(I + J)_+ = I_+ + J_+$: Let z, w be the open central projections for I, J and $a \in (I + J)_+$. Solve the interpolation problem $a(1 - w) \leqq x \leqq az$, and note $x \in I_+$, $a - x \in J_+$. (This result will be generalized below (3.48).)

For (N4) we need a definition. If $p \in A^{**}$ is a closed projection and $h \in pA_{sa}^{**}p$, then h is called *q-continuous on p* ([7]) if $E_{(-\infty,t]}(h)$ and $E_{[t,\infty)}(h)$ are closed in A^{**}, $\forall t \in \mathbf{R}$, where the spectral projections are computed in $pA^{**}p$. Also h is called *strongly q-continuous on p* if in addition $E_{(-\infty,-t]}(h)$ and $E_{[t,\infty)}(h)$ are compact for $t > 0$.

3.43. THEOREM. *Let $p \in A^{**}$ be a closed projection and $h \in pA_{sa}^{**}p$.*

(a) *If h is strongly q-continuous on p, then $\exists \tilde{h} \in A_{sa}$ such that* $[\tilde{h}, p] = 0$, $p\tilde{h} = h$, *and* $\sigma(\tilde{h}) \subset \mathrm{co}(\sigma(h) \cup \{0\})$.

(b) *If A is σ-unital and h is q-continuous on p, then $\exists \tilde{h} \in M(A)_{sa}$ such that* $[\tilde{h}, p] = 0$, $p\tilde{h} = h$, *and* $\sigma(\tilde{h}) \subset \mathrm{co}(\sigma(h))$, *where the latter spectrum is computed in $pA^{**}p$.*

Proof. (a). Let $[s, t] = \mathrm{co}(\sigma(h) \cup \{0\})$. Let $x = h + s(1 - p)$, $y = h + t(1 -_q p)$. It is easy to see that x is strongly q-usc, y is strongly q-lsc, and $y \leqq x$. Thus 3.16 and 2.50 imply that the interpolation problem $x \leqq \tilde{h} \leqq y$ can be solved for $\tilde{h} \in A_{sa}$.

$$s(1 - p) \leqq \tilde{h} - h \leqq t(1 - p) \Rightarrow [\tilde{h}, p] = 0.$$

(b) is proved in the same way as (a) based on 3.40.

3.44. *Remark-Example.* This result is sharp, even if 3.40 is not, since every element of $p\{x \in A_{sa}:[x, p] = 0\}$ is strongly q-continuous on p and every element of

$$p\{x \in M(A)_{sa}:[x, p] = 0\}$$

is q-continuous on p.

Recall that $pA_{sa}p$ can be identified with the set of continuous affine functionals vanishing at 0 on the closed face of $\Delta(A)$ corresponding to p. Not every element of $pA_{sa}p$ need be q-continuous on p. Thus, for an h not q-continuous on p, it follows from 3.16 that either $h + t(1 - p)$ fails to be in $\overline{A^m_{sa}}$ no matter how large t is, or $h + s(1 - p) \notin (A_{sa})^-_m$ no matter how small s.

The example is very simple. Let $A = E_2$, which is unital. Let p be given by

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, n = 1, 2, \dots, \quad \text{and} \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $a \in A_{sa}$ be given by

$$a_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n = \infty, 1, 2, \dots,$$

and $h = pap$. Clearly $h \neq p\tilde{h}$ for $\tilde{h} \in A$ and $[\tilde{h}, p] = 0$. In this case $h + t(1 - p)$ is never lsc or usc. Note also that her$(1 - p)$ is a corner of an ideal.

By combining 2.39 (v) (c) with 3.26 (c), one can obtain a "weak" result: Let

$$C = \{y \in QM(A):yp = py\}.$$

Then for $h \in pA^{**}_{sa}p$, $h \in pC$ if and only if $\exists s, t \in \mathbf{R}$ such that

$$h + s(1 - p) \in [(\tilde{A}_{sa})_m]^- \quad \text{and} \quad h + t(1 - p) \in (\tilde{A}^m_{sa})^-.$$

We do not consider this a "weak" version of (N4) (or 3.43) because the characterization of pC cannot be stated solely in terms of the closed face of $\Delta(A)$ corresponding to p (so far as we know). Also 3.13 (v) shows that pC need not be norm closed.

3.45. COROLLARY. *If $p \in A^{**}$ is a closed projection, then*

$$\{h \in pA^{**}_{sa}p:h \text{ is strongly } q\text{-continuous on } p\}$$

is the real part of a C^*-algebra. If A is σ-unital, the same holds for

$$\{h \in pA^{**}_{sa}p:h \text{ is } q\text{-continuous on } p\}.$$

3.46. THEOREM. *Let B_0, B_1 be q-commuting hereditary C^*-subalgebras of A, $B = \text{her}(B_0 \cup B_1)$, and q_0, q_1, q the corresponding open projections.*

(a) *If $h \in B_{sa}$ and $[h, q_0] = 0$, then $h = h_0 + h_1$, where*

$$h_i \in (B_i)_{sa}, [h_i, q_0] = 0, \text{ and } \sigma(h_1) \subset \text{co}(\sigma(h) \cup \{0\}).$$

(b) *If A is σ-unital, $h \in M(A, B)_{sa}$, and $[h, q_0] = 0$, then $h = h_0 + h_1$ where*

$$h_i \in M(A, B_i)_{sa}, [h_i, q_0] = 0, \text{ and}$$
$$\sigma(h_1) \subset \text{co}(\sigma(h) \cup \{0\}).$$

Proof. Let $[s, t] = \text{co}(\sigma(h) \cup \{0\})$. Both parts are proved by solving the interpolation problem,

$$(1 - q_0)h + sq_0q_1 \leqq h_1 \leqq (1 - q_0)h + tq_0q_1.$$

Either 3.40 or 3.16 applies, since h is q-continuous (strongly in part (a)).

The following are special situations where 3.46 can be applied, each more general than the next. In all of these cases 3.46 (a) is trivial.

(1) B_0 is an ideal of B: In this case the hypothesis $[h, q_0] = 0$ is automatic. That $B = B_0 + B_1$ can be seen by elementary arguments. This special case of 3.46 (b) becomes: If A is a σ-unital C^*-algebra, B_0 and B_1 are hereditary C^*-subalgebras, and B_0 is an ideal of $\text{her}(B_0 \cup B_1)$, then

$$M(A, B_0 + B_1) = N(A, B_0) + M(A, B_1).$$

(2) B_0 is an ideal of A and B_1 any hereditary C^*-subalgebra.

(3) B_0 and B_1 are both ideals of A.

A forthcoming paper by J. Mingo, who told us about the problem, will give a more elementary proof of the result in situation (3):

(3) $M(A, I_1 + I_2) = M(A, I_1) + M(A, I_2)$.

For quasi-multipliers we have a result for situation (1).

3.47. THEOREM. *If A is a σ-unital C^*-algebra, B_0 and B_1 are hereditary C^*-subalgebras such that B_0 is an ideal of $B = \text{her}(B_0 \cup B_1)$, and $h \in QM(A, B)_{sa}$, then $h = h_0 + h_1$, where*

$$h_i \in QM(A, B_i) \text{ and } \sigma(h_1) \subset \text{co}(\sigma(h) \cup \{0\}).$$

Proof. There is an ideal I of A such that $B_0 = B \cap I$. Let z be the open central projection for I. We use the same interpolation problem as in 3.46 but note that

$$y = (1 - q_0)h + tq_0q_1 = (1 - z)h + tzq_1.$$

Since h is no longer q-continuous, we need to give a direct proof that $y \in (\widetilde{A}_{sa}^m)^-$. Then 3.26 (c) applies.

Let $\varphi_\alpha \to \varphi$ in $S(A)$. Passing to a subnet, we may assume $z\varphi_\alpha \to \theta$, $(1 - z)\varphi_\alpha \to \psi$, where $\varphi = \theta + \psi$. Moreover,

$$\|\varphi_\alpha\|, \|\varphi\| = 1 \Rightarrow \|z\varphi_\alpha\| \to \|\theta\|, \|(1 - z)\varphi_\alpha\| \to \|\psi\|.$$

Clearly ψ vanishes on I. Thus

$$\psi(y) = \psi(h) = \lim[(1 - z)\varphi_\alpha](h) = \lim[(1 - z)\varphi_\alpha](y).$$

Also

$$\theta(y) \leqq t\|\theta_{|B_1}\|$$

(since $y \in B_1^{**}$)

$$\leqq t\ \underline{\lim}\|z\varphi_{\alpha|B_1}\| = t\ \underline{\lim}(z\varphi_\alpha)(q_1) = \underline{\lim}(z\varphi_\alpha)(y).$$

Thus

$$\varphi(y) \leqq \underline{\lim}\ \varphi_\alpha(y).$$

3.48. THEOREM. *Let B_1 and B_2 be q-commuting hereditary C^*-subalgebras of A, $B =$ her$(B_1 \cup B_2)$, and q_1, q_2, q the corresponding open projections.*

(a) *If $x \in B_{sa}$ and $[x, q_1] = [x, q_2] = 0$, then $x = x_1 + x_2$ with $x_i \in (B_i)_{sa}$, $[x_i, q_j] = 0$, and*

$$\sigma(x_1), \sigma(x_2) \subset \text{co}(\sigma(x) \cup \{0\}).$$

(b) *If A is σ-unital, $x \in M(A, B)_{sa}$, and $[x, q_1] = [x, q_2] = 0$, then $x = x_1 + x_2$ with $x_i \in M(A, B_i)_{sa}$, $[x_i, q_j] = 0$, and*

$$\sigma(x_i) \in \text{co}(\sigma(x) \cup \{0\}).$$

Proof. Let $[s, t] = \text{co}(\sigma(x) \cup \{0\})$. Both parts are proved by solving the interpolation problem,

$$k = (1 - q_2)x + q_1q_2[(x - t) \vee s] \leqq x_1$$
$$\leqq (1 - q_2)x + q_1q_2[(x - s) \wedge t] = h.$$

Clearly $[h, k] = 0$, and

$$x - s \geqq x \geqq s, t \geqq x \geqq x - t (s \leqq 0, t \geqq 0)$$
$$\Rightarrow h \geqq k \Rightarrow h \overset{q}{\geqq} k.$$

We need to show that h is q-lsc and k q-usc (strongly in case (a)). Let $p = E_{(-\infty,a]}(h)$. For $a < 0$,

$$p = E_{(-\infty,a]}(x) \cdot (1 - q_2),$$

which has the required properties. For $a \geqq t$, $p = 1$. For $0 \leqq a < t$,

$$p = [E_{(-\infty,a]}(x) \cdot (1 - q_2)] \vee [E_{(-\infty,a+s]}(x)] \vee (1 - q_1).$$

(By [**1**] the sup of finitely many commuting closed projections is closed.) The proof for k is similar, and hence 3.16 or 3.40 applies.

Again our result for the "weak" case is weaker.

3.49. THEOREM. *If A is a σ-unital C^*-algebra, B_1 and B_2 are hereditary C^*-subalgebras, and B_1 and B_2 are both ideals of $B = \text{her}(B_1 \cup B_2)$, then*

$$QM(A, B)_+ = QM(A, B_1)_+ + QM(A, B_2)_+.$$

Remark. Of course $B = B_1 + B_2$, and this result includes the case where B_1 and B_2 are ideals of A. Without the "+'s" 3.49 would follow from 3.47.

Proof. There are ideals I_1 and I_2 of A such that $B_i = B \cap I_i$. Let z_1, z_2 be the corresponding open central projections and $x \in QM(A, B)_+$. By 2.18 (c) and 3.26 (c) we can solve:

$$(1 - z_2)x \leqq x_1 \leqq z_1 x.$$

3.50. LEMMA. *If I and J are ideals of a C^*-algebra A and $x \in I + J$, then $x = i + j$, $i \in I$, $j \in J$, $\|i\|, \|j\| \leqq \|x\|$.*

Proof. Write $x = uh$ (polar decomposition), $u \in A^{**}$, $h \in I + J$. By Størmer [**32**], $h = h_1 + h_2$, $h_1 \in I_+$, $h_2 \in J_+$. $uh \in A \Rightarrow uh_1, uh_2 \in A$, since $h_1, h_2 \in (hA)^-$. It follows that $i = uh_1 \in I$ and $j = uh_2 \in J$. ($I = A \cap I^{**}$, for example.)

3.51. *Remark*. With 3.50 we can derive results for non-self-adjoint operators from 3.48. With the hypotheses of 3.48 each of the C^*-algebras

$$C_1 = \{x \in B:[x, q_1] = [x, q_2] = 0\} \quad \text{and}$$

$$C_2 = \{x \in M(A, B):[x, q_1] = [x, q_2] = 0\}$$

is the sum of two ideals:

$$C_1 = C_1 \cap B_1 + C_1 \cap B_2 \quad \text{and}$$

$$C_2 = C_2 \cap M(A, B_1) + C_2 \cap M(A, B_2).$$

A similar trick could be used to supplement 3.46. For completeness we also consider analogous results for non-self-adjoint left or quasi-multipliers. For I an ideal of A, let

$$LM(A, I) = LM(A) \cap I^{**} \subset A^{**}.$$

Suppose $x \in LM(A, I_1 + I_2)$. If $I_1 + I_2$ is σ-unital, we can easily show $x = x_1 + x_2$, $x_i \in LM(A, I_i)$, $\|x_i\| \leqq \|x\|$. To do this, apply Urysohn's lemma to $I_1 + I_2$, obtaining $h \in M(I_1 + I_2, I_1)$ such that $0 \leqq h \leqq 1$ and $1 - h \in M(I_1 + I_2, I_2)$. (The existence of h follows from [**30**] ((N3)) or from 3.31.) Let $x_1 = hx$, $x_2 = (1 - h)x$. (That x_i is in $LM(A)$ follows from $xA \subset I_1 + I_2$.) For quasi-multipliers the proof is more elaborate.

3.52. THEOREM. *Let I_1 and I_2 be ideals of a C^*-algebra A, and assume A and $I_1 + I_2$ are σ-unital. Then if $x \in QM(A, I_1 + I_2)$, $x = x_1 + x_2$ with $x_i \in QM(A, I_i)$ and $\|x_i\| \leqq \|x\|$.*

Proof. Let e be a strictly positive element of A and

$$e_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

a strictly positive element of $A \otimes M_2$. Assume $\|x\| = 1$, and let

$$T = \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \in QM(A \otimes M_2).$$

In 4.9 of [**10**] (see also 4.10) we showed that $T = L^*L$, $L \in LM(A \otimes M_2)$. Since e_2Te_2 and e_2^2 have the same image in $A \otimes M_2/(I_1 + I_2) \otimes M_2$, the proof of 4.9 shows that L maps to 1 in $LM(A \otimes M_2/(I_1 + I_2) \otimes M_2)$. Then if

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad x = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}^* \begin{pmatrix} L_{12} \\ L_{22} \end{pmatrix},$$

where both columns are isometries, and $L_{12}, L_{21} \in LM(A, I_1 + I_2)$. Now choose a Urysohn element $h \in M(I_1 + I_2, I_1)$ as in 3.51. Then hL_{12}, hL_{21}, $(1 - h)L_{12}$, $(1 - h)L_{21} \in LM(A)$. Take

$$x_1 = L_{11}^*(hL_{12}) + (L_{21}^*h)L_{22} \quad \text{and}$$

$$x_2 = L_{11}^*[(1 - h)L_{12}] + [L_{21}^*(1 - h)]L_{22}.$$

Remark. Even in the self-adjoint case 3.52 does not follow from 3.49, since $QM(A, I)$ need not be generated by $QM(A, I)_+$. For example, consider $A = E_1$ and $I = \{x \in E_1 : x_\infty = 0\}$. Then $QM(A, I)_+ \subset M(A)$ but $QM(A, I) \not\subset M(A)$.

3.53. *Example.* In the decompositions, $x = x_1 + x_2$, obtained in 3.46-3.52, we were able to impose conditions on both x_1 and x_2 if $[x, q_1] = [x, q_2] = 0$; but if only $[x, q_2] = 0$, we can impose conditions only on x_1: Let

$$A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \widetilde{\mathscr{K}}, b, c \in \mathscr{K} \right\}.$$

$(A \cong \widetilde{E}_6)$. Let

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A : a \in \mathscr{K} \right\} \quad \text{and}$$

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A : b = c = d = 0 \right\}.$$

Then I is an ideal, B is hereditary, and $A = B + I$. Also A and B are unital, so that $A = M(A) = QM(A)$. Let $K \in \mathcal{K}$ be positive and one-one, and

$$x = \begin{pmatrix} 1 & K^{1/2} \\ K^{1/2} & K \end{pmatrix} \in A_+.$$

Then $x \notin B_+ + I_+$. In fact if $x = b + i$, $b \in B_+$, $i \in I_+$, then necessarily

$$b = \begin{pmatrix} 1 - L & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad i = \begin{pmatrix} L & K^{1/2} \\ K^{1/2} & K \end{pmatrix}$$

for some $L \in \mathcal{K}$ such that $0 \leqq L \leqq 1$. $i \geqq 0 \Rightarrow K^{1/2} = L^{1/2}SK^{1/2}$ for some $S \in B(H)$ with $\|S\| \leqq 1$. This implies $L^{1/2}S = 1$, which is impossible, since $L \in \mathcal{K}$. If $\|K\| < 1$, it can also be shown that

$$x - \left(\frac{1 + \|K\|}{2}\right) \neq y_1 + y_2, y_1 \in B, y_2 \in I,$$

$$\|y_i\| \leqq \left\|x - \left(\frac{1 + \|K\|}{2}\right)\right\|.$$

3.F. *A problem on commuting closed projections*. In the context of 3.46, if $h \in M(A, B)$ does not commute with either q_0 or q_1, we certainly can not expect to prove that $h \in M(A, B_0) + M(A, B_1)$. However, there is a sensible problem, which is explained by the following definition. Let B_1 and B_2 be q-commuting hereditary C^*-subalgebras of A and $B = \text{her}(B_1 \cup B_2)$. We say that (B_1, B_2) satisfies (C) if $M(A, B_1)$ q-commutes with $M(A, B_2)$ and

$$M(A, B) = \text{her}_{M(A)}(M(A, B_1) \cup M(A, B_2)).$$

There is another description. A theorem of topology states: If X is a normal space and $F_1, F_2 \subset X$ are closed, then

$$(F_1 \cap F_2)^{-\beta} = \bar{F}_1^{\beta} \cap \bar{F}_2^{\beta},$$

where "$-\beta$" denotes closure in the Stone-Čech compactification. This theorem can sometimes be applied in the context of fibre bundles. The most obvious non-commutative analogue is explained by the following definition. Let $p_1, p_2 \in A^{**}$ be closed projections such that $[p_1, p_2] = 0$. We say that (p_1, p_2) satisfies (C′) if

$$[\bar{p}_1^M, \bar{p}_2^M] = 0 \text{ and } (p_1p_2)^{-M} = \bar{p}_1^M\bar{p}_2^M.$$

If $B_i = \text{her}(1 - p_i)$, then $[p_1, p_2] = 0 \Leftrightarrow B_1$ and B_2 q-commute. Also (given $[p_1, p_2] = 0$)

$$\text{her}(B_1 \cup B_2) = B = \text{her}(1 - p_1p_2).$$

To see that (C) and (C′) are equivalent, note that

$$\text{her}_{M(A)}(1 - \bar{p}_i^M) = M(A, B_i) \quad \text{and}$$
$$\text{her}_{M(A)}(1 - (p_1p_2)^{-M}) = M(A, B).$$

(For $p \in M(A)^{**}$,

$$\text{her}_{M(A)}(1 - \bar{p}^M) = \{x \in M(A):xp = px = 0\}.$$

When $p \in A^{**} \subset M(A)^{**}$, the computation of xp and px can be done in A^{**}.) Thus $[\bar{p}_1^M, \bar{p}_2^M] = 0 \Leftrightarrow M(A, B_1)$ and $M(A, B_2)$ q-commute; and if $[\bar{p}_1^M, \bar{p}_2^M] = 0$, then

$$\text{her}_{M(A)}(M(A, B_1) \cup M(A, B_2)) = \text{her}_{M(A)}(1 - \bar{p}_1^M\bar{p}_2^M),$$

so that

$$M(A, B) = \text{her}_{M(A)}(M(A, B_1) \cup M(A, B_2))$$

if and only if $(p_1p_2)^{-M} = \bar{p}_1^M\bar{p}_2^M$.

3.33 says that (p_1, p_2) satisfies (C′) whenever $p_1p_2 = 0$, for A σ-unital. Also if B_1 is an ideal of B, $M(A, B_1)$ is an ideal of $M(A, B)$; so that $M(A, B_1)$ and $M(A, B_2)$ certainly q-commute. Thus 3.46, specialized to situation (1), implies that (B_1, B_2) satisfies (C) whenever B_1 or B_2 is an ideal of $B = \text{her}(B_1 \cup B_2)$. We will prove some other positive results, but in general (C) and (C′) are false, even for nice algebras. Recall that a projection $p \in A^{**}$ is called regular (**[33]**) if $\|xp\| = \|x\bar{p}\|$, $\forall x \in A$.

3.54. Proposition. *Let B be a hereditary C^*-subalgebra of A and p a projection in $B^{**} \subset A^{**}$. Then $\bar{p}^B \leqq \bar{p}^A$ in the following two cases:*

(i) *B is a corner of an ideal of A.*

(ii) *p is regular relative to B.*

Proof. (i) Let B be a corner of the ideal I. It is enough to show $\bar{p}^B \leqq \bar{p}^I$ and $\bar{p}^I \leqq \bar{p}^A$. For the first let $q \in M(I)$ be the open projection corresponding to B. Then for $x \in I$,

$$xp = 0 \Leftrightarrow xqp = 0 \Leftrightarrow qx^*xqp = 0 \Leftrightarrow (qx^*xq) \cdot \bar{p}^B = 0$$

(since $qx^*xq \in B$)

$$\Leftrightarrow xq\bar{p}^B = 0 \Leftrightarrow x\bar{p}^B = 0.$$

For the second, let $x \in A$. Then

$$xp = 0 \Leftrightarrow I(xp) = 0 \Leftrightarrow (Ix)p = 0 \Leftrightarrow (Ix)\bar{p}^I = 0 \Leftrightarrow x\bar{p}^I = 0.$$

(For $Ix\bar{p}^I = 0 \Rightarrow x\bar{p}^I = 0$, note that $x\bar{p}^I \in I^{**}$.)

(ii) By Theorem 6.1 and Lemma 3.5 of **[20]** p regular relative to

$$B \Rightarrow \{\varphi \in S(B):\text{supp } \varphi \leqq p\}$$

is weak* dense in

$$\{\varphi \in S(B):\text{supp } \varphi \leqq \bar{p}^B\}.$$

(We are here using the equivalence of (3) and (4) of 6.1, which is valid for arbitrary B. Effros' proof of (2) $\Rightarrow$ (3) assumes a unital algebra.) For $\varphi \in S(B)$, let $\tilde{\varphi}$ be the unique element of $S(A)$ such that $\tilde{\varphi}_{|B} = \varphi$. If $\varphi_\alpha \to \varphi$ in $S(B)$, then the uniqueness of norm-preserving extension implies $\tilde{\varphi}_\alpha \to \tilde{\varphi}$ in $S(A)$. For $\varphi \in S(B)$ the support projection of $\tilde{\varphi}$ in A^{**} is the same as the support projection of φ in B^{**}. Since

$$\bar{p}^A \geqq \text{supp } \psi, \forall \psi \in \{\theta \in S(A):\text{supp } \theta \leqq p\}^{-w^*},$$

the result follows.

3.55. THEOREM. *Let $p_1, p_2 \in A^{**\prime}$ be closed projections such that $[p_1, p_2] = 0$. Assume that $B = \text{her}(1 - p_1p_2)$ is σ-unital and p_1, p_2 are regular relative to $M(A)$ (as elements of $M(A)^{**} \supset A^{**}$). Then (p_1, p_2) satisfies* (C′).

Proof. For $x \in M(A)_+$, $xp_1 = 0$ or $xp_2 = 0 \Rightarrow xp_1p_2 = 0 \Rightarrow x \in M(A, B)$. Consider $p_i' = p_i - p_1p_2 \in B^{**}$, which is closed relative to B. Then $M(A, B_i)$ (notation as above) can be identified with

$$\text{her}_{M(A,B)}(1 - (p_i')^{-M(A,B)}).$$

Claim. p_i regular relative to $M(A) \Rightarrow p_i'$ regular relative to $M(A, B)$.

Proof of claim. Let $\varphi \in S(M(A, B)) \subset S(M(A))$ such that

$$\text{supp } \varphi \leqq \bar{p}_i'^{M(A,B)} \leqq \bar{p}_i^{M(A)}.$$

Then by [**20**], there are $\varphi_\alpha \in S(M(A))$ such that supp $\varphi_\alpha \leqq p_i$ and $\varphi_\alpha \to \varphi$ weak*. If $q \in M(A)^{**}$ is the open projection corresponding to $M(A, B) \subset M(A)$, then the facts that φ is supported by q and $\|\varphi\| = 1$ imply

$$\|\varphi_\alpha - q\varphi_\alpha q\| \to 0.$$

Hence $q\varphi_\alpha q \to \varphi$ weak*. But

$$\text{supp } \varphi_\alpha \leqq p_i \Rightarrow \text{supp } q\varphi_\alpha q \leqq p_i',$$

since the component of q on $A^{**} \subset M(A)^{**}$ is $1 - p_1p_2$. This proves the claim by [**20**, Theorem 6.1].

Now consider $B \subset M(A, B) \subset M(B)$. $M(A, B)$ is hereditary in $M(B)$, since

$$M(A, B)M(B)M(A, B) \subset M(A, B).$$

Since p_1' and p_2' are orthogonal closed projections relative to B, 3.33 implies

$$(p_1')^{-M(B)} \cdot (p_2')^{-M(B)} = 0.$$

By 3.54 (ii), this implies

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$(p_1')^{-M(A,B)} \cdot (p_2')^{-M(A,B)} = 0.$$

Since

$$\bar{p}_i^{M(A)} = 1 - q + (p_i')^{-M(A,B)} = (p_1 p_2)^{-M(A)} + (p_i')^{-M(A,B)},$$

this shows that

$$[\bar{p}_1^{M(A)}, \bar{p}_2^{M(A)}] = 0 \quad \text{and } \bar{p}_1^{M(A)} \cdot \bar{p}_2^{M(A)} = (p_1 p_2)^{-M(A)}.$$

3.56. *Remark*. Even, if B is an ideal of A, $M(A, B)$ need not be an ideal of $M(B)$. If it were an ideal, 3.54 (i) could be used to prove (C′) without any regularity assumption; but in one of the counterexamples to (C′) below, B is an ideal. (Cf. 2.57 (i).)

3.57. LEMMA. *Let $p \in A^{**}$ be a closed projection. Then p is regular relative to $M(A)$ in the following cases.*

(i) *There are a closed central projection $z \in A^{**}$ and a projection $q \in M(A)$, such that $p = zq$.*

(ii) her$(1 - p)$ *is an ideal of a corner of A.*

Proof. (i) z is also central in $M(A)^{**} \supset A^{**}$, and we can forget A. Let $B = qM(A)q$ and $x \in M(A)$. Then

$$\|xp\| = \|xqp\| = \| |xq|p\| = \| |xq|\bar{p}^B\|,$$

since $|xq| \in B$, p is central in B^{**}, and central projections are always regular. It is routine to check that $\bar{p}^B = \bar{p}^{M(A)}$. Thus

$$\|xp\| = \| |xq|\bar{p}^M\| = \|x\bar{p}^M\|.$$

(ii) Let q be a projection in $M(A)$ such that her$(1 - p)$ is an ideal of her(q). Then there is an open central projection $w \in A^{**}$ such that $(1 - p) = wq \Rightarrow p = 1 - wq = 1 - q + zq$, where $z = 1 - w$. Here $1 - q$ is regarded as an element of $M(A) \subset A^{**}$. It is easy to see that $\bar{p}^M = (1 - q)^{-M} + (zq)^{-M}$, where $(1 - q)^{-M} = 1 - q$ regarded as an element of $M(A)^{**}$. By the criterion of [**30**] p is regular if and only if $M(A) \ni x \geqq p \Rightarrow x \geqq \bar{p}^M$. (The first inequality can be computed in A^{**}, since $p \in A^{**}$, but the second is in $M(A)^{**}$.) But

$$x \geqq p = (1 - q) + zq \Rightarrow x - (1 - q) \geqq zq$$

$$\Rightarrow x - (1 - q) \geqq (zq)^{-M}$$

(by (i))

$$\Rightarrow x \geqq (1 - q)^{-M} + (zq)^{-M} = \bar{p}^M.$$

3.58. COROLLARY. *Let $p_1, p_2 \in A^{**}$ be closed projections such that $[p_1, p_2] = 0$. Then (p_1, p_2) satisfies* (C′) *in these cases*:

(a) her$(1 - p_1p_2)$ *is σ-unital and each p_i satisfies* (i) *or* (ii) *of* 3.57.

(b) *p_1 or p_2 is compact.*

Proof. (a) is clear from 3.57 and 3.55.

(b) is also easy. p_1 compact implies p_1 and p_1p_2 are already closed relative to $M(A)$. Then the computations needed to verify (C′) can be done in the A^{**} component of

$$M(A)^{**} = A^{**} \oplus (M(A)/A)^{**}.$$

Remarks. (i) We will see that p_1p_2 compact $\not\Rightarrow$ (C′).

(ii) With regard to (a), it is actually sufficient for only one of p_1, p_2 to satisfy 3.57 (ii).

3.59. *Examples.* Let $A = E_1$.

(i) Let

$$v_n = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_{n+1}, \quad w_n = \frac{1}{\sqrt{2}}e_1 - \frac{1}{\sqrt{2}}e_{n+1},$$

and p any projection such that $p \geqq e_1 \times e_1$. Define p_1 and p_2 by

$$(p_1)_n = v_n \times v_n, \quad (p_2)_n = w_n \times w_n, \quad n = 1, 2, \dots,$$

and $(p_1)_\infty = (p_2)_\infty = p$. Then $[\bar{p}_1^M, \bar{p}_2^M] = 0$ but

$$(p_1p_2)^{-M} \neq \bar{p}_1^M \cdot \bar{p}_2^M.$$

First we show that $M(A, B_1)$ and $M(A, B_2)$ have the same image in $M(A)/A$. To see this, note that, for $x \in M(A)$, $x \in M(A, B_1) \Leftrightarrow xp_1 = p_1x = 0 \Leftrightarrow x_nv_n = x_n^*v_n = 0$ and $x_\infty p = px_\infty = 0$. But

$$x_\infty p = 0 \Rightarrow x_\infty e_1 = 0 \Rightarrow \|x_ne_1\| \to 0,$$

since $x_n \to x_\infty$ strongly. This and

$$x_nv_n = 0 \Rightarrow \|x_ne_{n+1}\| \to 0.$$

Also $\|x_n^*e_{n+1}\|$, $\|x_n^*e_1\| \to 0$. Then we can find $a_n \in \mathscr{K}$ such that $\|a_n\| \to 0$ and $(x_n + a_n)e_{n+1}$, $(x_n + a_n)^*e_{n+1}$, $(x_n + a_n)e_1$, and $(x_n + a_n)^*e_1$ are all 0. Thus we have found $a \in A$ such that $x + a \in M(A, B_2)$. This and symmetry show that $\bar{p}_1^M$ and $\bar{p}_2^M$ have the same image in $(M(A)/A)^{**}$. Since their components in A^{**} are p_1 and p_2,

$$[\bar{p}_1^M, \bar{p}_2^M] = 0.$$

Now

$$x \in M(A, B) \Leftrightarrow xp_1p_2 = p_1p_2x = 0 \Leftrightarrow x_\infty p = px_\infty = 0.$$

But by the above

$$x \in \text{her}(M(A, B_1) \cup M(A, B_2)) \Rightarrow \|x_ne_{n+1}\| \to 0.$$

Clearly $\exists x \in M(A)$ such that $x_\infty = 0$ and $\|x_ne_{n+1}\| \not\to 0$. Thus (C) and (C′) fail.

(ii) Let v_n and p be as in (i) and

$$w_n = -\frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_{n+1} + \frac{1}{\sqrt{3}}e_{n+2}.$$

Again take $(p_1)_n = v_n \times v_n$, $(p_2)_n = w_n \times w_n$, $(p_1)_\infty = (p_2)_\infty = p$. Then

$$[\bar{p}_1^M, \bar{p}_2^M] \neq 0.$$

We prove this by showing that 3.28 fails; i.e., $\exists b \in M(A, B_1)$, $c \in M(A, B_2)$, and $\epsilon > 0$ such that $\forall x \in M(A, B_1) \cap M(A, B_2)$ with $0 \leqq x \leqq 1$,

$$||b(1 - x)c|| \geqq \epsilon.$$

Now as in (i),

$$x \in M(A, B_1) \Rightarrow ||x_n e_{n+1}||, ||x_n^* e_{n+1}|| \rightarrow 0.$$

Similarly

$$x \in M(A, B_2) \Rightarrow ||x_n(e_{n+1} + e_{n+2})||, ||x_n^*(e_{n+1} + e_{n+2})|| \rightarrow 0.$$

Define b by

$$b_n = e_{n+2} \times e_{n+2}, n = 1, 2, \dots, b_\infty = 0,$$

and c by

$$c_n = \left(\frac{1}{\sqrt{2}}e_{n+1} - \frac{1}{\sqrt{2}}e_{n+2}\right) \times \left(\frac{1}{\sqrt{2}}e_{n+1} - \frac{1}{\sqrt{2}}e_{n+2}\right),$$

$$n = 1, 2, \dots, c_\infty = 0.$$

Then $||b_n c_n|| = 1/\sqrt{2} > 0$. From above,

$$x \in M(A, B_1) \cap M(A, B_2) \Rightarrow ||x_n e_{n+1}||, ||x_n e_{n+2}|| \rightarrow 0$$

$$\Rightarrow ||x_n c_n|| \rightarrow 0 \Rightarrow ||b(1 - x)c|| \geqq \frac{1}{\sqrt{2}}.$$

Note that if we take $p = 1$ in (i) or (ii), then

$$B = \{a \in A : a_\infty p = pa_\infty = 0\}$$

is an ideal of A. Also B_1 and B_2 are corners of the ideal B in this case. If we take $p = e_1 \times e_1$, $p_1 p_2$ is compact. Note also that although p_1 and p_2 are not regular, there does exist a constant K such that $||x\bar{p}_i^M|| \leqq K||xp_i||$. For the p of example 3.13 not even this is true.

4. Results on $T \mapsto T^*T$.

4.A. *Basic results.*

4.1. PROPOSITION. (a) $T \in RM(A) \Rightarrow T^*T \in A_+^m$. *If A is σ-unital,*

$$T \in RM(A) \Rightarrow T^*T \in A_+^\sigma.$$

(b) $T \in QM(A) \Rightarrow T^*T \in QM(A)_+^m$. *If A is σ-unital,*

$$T \in QM(A) \Rightarrow T^*T \in QM(A)_+^\sigma.$$

Proof. Let (e_α) be an approximate identity of A, sequential if A is σ-unital.

(a). $T \in RM(A) \Rightarrow T^*e_\alpha T \in A_+$. Clearly $T^*e_\alpha T \nearrow T^*T$.

(b). $T \in QM(A) \Rightarrow T^*e_\alpha T \in QM(A)_+$. Again $T^*e_\alpha T \nearrow T^*T$.

4.2. PROPOSITION. *Let A be a σ-unital C^*-algebra and $h \in A_+^{**}$. Then* (i)-(iv) *are equivalent and* (i′)-(iv′) *are equivalent.*

(i) $h \in \overline{A_{sa}^m}$ *and is separable* (2.16).

(ii) $h \in A_+^\sigma$.

(iii) $h \in \overline{A_+^\sigma}$.

(iv) $h \in \overline{A_{sa}^\sigma}$.

(i′) $h \in (\widetilde{A}_{sa}^m)^-$ *and is separable.*

(ii′) $h \in QM(A)_+^\sigma$.

(iii′) $h \in (QM(A)_+^\sigma)^-$.

(iv′) $h \in (QM(A)_{sa}^\sigma)^-$.

Proof. (i) $\Rightarrow$ (ii): There is a separable C^*-subalgebra B of A such that $h \in B^{**}$ and her$(B) = A$. By 2.14 $h \in \overline{B_{sa}^m}$. Then 3.24 (a) implies $h \in B_+^\sigma \subset A_+^\sigma$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) is clear from 2.16.

The other half of 4.2 is the same except that we use 3.26 (a) instead of 3.24 (a) and observe that

$$\text{her}(B) = A \Rightarrow QM(B) \subset QM(A).$$

4.3. PROPOSITION. *If e is a strictly positive element of a C^*-algebra A and $h \in A_+^{**}$, then $h = T^*T$ for some $T \in QM(A)$ if and only if $ehe = R^*R$ for some $R \in RM(A)$.*

Proof. If $h = T^*T$, then $ehe = (Te)^*(Te)$; and $T \in QM(A) \Rightarrow Te \in RM(A)$.

If $ehe = R^*R$, $R \in RM(A)$, then $R^*R \leqq \|h\|e^2$. Therefore $R = Te$ for some $T \in A^{**}$.

$$AR \subset A \Rightarrow ATe \subset A \Rightarrow AT(eA) \subset A \Rightarrow ATA \subset A,$$

since $(eA)^- = A$. Therefore $T \in QM(A)$, and $eT^*Te = ehe \Rightarrow T^*T = h$.

4.4. THEOREM. *Let A be a stable σ-unital C^*-algebra and $h \in A_+^{**}$.*

(a) *$h = T^*T$ for some $T \in RM(A)$ if and only if h satisfies* 4.2 (i)-(iv). *In particular, if A is separable, this is so if and only if h is strongly* lsc.

(b) *$h = T^*T$ for some $T \in QM(A)$ if and only if h satisfies* 4.2 (i′)-(iv′). *In particular, if A is separable, this is so if and only if h is weakly* lsc.

Proof. 4.1 implies the necessity of the conditions.

(a). If $h \in A_+^\sigma$, write

$$h = \sum_1^\infty a_n, \quad a_n \in A_+.$$

Since A is stable, there are isometries $U_n \in M(A)$ such that $U_n^* U_m = 0$ for $n \neq m$ and $\sum U_n U_n^* = 1$ with convergence in the strict topology of $M(A)$. (To see this, write $A = B \otimes \mathscr{K}$ so that $M(B) \otimes B(H) \subset M(A)$ by [7]. Choose the U_n's in $1 \otimes B(H)$.) Then it is easy to check that $\sum_1^\infty U_n a_n^{1/2}$ converges σ-strongly and right strictly to a $T \in A^{**}$ such that $T^*T = h$. It follows that $T \in RM(A)$.

(b). Let e be strictly positive in A. Obviously h separable $\Rightarrow$ ehe separable, so that 2.4 $\Rightarrow$ ehe satisfies 4.2 (i). The result follows from (a) and 4.3.

4.5. COROLLARY. *If A is σ-unital and stable, then*

$$\{T^*T : T \in RM(A)\} \text{ and } \{T^*T : T \in QM(A)\}$$

are norm closed.

4.6. *Question.* Does the conclusion of 4.5 hold if the stability hypothesis is dropped?

Remark. If the A of 4.4 is not separable, there may be elements of A_+^m not of the form T^*T, $T \in QM(A)$. For example, h could be an open projection such that her(h) is not σ-unital (2.16).

4.7. COROLLARY. *If A is σ-unital and stable, $T \in QM(A)$, and $T^*T \in \overline{A_+^m}$, then $\exists R \in RM(A)$ such that $R^*R = T^*T$.*

Remark. This is false if A is not stable by 5.F below.

4.8. PROPOSITION. *If A is a σ-unital C^*-algebra and $0 < \epsilon \leqq h \in A^{**}$, then $h = T^*T$ for an invertible $T \in RM(A)$ if and only if $h^{-1} \in QM(A)$.*

Proof. If $h^{-1} \in QM(A)$, then $h^{-1} = L^*L$ for an invertible $L \in LM(A)$ by 4.8 of [**10**]. Then take $T = (L^{-1})^*$. L^{-1} is in $LM(A)$ by 4.1 of [**10**].

If $h = T^*T$ for $T \in RM(A)$ and invertible (in A^{**}), then

$$h^{-1} = T^{-1}(T^*)^{-1} = [(T^*)^{-1}]^*(T^*)^{-1}.$$

By 4.1 of [**10**], $(T^*)^{-1} \in LM(A)$, and this implies $h^{-1} \in QM(A)$.

Example. It is easy to use 4.8 and 4.4 to construct examples where $h \geqq \epsilon > 0$, $h = T^*T$ for some $T \in RM(A)$, but $h \neq T^*T$ for any invertible $T \in RM(A)$. A very simple example would be to take $A = E_1$, $h_n = 1$, $n = 1, 2, \ldots, h_\infty = 1/2$.

4.9. *Questions.* (i) If A is stable and σ-unital (or separable) and $0 < \epsilon \leqq h \in (\tilde{A}^m_{sa})^-$, is $h = T^*T$ for an invertible $T \in QM(A)$?

(ii) If in (i) we assume only that h is one-one (on the universal Hilbert space of A), can T be taken with dense range? (It will automatically be one-one.)

(i′) Same as (i) except drop the assumption that A is stable and add the assumption that $h = T^*T$ for some $T \in QM(A)$.

(ii′) Same as (ii) except drop the assumption that A is stable and add the assumption that $h = T^*T$ for some $T \in QM(A)$.

It will be shown in Section 5 that the answers to (i), (ii) are yes for $A = E_1$.

4.B. *Applications.* Let $\mathcal{Q}$ be the C^*-subalgebra of A^{**} generated by $QM(A)$ and $\tilde{\mathcal{B}}_0$ the norm closed real vector space generated by A^m_{sa}, or equivalently ([**5**, Proposition 2.6) by $\tilde{A}^m_{sa}$. It was shown by Combes [**15**] that $\tilde{\mathcal{B}}_0$ is a Jordan algebra. By 4.15 of [**10**] $\mathcal{Q} \subset \tilde{\mathcal{B}}_0 + i\tilde{\mathcal{B}}_0$. This implies that the atomic representation of A is faithful on $\mathcal{Q}$; but an arbitrary faithful representation of A, though it is isometric on $QM(A)$, need not be faithful on $\mathcal{Q}$. This is shown by the example of Fillmore and Mingo alluded to in 2.23 (ii).

4.10. THEOREM. *If A is a separable C^*-algebra, then $\tilde{\mathcal{B}}_0 + i\tilde{\mathcal{B}}_0$ is a C^*-algebra. If A is also stable, then $\tilde{\mathcal{B}}_0 + i\tilde{\mathcal{B}}_0 = \mathcal{Q}$.*

Proof. First assume A stable. It is an easy consequence of 4.4 that $\tilde{\mathcal{B}}_0 \subset \mathcal{Q}$. Thus by 4.15 of [**10**], $\mathcal{Q} = \tilde{\mathcal{B}}_0 + i\tilde{\mathcal{B}}_0$.

For general A, consider $B = A \otimes \mathcal{K}$, and identify A with $A \otimes p$, where p is a rank one projection in $\mathcal{K}$. It is easy to see, and follows from 2.13, that

$$\tilde{\mathcal{B}}_0(A) = p\tilde{\mathcal{B}}_0(B)p.$$

Therefore $\tilde{\mathcal{B}}_0(A) + i\tilde{\mathcal{B}}_0(A) = p\mathcal{Q}(B)p$, which is a C^*-algebra, since $\mathcal{Q}(B)$ is a C^*-algebra and $p \in M(B) \subset \mathcal{Q}(B)$.

It is well known that the set of continuity points of any function with values in a metric space is a G_δ set and that for an lsc or usc (real) function on a compact Hausdorff space this G_δ set is dense. Since every element of $\tilde{\mathcal{B}}_0$ is the norm limit of a sequence $(f_n - g_n)$, $f_n, g_n \in A^m_{sa}$, it follows from the above and the Baire category theorem that the set of continuity points of an element of $\tilde{\mathcal{B}}_0$, regarded as a function on $\Delta(A)$, is a dense G_δ. D. Olesen told us that this observation might have applications in connection with crossed products. The two corollaries below are offered on the chance that they would facilitate such applications.

4.11. COROLLARY. *If A is a separable C^*-algebra and $V \subset \tilde{\mathcal{B}}_0 + i\tilde{\mathcal{B}}_0$ is norm separable, then $\Delta(A)$ contains a dense G_δ set of simultaneous continuity points for $C^*(V)$, the C^*-subalgebra of A^{**} generated by V. (Here, as above, elements of A^{**} are regarded as functions on $\Delta(A)$.)*

Proof. From the above it is obvious that any norm separable subset of $\widetilde{\mathscr{B}}_0 + i\widetilde{\mathscr{B}}_0$ has a dense G_δ set of simultaneous continuity points. The only point here is that $C^*(V)$, which is still separable, is still contained in $\widetilde{\mathscr{B}}_0 + i\widetilde{\mathscr{B}}_0$.

4.12. COROLLARY. *If A is a separable C^*-algebra and $x, y \in \widetilde{\mathscr{B}}_0 + i\widetilde{\mathscr{B}}_0$, then the map $\varphi \mapsto x\varphi y = \varphi(x \cdot y)$ regarded as a map from $\Delta(A)$ to A^* (weak* topologies), has a dense G_δ set of continuity points.*

Proof. The map takes values in a bounded subset of A^*, which is metrizable for the weak* topology. If $\{a_n : n = 1, 2 \dots\}$ is a dense subset of A, then φ_0 is a continuity point if and only if it is a continuity point for each of the maps $\varphi \mapsto \varphi(xa_ny)$. Since $xa_ny \in \widetilde{\mathscr{B}}_0 + i\widetilde{\mathscr{B}}_0$, each of these maps has a dense G_δ set of continuity points.

It should be noted that if A is non-unital, $S(A)$ is a dense G_δ in $\Delta(A)$, so that it is unimportant whether the conclusions of 4.11 and 4.12 are stated in terms of $S(A)$ or $\Delta(A)$. Also for every Borel function F, there is a dense G_δ set Δ_0 such that $F_{|\Delta_0}$ is continuous. Any application of 4.11 or 4.12 would have to hinge on the distinction between "$F_{|\Delta_0}$ is continuous" and "F is continuous at each point of Δ_0".

4.C. *Density theorems, mainly for stable algebras.* Suppose A is stable and σ-unital, $h_1, h_2 \in A_+^\sigma$, and $h_1 \geqq h_2$. If $B = A \otimes c$, define $k \in B_+^\sigma$ by $k_n = h_1$, $n = 1, 2, \dots, k_\infty = h_2$. By 4.4 there is $T \in RM(B)$ such that $T^*T = k$, and this means there are $T_n \in RM(A)$, $n = \infty, 1, 2, \dots$, such that $T_n^*T_n = h_1$, $n < \infty$, $T_\infty^* T_\infty = h_2$, and $T_n \to T_\infty$ right strictly. This observation is the basis for 4.C. It turned out that most of the theorems could be proved *ab ovo*, but 4.25 and 4.26 seem to depend non-trivially on 4.2, 4.4, and 3.41 (b).

4.13. LEMMA. *Let A be a stable C^*-algebra.*

(a) $\{U \in M(A) : U^*U = 1\}$ *is right strictly dense in*

$$\{T \in M(A) : \|T\| \leqq 1\}.$$

(b) $\{U \in M(A) : U^*U = UU^* = 1\}$ *is quasi-strictly dense in*

$$\{T \in M(A) : \|T\| \leqq 1\}.$$

Proof. We can find $V_n, W_n \in M(A)$ such that $V_n^*V_n = W_n^*W_n = 1$, $V_n^*W_n = 0$, $V_nV_n^* + W_nW_n^* = 1$, $V_n \to 1$ right strictly. To do this, write $A = B \otimes \mathscr{K}$, so that $1 \otimes B(H)$ embeds in $M(A)$ by [7]. On bounded subsets of $B(H)$ this embedding is continuous from the strong topology to the left strict topology.

(a). Now if $T \in M(A)$, $\|T\| \leqq 1$, let

$$U_n = V_nT + W_n(1 - T^*T)^{1/2}.$$

It is routine to check that $U_n^*U_n = 1$ and $U_n \to T$ right strictly.

(b). For $T \in M(A)$, $\|T\| \leqq 1$, let

$$U_n = V_n T V_n^* + V_n(1 - TT^*)^{1/2} W_n^* + W_n(1 - T^*T)^{1/2} V_n^* - W_n T^* W_n^*.$$

Since $A \subset M(A)$, $LM(A)$, $RM(A)$, $QM(A) \subset A^{**}$, A^* can be isometrically embedded in the Banach space duals $M(A)^*$, $LM(A)^*$, $RM(A)^*$, $QM(A)^*$. The following lemma is probably not new.

4.14. LEMMA. (a) *Every strictly continuous linear functional on $M(A)$ is in A^*.*

(b) *Every left strictly continuous linear functional on $LM(A)$ is in A^*.*

(c) *Every right strictly continuous linear functional on $RM(A)$ is in A^*.*

(d) *Every quasi-strictly continuous linear functional on $QM(A)$ is in A^*.*

Proof. Since all parts are similar, we prove only (c). Let f be right strictly continuous on $RM(A)$. Since the right strict topology is generated by the semi-norms $x \mapsto \|ax\|$, $a \in A$, there must be $a_1, \ldots, a_n \in A$ such that

$$|f(x)| \leqq \sum_1^n \|a_i x\|, \forall x \in RM(A).$$

Since $a_i RM(A) \subset A$, a standard use of the Hahn-Banach theorem yields $g_1, \ldots, g_n \in A^*$ such that

$$f(x) = \sum_1^n g_i(a_i x), \forall x \in RM(A).$$

If $h = \sum_1^n a_i g_i \in A^*$, then h maps to f under the embedding $A^* \to RM(A)^*$.

For $h \in A_+^{**}$ let

$$\mathscr{S}(h) = \{T \in RM(A): T^*T = h\},$$

$$\mathscr{T}(h) = \{T \in RM(A): T^*T \leqq h\},$$

$$\mathscr{S}'(h) = \{T \in QM(A): T^*T = h\}, \text{ and}$$

$$\mathscr{T}'(h) = \{T \in QM(A): T^*T \leqq h\}.$$

4.15. THEOREM. *Let A be a stable C^*-algebra.*

(a) *If $T \in RM(A)$ and $h = T^*T$, then*

$$\{UT: U \in M(A), U^*U = 1\}$$

is right strictly dense in $\mathscr{T}(h)$.

(b) *If $T \in QM(A)$ and $h = T^*T$, then*

$$\{UT: U \in M(A), U^*U = 1\}$$

is quasi-strictly dense in $\mathscr{T}'(h)$.

(c) *If* $T \in LM(A)$ *and* $h = T^*T$, *then*

$$\{UT:U \in M(A), U^*U = UU^* = 1\}$$

is quasi-strictly dense in $\mathscr{T}'(h)$.

Proof. (a). Let

$$\mathscr{T}_0 = \{xT:x \in M(A), \|x\| \leqq 1\}.$$

Since $U_\alpha \to x$ right strictly implies $U_\alpha T \to xT$ right strictly, it is enough, by 4.13 (a), to show that $\mathscr{T}_0$ is right strictly dense in $\mathscr{T}(h)$. But $\mathscr{T}_0$ and $\mathscr{T}(h)$ are both convex subsets of $RM(A)$. By 4.14 (c) it is enough to show

$$\sup \operatorname{Re} f_{|\mathscr{T}_0} = \sup \operatorname{Re} f_{|\mathscr{T}(h)}, \quad \forall f \in A^*;$$

i.e., it is enough to show $\mathscr{T}_0$ σ-weakly dense in $\mathscr{T}(h)$. Since

$$\mathscr{T}(h) \subset \{yT:y \in A^{**}, \|y\| \leqq 1\} \text{ and}$$

$$\mathscr{T}_0 \supset \{xT:x \in A, \|x\| \leqq 1\},$$

this follows from the Kaplansky density theorem.

(b) $U_\alpha \to x$ right strictly $\Rightarrow U_\alpha T \to xT$ right strictly $\Rightarrow U_\alpha T \to xT$ quasi-strictly. This and the use of 4.14 (d) instead of 4.14 (c) are the only differences from the proof of (a).

(c) Since $T \in LM(A)$, $U_\alpha \to x$ quasi-strictly $\Rightarrow U_\alpha T \to xT$ quasi-strictly. Thus we can use 4.13 (b) instead of 4.13 (a). Otherwise (c) is the same as (b).

4.16. COROLLARY. *Let A be a stable* C^**-algebra. If* $\mathscr{S}(h)$ ($\mathscr{S}'(h)$) *is non-empty, then* $\mathscr{S}(h)$ ($\mathscr{S}'(h)$) *is right strictly* (*quasi-strictly*) *dense in* $\mathscr{T}(h)$ ($\mathscr{T}'(h)$). *Also if* $\mathscr{S}(h)$ *is non-empty, then* $\mathscr{S}(h)$ *is quasi-strictly dense in* $\mathscr{T}'(h)$.

4.17. COROLLARY (strengthening of 4.13). *Let A be a stable* C^**-algebra.*

(a) $\{U \in M(A): U^*U = 1\}$ *is right strictly dense in*

$$\{S \in RM(A): \|S\| \leqq 1\}.$$

(b) $\{U \in M(A): U^*U = UU^* = 1\}$ *is quasi-strictly dense in*

$$\{S \in QM(A): \|S\| \leqq 1\}.$$

Proof. Put $T = 1$ in 4.15 (a) or (c).

It is equally interesting to consider the strict or left strict topologies of course, but note that the map $T \mapsto T^*T$ is left strict to quasi-strict continuous. Also, by [**5**], for $S \in QM(A)$, $S \in LM(A)$ if and only if $S^*S \in QM(A)$. With the help of 4.18 below results about other types of multipliers or other types of strict convergence can be derived from the above. 4.19 below is also a complement to the above; it sometimes allows $\{UT:U \in M(A), U^*U = 1\}$ to be replaced by $\{UT:U \in M(A), U^*U = UU^* = 1\}$ in 4.15 (a) or (b).

4.18. PROPOSITION. *If* $T \in A^{**}$, (T_α) *is a net in* A^{**}, $T_\alpha \to T$ *quasi-strictly*, $T_\alpha^* T_\alpha \to T^* T$ *quasi-strictly, and* $T \in LM(A)$ (*or more generally if* $TAT^* \subset \mathrm{her}_{A^{**}}(A)$), *then* $T_\alpha \to T$ *left strictly.*

Proof. Let $a \in A$. It is sufficient to show $|(T_\alpha - T)a|^2 \to 0$ in norm. Since

$$|(T_\alpha - T)a|^2 = a^* T_\alpha^* T_\alpha a + a^* T^* Ta - 2\mathrm{Re}\, a^* T_\alpha^* Ta \quad \text{and}$$

$$a^* T_\alpha^* T_\alpha a \to a^* T^* Ta$$

in norm, it is enough to show $a^* T_\alpha^* Ta \to a^* T^* Ta$ in norm. This last is obvious if $Ta \in A$; and with the help of Theorem 1.2 of [**3**], it is enough to have $Taa^*T^* \in \mathrm{her}_{A^{**}}(A)$.

4.19. PROPOSITION. *If* A *is stable,* $T \in A^{**}$, *and* $TT^* \in \mathrm{her}_{A^{**}}(A)$, *then* $\{UT:U \in M(A),\ U^*U = UU^* = 1\}$ *is right strictly dense in*

$$\{xT:x \in M(A),\ \|x\| \leqq 1\}.$$

Proof. If $x \in M(A)$, $\|x\| \leqq 1$, then by 4.13 (b) there is a net (U_α) of unitary multipliers such that $U_\alpha \to x$ quasi-strictly. Fix $a \in A$. Since $\forall b \in A$, $aU_\alpha b \to axb$ in norm, $aU_\alpha S \to axS$ in norm, $\forall S \in A \cdot A^{**}$. Since $\|U_\alpha\|$ is bounded, $aU_\alpha S \to axS$ in norm for all S in the norm closed right ideal of A^{**} generated by A. $TT^* \in \mathrm{her}_{A^{**}}(A)$ is equivalent to membership of T in this right ideal.

4.20. COROLLARY. *If* A *is a stable* C^**-algebra, then*

$$\{U \in M(A):U^*U = UU^* = 1\}$$

is left strictly dense in

$$\{U \in LM(A):U^*U = 1\}.$$

Proof. Combine 4.15 (c) for $T = 1$ with 4.18.

Similarly,

4.21. COROLLARY. *If* A *is a stable* C^**-algebra and* $h \in M(A)_+$, *then* $\{S \in M(A):S^*S = h\}$ *is left strictly dense in*

$$\{S \in LM(A):S^*S = h\}.$$

4.22. COROLLARY. *If* A *is a stable* C^**-algebra and* $a \in A$, *then*

$$\{Ua:U \in M(A),\ U^*U = UU^* = 1\}$$

is norm dense in

$$\{b \in A:b^*b = a^*a\}.$$

Proof. Let $b \in A$ such that $b^*b = a^*a$. By 4.15 (c) and 4.18 there is a net (U_α) of unitary multipliers such that $U_\alpha a \to b$ left strictly. Let $\epsilon > 0$ and choose $e \in A$ such that $0 \leqq e \leqq 1$ and

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$\|a(1 - e)\| < \epsilon, \quad \|b(1 - e)\| < \epsilon.$$

Then

$$\|U_\alpha a - b\| \leqq \|U_\alpha a - U_\alpha ae\| + \|(U_\alpha a - b)e\| + \|b - be\|$$
$$\leqq 2\epsilon + \|(U_\alpha a - b)e\|, \forall \alpha.$$

Since $\|(U_\alpha a - b)e\| \to 0$,

$$\overline{\lim}\|U_\alpha a - b\| \leqq 2\epsilon;$$

and the result follows.

The next density result will make use of a version of the stabilization theorem (Theorem 3.1 of [**9**]). We have been advised that many people do not realize that there is a relation between the stabilization theorems of [**9**] and those of Kasparov, Theorem 2 of [**23**] for example, and that we ought to clarify it. Since [**25**] has appeared, perhaps not much comment is necessary. [**9**] uses the setting of hereditary subalgebras and [**23**] the setting of right Hilbert modules. It was of course a significant advance when Kasparov introduced right Hilbert modules into *KK*-theory. Theorem 2 of [**23**] is more general than Theorem 3.1 of [**9**] in that it allows a group to operate and allows the real and "real" cases. (Also there is a minor difference in the σ-unitality hypotheses.) Otherwise they are equivalent. The most elementary way to see this is to note that an isomorphism between right Hilbert modules X and Y is the same as a suitable partial isometry in $L(X \oplus Y)$. $L(X \oplus Y)$ is $M(\mathscr{K}(X \oplus Y))$ and Theorem 3.1 of [**9**] (as well as Corollary 2.6, the other stabilization theorem of [**9**]) is an existence theorem for a partial isometry in a multiplier algebra. [**12**] and Theorem 2.5 of [**10**] may also help the reader understand the relation between different approaches to the stabilization theorems. Of course the results of Dixmier and Douady [**19**] are the basic theorems, and the others are generalizations. 4.23 below is a simple corollary of Theorem 3.1 of [**9**] and is proved in detail in order to show what we had in mind by formulating 3.1 of [**9**] in what may seem to be a special case.

4.23. THEOREM. *If C is a σ-unital C^*-algebra, p is a projection in $M(C)$ such that $A = \text{her}(p)$ generates C as an ideal, and A is stable, then C is stable and $\exists u \in M(C)$ such that $u^*u = 1$ and $uu^* = p$.*

Proof. It is enough to prove the existence of u. By 2.6 of [**9**], $\exists v \in M(C \otimes \mathscr{K})$ such that $v^*v = 1$ and $vv^* = p \otimes 1$. Let $q \in \mathscr{K}$ be a rank one projection, and identify C with $C \otimes q$. Since A is stable, $\exists w \in M(C)$ such that $w^*w = p \otimes q$ and $ww^* = p \otimes 1$. Then

$$x = w^*v[(1 - p) \otimes q]$$

is a partial isometry such that

$$x^*x = (1 - p) \otimes q \text{ and } xx^* \leqq p \otimes q.$$

Thus if $B = \text{her}(xx^*)$, then B is a corner of A isomorphic to $\text{her}(1 - p)$. Let

$$D = \text{her}_{C\otimes\mathscr{K}}(xx^* + p \otimes (1 - q)).$$

Then D is the C^*-algebra denoted by the same symbol in 3.1 of [**9**]. Hence by 3.1 of [**9**], $\exists y \in M(D) \subset M(C \otimes \mathscr{K})$ such that

$$y^*y = xx^* + p \otimes (1 - q) \quad \text{and} \quad yy^* = p \otimes (1 - q).$$

Since A is stable, $\exists z \in M(C \otimes \mathscr{K})$ such that $z^*z = p \otimes q$ and $zz^* = p \otimes (1 - q)$. Let $u = z^*y(x + z)$. Then $u^*u = 1 \otimes q$ and $uu^* = p \otimes q$, as desired, given the identification of C with $C \otimes q$.

4.24. THEOREM. *Let A and B be σ-unital C^*-algebras, A stable, and X an $A - B$ Hilbert bimodule such that* $\text{span}(\langle X, X\rangle_B)$ *is dense in B. Then* $\{V \in M(X): V^*V = 1\}$ *is left strictly dense in*

$$\{V \in LM(X): V^*V = 1\}.$$

Remarks. (i) This is related to 4.20 and we have in mind the following potential application. Suppose A is as above and $S, T \in A^{**}$ such that $S^*S = T^*T$. Then $S = UT$ where $U^*U = r$, the range projection of T. If r is open, let $B = \text{her}(r)$ and $X = (AB)^-$. Then $U \in X^{**} \subset A^{**}$, and in some situations it may be possible to prove $U \in LM(X)$. (By 4.4 of [**5**], $U \in QM(X) \Rightarrow U \in LM(X)$.)

(ii) The σ-unitality hypothesis is a little too strong. It would be sufficient in 4.24 to have only B σ-unital and in 4.23 to have only $(1 - p)C(1 - p)$, instead of C, σ-unital. One way to see this is to use Lemma 1.7 of [**25**].

Proof. Let

$$L = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

be the linking algebra of X. (This is a slight generalization, found in [**31**], of the linking algebra of [**12**].) Let

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M(L).$$

By 4.23 L is stable and $\exists u \in M(L)$ such that $u^*u = 1$ and $uu^* = p$. Let $V \in LM(X)$ such that $V^*V = 1$. Since

$$LM(X) \cong pLM(L)(1 - p),$$

we can regard V as an element of $LM(L)$ such that $V^*V = 1 - p$, $VV^* \leqq p$. By 4.17 (b) there is a net (W_α) of unitaries in $M(L)$ such that $W_\alpha \to u^*V$ quasi-strictly. Then

$$V_\alpha = uW_\alpha(1 - p) \to V$$

quasi-strictly. Since $V_\alpha^* V_\alpha = (1 - p) = V^*V$, 4.18 implies $V_\alpha \to V$ left strictly. Since $V_\alpha = pV_\alpha(1 - p)$, we can regard V_α as an element of $M(X)$.

4.25. THEOREM. *If A is a stable and σ-unital C^*-algebra and $h \in QM(A)_+^\sigma$ (cf. 4.2), then $\mathscr{S}'(h)$ is right strictly dense in $\mathscr{T}'(h)$.*

Remark. This seems unnatural since we are using the right strict topology on sets of quasi-mulitipliers, but it does make sense. The right strict topology can be regarded as a topology on all of A^{**}.

Proof. Let $T \in \mathscr{T}'(h)$ and (e_n) an approximate identity of A. Then $e_nT \in LM(A) \cap \mathscr{T}'(h)$ and $e_nT \to T$ right strictly. Therefore we may assume $T \in LM(A)$. Thus $T^*T \in QM(A)$, $T^*T \leqq h$, and 4.2 imply $h - T^*T \in QM(A)_+^\sigma$. By 4.4, $\exists S_0 \in QM(A)$ such that $S_0^*S_0 = h - T^*T$. Let V_n, W_n be as in the proof of 4.13 and $S_n = V_nT + W_nS_0$. Then $S_n \in \mathscr{S}'(h)$ and $S_n \to T$ right strictly.

4.26. THEOREM. *If A is an arbitrary C^*-algebra and $h \in \overline{A_+^m}$ then $\{a \in A : a^*a \leqq h\}$ is left strictly dense in*

$$\{T \in LM(A) : T^*T \leqq h\}.$$

Proof. Since both sets are convex subsets of $LM(A)$, by 4.14 (b) it is enough to show the first is σ-weakly dense in the second. This follows from 3.41 (b).

5. Examples.

5.A. $\mathscr{K}$. It is well known that for $A = \mathscr{K}$, $A^{**} = M(A) = B(H)$. Let $\pi : B(H) \to B(H)/\mathscr{K}$ be the quotient map. Clearly

$$h \in \mathscr{K}_{sa}^m \Rightarrow \exists \mathscr{K} \ni K \leqq h \Rightarrow \pi(h) \geqq 0.$$

Therefore $h \in (\mathscr{K}_{sa}^m)^-$ (h strongly lsc) implies $\pi(h) \geqq 0$. Conversely, $h \geqq 0 \Rightarrow h \in \mathscr{K}_+^m$, since $h^{1/2}P_nh^{1/2} \nearrow h$, where the P_n's are suitable finite rank projections. It follows that $\pi(h) \geqq 0 \Rightarrow h \in \mathscr{K}_{sa}^m$. Conclusions: $h \in B(H)_{sa}$ is strongly lsc if and only if $\pi(h) \geqq 0$. Every element of $B(H)_{sa}$ is middle lsc. Since there is only one interesting type of semicontinuity for this example, we will write "lsc" for "strongly lsc".

The interpolation result 3.16 becomes:

5.1. If $h_1 \geqq h_2$, $\exists K_1 \in \mathscr{K}$ with $K_1 \leqq h_1$, and $\exists K_2 \in \mathscr{K}$ with $K_2 \geqq h_2$, then $\exists K \in \mathscr{K}$ with $h_1 \geqq K \geqq h_2$.

5.2. *Exercise.* Give a direct proof of 5.1.

We will have occasion, even for $A = \mathscr{K}$, to use something usually proved in abstract situations by Dini's theorem. It seems unaesthetic to rely on such methods for what should be a concrete example.

5.3. *Exercise*. Prove the following without Dini's theorem: If $h_\alpha \nearrow h$, where h_α and h are lsc, $K \in \mathscr{K}$, $K \leqq h$, and $\epsilon > 0$, then $K \leqq h_\alpha + \epsilon$ for α sufficiently large.

Solutions for these exercises are given at the end of 5.A.

5.4. LEMMA. *Assume P is a finite rank projection, $K \in \mathscr{K}$, and $0 \leqq K \leqq P + (1 - P)/2$. Then $\exists K' \in \mathscr{K}$ such that $0 \leqq K' \leqq (1 - P)/2$ and $K \leqq P + K'$.*

Proof. Represent operators as 2×2 matrices relative to $H = PH \oplus (1 - P)H$. Let

$$K = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}.$$

Then from

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \leqq \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

follows

$$b = (1 - a)^{1/2} t \left(\frac{1}{2} - c\right)^{1/2}, \ \|t\| \leqq 1.$$

If q is the (finite rank) range projection of $1 - a$, then $b = qb$ and we may assume $t = qt$. Write $(1 - a)^{-1}$ for the inverse of $1 - a$ in $qB(H)q$. (The key point in this whole proof is that $1 - a$ has closed range.) Then

$$(1 - a)^{-1/2} b = t\left(\frac{1}{2} - c\right)^{1/2} \Rightarrow b^*(1 - a)^{-1} b \leqq \frac{1}{2} - c.$$

Take $K' = c + b^*(1 - a)^{-1} b$.

5.5. THEOREM. *If $K, L \in \mathscr{K}$, $0 \leqq K, L \leqq 1$, then $\exists S \in \mathscr{K}$ such that $K, L \leqq S \leqq 1$.*

Proof. $K \leqq 1$ implies there is a finite rank projection P' such that $K \leqq P' + (1 - P')/2$. Similarly, there is P'' such that $L \leqq P'' + (1 - P'')/2$. Choose a finite rank projection $P_1 \geqq P', P''$. By 5.4, $\exists K_1, L_1 \in \mathscr{K}$ such that $0 \leqq K_1, L_1 \leqq (1 - P_1)/2$, $K \leqq P_1 + K_1$, and $L \leqq P_1 + L_1$. Continue this procedure. (The next step is to find a finite rank projection $P_2 \leqq 1 - P_1$ such that $K_1, L_1 \leqq P_2/2 + (1 - P_1 - P_2)/4$.) We obtain a sequence (P_n) of mutually orthogonal finite rank projections, and compact operators K_n, L_n such that

$$0 \leqq K_n, L_n \leqq 2^{-n}(1 - P_1 - \ldots - P_n),$$

$$K \leqq P_1 + \frac{1}{2}P_2 + \ldots + 2^{1-n}P_n + K_n, \text{ and}$$

$$L \leqq P_1 + \ldots + 2^{1-n}P_n + L_n.$$

Then take $S = \sum_1^\infty 2^{1-n}P_n$.

5.6. THEOREM. *Let $h \in B(H)_{sa}$ be* lsc.

(a) $\mathscr{A} = \{K \in \mathscr{K}: K \leqq h\}$ *is directed upward if and only if $h \in \mathscr{K}$ or h is Fredholm.*

(b) *If $h \geqq 0$, then $\mathscr{A} = \{K \in \mathscr{K}: 0 \leqq K \leqq h\}$ is directed upward if and only if Ph is either compact or invertible as an element of $B(PH)$, where $P = E_{(0,\infty)}(h)$.*

Proof. (a). If $h \in \mathscr{K}$, $\mathscr{A}$ has a largest element. Assume h is Fredholm and $K, L \in \mathscr{A}$. We need to find $S \in \mathscr{K}$ such that $K, L \leqq S \leqq h$. Since the problem is unchanged if we add the same compact operator to each of K, L, h, we may assume $K, L \geqq 0$ and $h \geqq \epsilon > 0$. Then with

$$K' = h^{-1/2}Kh^{-1/2}, \quad L' = h^{-1/2}Lh^{-1/2},$$

we have $0 \leqq K', L' \leqq 1$. By 5.5, $\exists S' \in \mathscr{K}$ such that $K', L' \leqq S' \leqq 1$. Let $S = h^{1/2}S'h^{1/2}$.

If h is neither Fredholm nor compact, then some compact perturbation of h has infinite dimensional kernel (Weyl-von Neumann theorem). By a further compact perturbation, we may assume $h = h_1 \oplus h_2 \geqq 0$, relative to $H = H_1 \oplus H_2$, h_i positive and one-one, $h_1 \notin \mathscr{K}$, $h_2 \in \mathscr{K}$, and H_1, H_2 both infinite dimensional. Now just as in 3.23 (i) and (ii), we can find projections $P = 0 \oplus 1$ and Q such that $Q - P \in \mathscr{K}$ and $P \vee Q = 1$. (Here "$\vee$" refers to the lattice of projections, but it follows that $P, Q \leqq S' \leqq 1 \Rightarrow S' = 1$.) If $K = h^{1/2}Ph^{1/2}$ and $L = h^{1/2}Qh^{1/2}$, then $K, L \in \mathscr{A}$. If $K, L \leqq S \leqq h$, then $S = h^{1/2}S'h^{1/2}$ for some S' with $0 \leqq S' \leqq 1$. Since h is one-one,

$$h^{1/2}S'h^{1/2} \geqq h^{1/2}Ph^{1/2},$$

$$h^{1/2}Qh^{1/2} \Rightarrow S' \geqq P,$$

$$Q \Rightarrow S' = 1 \Rightarrow S = h.$$

Thus $S \notin \mathscr{K}$.

(b). By replacing H with PH, we may assume h one-one.

The fact that $\mathscr{A}$ is directed upward if h is compact or invertible is proved as in part (a).

If h is not compact or invertible, choose $C \in \mathscr{K}$ such that $h + C = h_1 \oplus h_2$ as in the proof of (a), and choose P, Q as above. Then as above, $h^{1/2}Ph^{1/2}$, $h^{1/2}Qh^{1/2} \in \mathscr{A}$ and there does not exist $S \in \mathscr{A}$ such that $h^{1/2}Ph^{1/2}, h^{1/2}Qh^{1/2} \leqq S$.

5.7. LEMMA. *Assume* $a \in \mathcal{K}$, h_1 *is* lsc, h_2 *is* usc, $h_1 - h_2$ *is Fredholm, and* $a, h_2 \leqq h_1$. *Then* $\exists x \in \mathcal{K}$ *such that* $a, h_2 \leqq x \leqq h_1$.

Proof. Since $\pi(h_1 - h_2) \geqq 0$, $\pi(h_1 - h_2) \geqq \epsilon > 0$, for some ϵ. Let P be the kernel projection of $h_1 - h_2$, so that P has finite rank, and represent operators by 2×2 matrices relative to $H = (1 - P)H \oplus PH$. Write

$$a - h_2 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \leqq \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} = h_1 - h_2,$$

where the above and $h_1 - h_2 \geqq 0$ imply u positive and invertible. Then

$$B = (u - A)^{1/2} t(-C)^{1/2} \quad \text{with } \|t\| \leqq 1.$$

If Q is the (finite rank) range projection of C, then $B = BQ$ and we may assume $t = tQ$. As in the proof of 5.4, write $(-C)^{-1}$ for the inverse of $(-C)$ in $QB(H)Q$ and deduce $B(-C)^{-1}B^* \leqq u - A$. Let

$$v = A + B(-C)^{-1}B^* \leqq u.$$

Then if

$$y = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix},$$

$y \geqq a - h_2$ and $\pi(y) = \pi(a - h_2) = -\pi(h_2)$. Let $s' = u^{-1/2} v u^{-1/2} \leqq 1$. Since

$$\pi\left(\begin{pmatrix} s' & 0 \\ 0 & 0 \end{pmatrix}\right) = -\pi(h_1 - h_2)^{-1/2} \pi(h_2) \pi(h_1 - h_2)^{-1/2},$$

which is positive, $(s')_-$ is compact. Let $s = (s')_+$ and

$$x = h_2 + \begin{pmatrix} u^{1/2} s u^{1/2} & 0 \\ 0 & 0 \end{pmatrix} = h_2 + (h_1 - h_2)^{1/2} \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} (h_1 - h_2)^{1/2}.$$

$s \leqq 1 \Rightarrow x \leqq h_1$ and $s \geqq 0 \Rightarrow x \geqq h_2$. Also,

$$s \geqq s' \Rightarrow u^{1/2} s u^{1/2} \geqq v \Rightarrow x \geqq h_2 + y \geqq h_2 + (a - h_2) = a.$$

Finally, since $s - s'$ is compact, $\pi(x) = \pi(h_2) + \pi(y) = 0$.

5.8. THEOREM. *If* h_2, h_3 *are* usc, h_1 *is* lsc, *and* $h_1 \geqq h_2, h_3$, *then* $\exists x \in \mathcal{K}$ *such that* $h_1 \geqq x \geqq h_2, h_3$ *provided either* $h_1 - h_2$ *or* $h_1 - h_3$ *is Fredholm.*

Proof. If $h_1 - h_2$ is Fredholm, choose $a \in \mathcal{K}$ such that $h_1 \geqq a \geqq h_3$ (5.1), and apply 5.7.

5.9. LEMMA. (a) *If* $x \in \mathcal{K}$, $h \geqq 0$, k *is* lsc, $h + k$ *is Fredholm and* $x \leqq h + k$, *then* $\exists a \in \mathcal{K}$ *such that* $0 \leqq a \leqq h$ *and* $x \leqq a + k$.

(b) *If* $x \in \mathcal{K}$, $h, k \geqq 0$, $h + k$ *is Fredholm, and* $x \leqq h + k$, *then* $\exists a, b \in \mathcal{K}$ *such that* $0 \leqq a \leqq h$, $0 \leqq b \leqq k$, *and* $x \leqq a + b$.

(c) *If* $x \in \mathcal{K}$, $h_n \geqq 0$ *for* $n = 1, 2, \ldots, \sum_1^\infty h_n$ *converges strongly to a bounded Fredholm operator, and* $x \leqq \sum_1^\infty h_n$, *then* $\exists a_n \in \mathcal{K}$ *such that* $0 \leqq a_n \leqq h_n$ *and* $x \leqq \sum_1^\infty a_n$.

Proof. (a). Use 5.7 to find $a \in \mathcal{K}$ such that $0, x - k \leqq a \leqq h$.

(b) Use (a) to find $a_0, b_0 \in \mathcal{K}$ such that $0 \leqq a_0 \leqq h$, $0 \leqq b_0 \leqq k$, $x \leqq a_0 + k$, and $x \leqq h + b_0$. Then

$$x \leqq \frac{1}{2}(a_0 + b_0) + \frac{1}{2}(h + k), \quad \text{and}$$

$$x_1 = x - \frac{1}{2}(a_0 + b_0) \leqq \frac{1}{2}h + \frac{1}{2}k.$$

Repeat this construction recursively: We obtain $a_n, b_n \in \mathcal{K}$ such that

$$0 \leqq a_n \leqq 2^{-n}h, 0 \leqq b_n \leqq 2^{-n}k, \text{ and}$$

$$x_n = x - \frac{1}{2}\sum_0^{n-1} a_i + b_i \leqq 2^{-n}h + 2^{-n}k.$$

Then take

$$a = \frac{1}{2}\sum_0^\infty a_i, b = \frac{1}{2}\sum_0^\infty b_i.$$

(c). Choose $t_n > 0$, $n = 1, 2 \ldots$, such that

$$\sum_1^\infty t_n = 1 \text{ and } \sum_1^\infty t_n \|h_n\| < \infty.$$

(To see that this is possible consider

$$t'_n = \min(2^{-n}, 2^{-n}\|h_n\|^{-1}).)$$

By a method similar to the proof of (b), we can find $a_{nm} \in \mathcal{K}$ such that

$$0 \leqq a_{nm} \leqq (1 - t_n)^m h_n, \ m \geqq 0, \text{ and}$$

$$x - \sum_{i=0}^{m-1} \sum_{n=1}^\infty t_n a_{ni} \leqq \sum_1^\infty (1 - t_n)^m h_n.$$

(Note that $\sum_{n=1}^\infty t_n a_{ni}$ converges in norm to a compact operator.) Then take

$$a_n = \sum_{i=0}^\infty t_n a_{ni}$$

(norm convergent sum). The double sum

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$\sum_{i,n=(0,1)}^{(\infty,\infty)} t_n a_{ni}$$

converges strongly (to $\sum_1^\infty a_n$), since

$$0 \leqq t_n a_{ni} \leqq t_n (1 - t_n)^i h_n$$

and $\sum_1^\infty h_n$ converges strongly. Also for $v \in H$, the dominated convergence theorem shows that

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} (1 - t_n)^m (h_n v, v) = 0.$$

Thus we may take strong limits on both sides of the basic inequality to deduce

$$x - \sum_1^\infty a_n \leqq 0.$$

Remark. If $\sum_1^\infty \|h_n\| < \infty$, then necessarily $\sum_1^\infty a_n$ converges in norm to a compact operator. Otherwise it may not be possible to achieve $\sum_1^\infty a_n \in \mathcal{K}$. For example let $\{e_n : n = 1, 2, \dots\}$ be an orthonormal basis for H, let h_n be the projection on span (e_n), and let x be an appropriate rank one projection.

5.10. COROLLARY. *If* $x \leqq h + \epsilon$, $x \in \mathcal{K}$, $\epsilon > 0$, *and h is positive, then* $\exists a \in \mathcal{K}$ *such that* $0 \leqq a \leqq h$ *and* $x \leqq a + \epsilon$.

Proof. Take $k = \epsilon$ in 5.9 (a).

Solutions to exercises. 5.2. There are positive $c_1, c_2 \in \mathcal{K}$ such that $h_1 + c_1 \geqq 0$, $h_2 - c_2 \leqq 0$. Since $0 \leqq c_2 - h_2 \leqq h_1 - h_2 + c_1 + c_2$, there is $t \in B(H)$ such that $0 \leqq t \leqq 1$ and

$$(c_2 - h_2) = (h_1 - h_2 + c_1 + c_2)^{1/2} t (h_1 - h_2 + c_1 + c_2)^{1/2}.$$

Take

$$K = h_2 + (h_1 - h_2)^{1/2} t (h_1 - h_2)^{1/2}$$

and compute $\pi(K) = 0$.

5.3. By adding the same compact operator to K, h, and h_α, we reduce to the case $K, h_\alpha \geqq 0$ $(\alpha \geqq \alpha_0)$. There is a finite rank operator L such that $0 \leqq L \leqq K \leqq L + \epsilon/2$. $0 \leqq L \leqq h \Rightarrow \exists t$ such that $0 \leqq t \leqq 1$ and $L = h^{1/2} t h^{1/2}$. Also t may be assumed finite rank.

$$h_\alpha^{1/2} \to h^{1/2} \text{ strongly}$$

$$\Rightarrow h_\alpha^{1/2} t h_\alpha^{1/2} \to h^{1/2} t h^{1/2} \text{ in norm}$$

$$\Rightarrow L \leqq h_\alpha^{1/2} t h_\alpha^{1/2} + \frac{\epsilon}{2} \leqq h_\alpha + \frac{\epsilon}{2}, \alpha \geqq \alpha_1.$$

5.B. $\widetilde{\mathcal{K}}$. If $A = \widetilde{\mathcal{K}}$, then $A^{**} \cong \mathcal{K}^{**} \oplus \mathbf{C} = B(H) \oplus \mathbf{C}$. We will denote a typical element of $\widetilde{\mathcal{K}}_{sa}^{**}$ by (h, λ), $h \in B(H)_{sa}$, $\lambda \in \mathbf{R}$. Since $\widetilde{\mathcal{K}}$ is unital, there is only one kind of semicontinuity, and $\widetilde{\mathcal{K}}_{sa}^m$ is closed under translation by scalars. Thus (h, λ) is lsc $\Leftrightarrow$ $(h - \lambda, 0)$ is lsc $\Leftrightarrow$ $h - \lambda$ is lsc in $B(H)$, by 2.14. The criterion is: (h, λ) is lsc $\Leftrightarrow$ $\pi(h - \lambda) \geqq 0$.

Referring to 3.23, we observe that since (D1) fails for $\mathcal{K}$ it must fail for $\widetilde{\mathcal{K}}$. (If $h \in B(H)_{sa}$ is lsc and $\mathcal{K} \ni x, y \leqq h$, then $a \in \widetilde{\mathcal{K}}$, $(x, 0), (y, 0) \leqq a \leqq (h, 0) \Rightarrow a \in \mathcal{K}$.) Since $\widetilde{\mathcal{K}}$ is unital, (D3) also must fail for $\widetilde{\mathcal{K}}$, though 5.10 showed (D3) is true for $\mathcal{K}$. The next result makes the facts about (D1) fairly clear.

5.11. THEOREM. *Assume* (x_1, λ_1) *and* $(x_2, \lambda_2) \in \widetilde{\mathcal{K}}$, (h, λ) *is* lsc, *and* $(x_1, \lambda_1), (x_2, \lambda_2) \leqq (h, \lambda)$. *Then* $\exists a \in \widetilde{\mathcal{K}}$ *such that* $(x_1, \lambda_1), (x_2, \lambda_2) \leqq a \leqq (h, \lambda)$ *unless* $\lambda = \lambda_1 = \lambda_2$.

Proof. We may assume $\lambda_1 < \lambda$. We seek a solution in the form $a = (y + \lambda, \lambda)$, $y \in \mathcal{K}$. The problem becomes: $x_1 - \lambda, x_2 - \lambda \leqq y \leqq h - \lambda$. By 5.1 we can find $b \in \mathcal{K}$ such that $x_2 - \lambda \leqq b \leqq h - \lambda$. (Recall $x_2 - \lambda_2 \in \mathcal{K}$.) Then the problem $x_1 - \lambda, b \leqq y \leqq h - \lambda$ is a special case of 5.7.

5.C. E_1. Let $A = E_1$ and recall that elements of A^{**} can be identified with bounded collections $\{h_n : 1 \leqq n \leqq \infty, h_n \in B(H)\}$. The discussion in 5.A applies to each h_n.

5.12. THEOREM. *Let* $h \in A_+^{**}$.

(a) *Assume that for every finite rank projection* $P \in B(H)$ *and* $\epsilon > 0$, $\exists N$ *such that* $h_\infty^{1/2} P h_\infty^{1/2} \leqq h_n + \epsilon$, $\forall n \geqq N$. *Then there is a sequence* (U_n) *of isometries such that* $h_n^{1/2} U_n^* \to h_\infty^{1/2}$ *strongly.*

(b) *Assume that for every finite rank projection* $P \in B(H)$ *and* $\epsilon > 0$, $\exists N$ *such that* $P h_\infty P \leqq P h_n P + \epsilon$, $\forall n \geqq N$. *Then there is a sequence* (U_n) *of unitaries such that* $U_n h_n^{1/2} \to h_\infty^{1/2}$ *weakly.*

Proof. (a). Choose finite rank projections P_k and $\epsilon_k > 0$ such that $P_k \nearrow 1$ and $\epsilon_k \searrow 0$. Choose $N_1 < N_2 < \ldots$ so that for $n \geqq N_k$,

$$h_\infty^{1/2} P_k h_\infty^{1/2} \leqq h_n + \epsilon_k.$$

Then for $N_k \leqq n < N_{k+1}$ write

$$P_k h_\infty^{1/2} = A_n(\epsilon_k + h_n)^{1/2}, \ \|A_n\| \leqq 1, \ P_k A_n = A_n.$$

There is an isometry U_n such that $P_k U_n = A_n$. Choose $U_n = 1$ for $n < N_1$. Then

$$\|P_k h_\infty^{1/2} - P_k U_n h_n^{1/2}\| \leqq \epsilon_k^{1/2} \text{ for } N_k \leqq n < N_{k+1},$$

since $\|(\epsilon_k + h_n)^{1/2} - h_n^{1/2}\| \leqq \epsilon_k^{1/2}$. Since

$$\|P_k h_\infty^{1/2} - P_k U_n h_n^{1/2}\| \leqq \|P_{k'} h_\infty^{1/2} - P_{k'} U_n h_n^{1/2}\| \text{ for } k < k',$$

we have that

$$P_k U_n h_n^{1/2} \to P_k h_\infty^{1/2}$$

in norm for each fixed k.

(b). Let P_k and ϵ_k be as in (a), and choose $N_1 < N_2 < \ldots$ such that

$$P_k h_\infty P_k \leqq P_k(h_n + \epsilon_k)P_k, \forall n \geqq N_k.$$

For $N_k \leqq n < N_{k+1}$ write

$$h_\infty^{1/2} P_k = A_n(h_n + \epsilon_k)^{1/2} P_k, \ \|A_n\| \leqq 1.$$

Let Q_n be the range projection of $h_n^{1/2} P_k$, and choose a unitary U_n such that $P_k U_n Q_n = P_k A_n Q_n$. Then

$$P_k U_n h_n^{1/2} P_k = P_k A_n h_n^{1/2} P_k \Rightarrow \|P_k(U_n h_n^{1/2} - h_\infty^{1/2})P_k\| \leqq \epsilon_k^{1/2},$$

$$N_k \leqq n < N_{k+1}.$$

This implies

$$P_k U_n h_n^{1/2} P_k \to P_k h_\infty^{1/2} P_k$$

in norm for each fixed k.

5.13. Criterion for strong semicontinuity. *If* $h \in A_{sa}^{**}$, *then* $h \in \overline{A_{sa}^m}$ *if and only if*

(i) *Each* h_n *is* lsc, $1 \leqq n \leqq \infty$.

(ii) *If* $K \in \mathcal{K}$, $K \leqq h_\infty$, *and* $\epsilon > 0$, *then* $\exists N$ *such that* $K \leqq h_n + \epsilon$, $\forall n \geqq N$.

Proof. First assume $h \in \overline{A_{sa}^m}$. Then it is obvious that each $h_n \in (\mathcal{K}_{sa}^m)^-$, $1 \leqq n \leqq \infty$. Let K and ϵ be as in (ii). Choose a net (a_α) in A such that $a_\alpha \nearrow h + \epsilon/3$. Then $(a_\alpha)_\infty \nearrow h_\infty + \epsilon/3$. By 5.3, $K \leqq (a_\alpha)_\infty + \epsilon/3$ for α sufficiently large. Fix such an α. Since $a_\alpha \in A$, $\exists N$ such that

$$(a_\alpha)_\infty \leqq (a_\alpha)_n + \frac{\epsilon}{3}, \forall n \geqq N.$$

Then for $n \geqq N$,

$$K \leqq (a_\alpha)_n + \frac{2\epsilon}{3} \leqq h_n + \frac{\epsilon}{3} + \frac{2\epsilon}{3}.$$

Now assume (i) and (ii). We need to prove $h \in \overline{A_{sa}^m}$, and it is obviously permissible to replace h by $h + a$ for $a \in A$. Thus, choosing $a_n = a_\infty = (h_\infty)_- \in \mathcal{K}$, we can reduce to the case $h_\infty \geqq 0$. Now by taking $K = 0$ in (ii), we see that

$$\|(h_n)_-\| \to 0 \text{ as } n \to \infty.$$

Thus, by replacing h by $h + b$, $b_n = (h_n)_-$, $b_\infty = 0$, we reduce to the case $h \geqq 0$. Now (ii) implies the hypothesis of 5.12 (a). Take U_n as in 5.12 (a), and let

$$R_n = U_n h_n^{1/2}, \ n = 1, 2, \dots, R_\infty = h_\infty^{1/2}.$$

Then $R \in RM(A)$ and $R^*R = h$. By the (trivial) Proposition 4.1, $h \in A_+^m \subset \overline{A_{sa}^m}$.

Remarks. (i) By 5.3 it is sufficient to verify 5.13 (ii) only for each element of a sequence (K_n) such that $K_n \nearrow h_\infty$.

(ii) Since for $a \in A$, $a_n \to a_\infty$ in norm, one might have guessed that the criterion for h to be strongly lsc would be $h_\infty \leqq h_n + \epsilon$, $n \geqq N$. This is correct whenever $h_\infty \in \mathcal{K}$, but in general it is too strong a requirement. For example, it is not always true for h an open projection.

5.14. Criterion for weak semicontinuity. *If* $h \in A_{sa}^{**}$, *then* $h \in (\widetilde{A}_{sa}^m)^-$ *if and only if for every finite rank projection* P *and* $\epsilon > 0$, $\exists N$ *such that* $Ph_\infty P \leqq Ph_nP + \epsilon$, $\forall n \geqq N$.

Proof. First assume $h \in (\widetilde{A}_{sa}^m)^-$ and let P be given. Define $a \in A$ by $a_n = P$, $n = \infty, 1, 2, \dots$. By 2.4, $a^*ha \in \overline{A_{sa}^m}$. Since $(a^*ha)_\infty = Ph_\infty P \in \mathcal{K}$, 5.13 (ii) implies that $\forall \epsilon > 0$, $\exists N$ as desired.

Now assume h satisfies the criterion. It is clearly permissible to replace h by $h + \lambda$, $\lambda \in \mathbf{R}$, and therefore we may assume $h \geqq 0$. Then 5.12 (b) applies, and we define $T \in A^{**}$ by $T_\infty = h_\infty^{1/2}$, $T_n = U_n h_n^{1/2}$, $n = 1, 2, \dots, U_n$ as in 5.12 (b). Then $T \in QM(A)$, $T^*T = h$, and 4.1 implies $h \in (\widetilde{A}_{sa}^m)^-$.

5.15. *Remarks*. (i) The following alternative criterion follows from 5.14: $h \in (\widetilde{A}_{sa}^m)^-$ if and only if $h_\infty \leqq k$ for every weak cluster point k of (h_n).

(ii) The fact that the U_n's in 5.12 (b) are unitary gives a positive answer to the questions in 4.9 for this example.

5.16. Criterion for middle semicontinuity. *If* $h \in A_{sa}^{**}$, *then the following are equivalent*:

(i) $h \in \widetilde{A}_{sa}^m$.

(ii) *There is a sequence* (Q_n) *in* $B(H)_+$ *such that* $Q_n \to 0$ *strongly and* $h_\infty \leqq h_n + Q_n$, $\forall n$.

(iii) $\exists x \in M(A)_{sa}$ *such that* $h + x$ *is* q-lsc.

Proof. (i) $\Rightarrow$ (ii): Let $\lambda > 0$ be such that $h + \lambda \geqq 0$ and $h + \lambda \in \overline{A_{sa}^m}$. Let (P_k) be a sequence of finite rank projections in $B(H)$ such that $P_k \nearrow 1$. Choose $N_1 < N_2 < \dots$ such that for $n \geqq N_k$,

$$(h_\infty + \lambda)^{1/2} P_k (h_\infty + \lambda)^{1/2} \leqq h_n + \lambda + \frac{1}{k}.$$

Therefore,

$$h_\infty \leqq h_n + \frac{1}{k} + (h_\infty + \lambda)^{1/2}(1 - P_k)(h_\infty + \lambda)^{1/2}, \ n \geqq N_k.$$

Let

$$Q_n = \frac{1}{k} + (h_\infty + \lambda)^{1/2}(1 - P_k)(h_\infty + \lambda)^{1/2} \text{ for } N_k \leqq n < N_{k+1}.$$

Choose Q_n to be a large scalar for $n < N_1$.

(ii) ⇒ (iii): Define $x \in M(A)_{sa}$ by $x_\infty = -h_\infty$ and $x_n = -h_\infty + Q_n$, $n = 1, 2, \ldots$. If $h' = h + x$, then $h'_\infty = 0$ and $h'_n \geqq 0$ for $n < \infty$. It is easy to see that this implies h' is q-lsc.

(iii) ⇒ (i): This is trivial, since

$$x \in M(A)_{sa} \Rightarrow -x \in \widetilde{A}^m_{sa} \quad \text{and}$$

$$h + x \ q\text{-lsc} \Rightarrow h + x \in \widetilde{A}^m_{sa}.$$

5.D. E_2 and E_4. E_2 is a corner of E_1. Thus by 2.13, for $h \in (E_2)^{**}_{sa}$, h is lsc relative to E_2 if and only if h is lsc relative to E_1. Since E_2 is unital, there is only one type of semicontinuity in E_2^{**}.

E_4 is a unital C^*-subalgebra of E_2. Thus by 2.14, for $h \in (E_4)^{**}_{sa}$, h is lsc relative to E_4 if and only if h is lsc relative to E_2. The criterion is: For $h \in A^{**}_{sa}$, $A = E_2$ or E_4, h is lsc if and only if $\forall \epsilon > 0$, $\exists N$ such that $h_\infty \leqq h_n + \epsilon$, $\forall n \geqq N$.

Remarks. (i) This criterion is also valid for $A = c \otimes M_n$, $n > 2$.

(ii) It was asserted in 2.D that for these algebras every lsc element is the sum of a multiplier and a q-lsc element. (Of course multipliers are q-continuous.) The proof of this is similar to, and easier than, that of 5.16 and will be left to the reader.

5.E. E_6. Recall that $E_6 = \mathscr{K} + \mathbf{C}p$ where $p \in B(H)$ is a projection of infinite rank and co-rank. Since $\mathscr{K}$ is an ideal of E_6 and $E_6/\mathscr{K} \cong \mathbf{C}$, $E_6^{**} \cong B(H) \oplus \mathbf{C}$. As in 5.B, we will represent elements of E_6^{**} by pairs (h, λ); and (h, λ) is strongly lsc if and only if $h - \lambda p$ is lsc in $B(H)$, since $\overline{A^m_{sa}}$ is invariant under translation by multiples of p.

To study middle and weak semicontinuity, we use the 2×2 matrix representation of $A = E_6$. Thus

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and $x \in A$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a \in \widetilde{\mathscr{K}}, \ b, c, d \in \mathscr{K}.$$

If

$$h = \left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \lambda \right) \in A_{sa}^{**},$$

then h is middle lsc if and only if

$$\left(\begin{pmatrix} a + t & b \\ b^* & c + t \end{pmatrix}, \lambda + t \right)$$

is strongly lsc for t sufficiently large. This is equivalent to

$$\pi\left(\begin{pmatrix} a - \lambda & b \\ b^* & c + t \end{pmatrix} \right) \geqq 0$$

for t sufficiently large. Since

$$-\|c\| + t \leqq c + t \leqq \|c\| + t,$$

we may as well just write

$$\pi\left(\begin{pmatrix} a - \lambda & b \\ b^* & t \end{pmatrix} \right) \geqq 0;$$

and this is equivalent to

$$\pi(bt^{-1}b^*) \leqq \pi(a - \lambda).$$

In other words, the criterion is:

$$\left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \lambda \right)$$

is middle lsc if and only if $\pi(bb^*) \leqq t\pi(a - \lambda)$ for t sufficiently large. Since this last is automatic if $\pi(a - \lambda) \geqq \epsilon > 0$, we conclude also:

$$\left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \lambda \right)$$

is weakly lsc if and only if $\pi(a - \lambda) \geqq 0$.

From the above or otherwise we see that

$$\left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \lambda \right) \in M(A)_{sa}$$

if and only if $\pi(a) = \lambda$ and $\pi(b) = 0$. Now suppose

$$h = \left(\begin{pmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{pmatrix}, \lambda_1 \right), \ k = \left(\begin{pmatrix} a_2 & b_2 \\ b_2^* & c_2 \end{pmatrix}, \lambda_2 \right),$$

h is middle lsc, k is middle usc, and $h - k \in \overline{A_+^m}$. We will show that this does not imply the existence of $x \in M(A)_{sa}$ such that $h \geqq x \geqq k$, as promised after 3.40. To show this, it is sufficient to consider the special case $\lambda_1 = \lambda_2 = 0$. Then we are given:

https://doi.org/10.4153/CJM-1988-038-5 Published online by Cambridge University Press

$$(1)\quad \left.\begin{array}{l}\pi(b_1b_1^*) \leqq t\pi(a_1)\\ \pi(b_2b_2^*) \leqq -t\pi(a_2)\end{array}\right\}, t > 0,$$

$$(2)\quad \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ b_1^* - b_2^* & c_1 - c_2 \end{pmatrix} \geqq 0.$$

(2) does imply $h - k \in A_+^m$, since $\lambda_1 = \lambda_2$. Also if $\pi(a_1)$, $-\pi(a_2) \geqq \epsilon > 0$, which we will assume, (1) is automatic. We require $x \in M(A)_{sa}$ such that

$$h \geqq x \geqq k \Rightarrow \pi(h) \geqq \pi(x) \geqq \pi(k).$$

Since $\lambda = 0$, this yields

$$(3)\quad \begin{pmatrix} \pi(a_2) & \pi(b_2) \\ \pi(b_2^*) & \pi(c_2) \end{pmatrix} \leqq \begin{pmatrix} 0 & 0 \\ 0 & \pi(c) \end{pmatrix} \leqq \begin{pmatrix} \pi(a_1) & \pi(b_1) \\ \pi(b_1^*) & \pi(c_1) \end{pmatrix}.$$

$$(3) \Rightarrow \pi(b_1^*)\pi(a_1)^{-1}\pi(b_1) \leqq \pi(c_1 - c) \text{ and}$$

$$\pi(b_2)^*\pi(-a_2)^{-1}\pi(b_2) \leqq \pi(c - c_2)$$

$$\Rightarrow \pi(b_1^*)\pi(a_1)^{-1}\pi(b_1) + \pi(b_2^*)\pi(-a_2)^{-1}\pi(b_2) \leqq \pi(c_1 - c_2).$$

Since the only obvious consequence of (2) is

$$\pi(b_1 - b_2)^*\pi(a_1 - a_2)^{-1}\pi(b_1 - b_2) \leqq \pi(c_1 - c_2),$$

it is obvious that there are counterexamples. Perhaps the easiest occurs if $a_1 = 1$, $a_2 = -1$, $b_1 = b_2 = 1$, and $c_1 = c_2 = 0$.

We have included a fair amount of detail, despite the fact that the conjecture demolished by this example may seem foolish, because we are hoping it will lead someone to discover a new theorem (a general theorem, not one just for E_6).

5.F. *E_3 and similar algebras.* Let $d = k + l$, $k, l > 0$, and let A be the C^*-algebra of convergent sequences in M_d with limit of the form

$$\begin{matrix} k \\ l \end{matrix}\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

A^{**} can be identified with the algebra of bounded collections

$$\{h_n : 1 \leqq n \leqq \infty, h_\infty \in M_k, h_n \in M_d, n = 1, 2 \dots\}.$$

For $k = 1$, A is analogous to E_6.

A is, in an obvious way, a subalgebra of E_1. By 2.14 the criterion for strong semicontinuity follows from that for E_1: $h \in A_{sa}^{**}$ is strongly lsc if and only if $\forall \epsilon > 0$, $\exists N$ such that

$$\begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} \leqq h_n + \epsilon, \forall n \geqq N.$$

One could get the criterion for weak semicontinuity by embedding A explicitly as a corner of E_1, but it is easier to work directly: $h \in A_{sa}^{**}$ is weakly lsc if and only if $\forall \epsilon > 0$, $\exists N$ such that $h_\infty \leqq a_n + \epsilon$, $\forall n \geqq N$, where

$$h_n = \begin{pmatrix} a_n & b_n \\ b_n^* & c_n \end{pmatrix}.$$

Proof. Define $e \in A$ by

$$e_\infty = 1 \in M_k, \; e_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} \in M_d.$$

Then e is strictly positive, and 2.4 implies that h is weakly lsc if and only if ehe is strongly lsc. By the above, ehe is strongly lsc if and only if $\forall \epsilon > 0$, $\exists N$ such that

$$\begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} \leqq \begin{pmatrix} a_n & \frac{1}{n} b_n \\ \frac{1}{n} b_n^* & \frac{1}{n^2} c_n \end{pmatrix} + \epsilon, \; \forall n \geqq N.$$

It is easy to see that this is equivalent to the criterion stated.

5.17. LEMMA. *$\forall \epsilon > 0$, $\exists \delta > 0$ such that: If M is any finite W^*-algebra, $h, t \in M$, $0 \leqq h \leqq 1$, and $h - \delta \leqq t^*t \leqq h$, then $\exists t' \in M$ with $\|t' - t\| < \epsilon$ and $t'^*t' = h$.*

Proof. Write $t = sh^{1/2}$, $\|s\| \leqq 1$. If $0 < \delta < 1$, let

$$q = E_{(\delta^{1/2},\infty)}(h).$$

Then

$$h^{1/2}s^*sh^{1/2} \geqq h - \delta \Rightarrow qh^{1/2}s^*sh^{1/2}q \geqq qh - q\delta$$

$$\Rightarrow qs^*sq \geqq q - \delta(h')^{-1} \geqq (1 - \delta^{1/2})q,$$

where $h' = qh$ and the inverse is taken in qMq. From the polar decomposition of sq, we see that there is $v_1 \in M$ with $v_1^*v_1 = q$ and

$$\|v_1 - sq\| \leqq \delta_1 = 1 - (1 - \delta^{1/2})^{1/2}.$$

Let $v \in M$ be a unitary such that $vq = v_1$, and let $t' = vh^{1/2}$. Then

$$\|t' - t\| \leqq \|(t' - t)q\| + \|(t' - t)(1 - q)\|$$

$$\leqq \|v_1 - sq\| + 2\|h^{1/2}(1 - q)\| \leqq \delta_1 + 2\delta^{1/4}.$$

Choose δ small enough that $\delta_1 + 2\delta^{1/4} < \epsilon$.

For $x \in (M_m)_{sa}$ denote the eigenvalues of x (with multiplicity) by $(\lambda_1(x), \ldots, \lambda_m(x))$, where $\lambda_1(x) \geqq \lambda_2(x) \geqq \ldots \lambda_m(x)$.

5.18. THEOREM. *Let* $h \in A_+^{**}$.
(a) $h = T^*T$ *for some* $T \in RM(A)$ *if and only if*

$$h \in A_+^m \quad \text{and} \quad \lambda_{l+1}\left[h_n - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix}\right] \to 0 \quad \text{as } n \to \infty.$$

(b) $h = T^*T$ *for some* $T \in QM(A)$ *if and only if* $h \in (\tilde{A}_{sa}^m)^-$ *and* $\lambda_{l+1}(a_n - h_\infty) \to 0$ *as* $n \to \infty$, *where*

$$h_n = \begin{pmatrix} a_n & b_n \\ b_n^* & c_n \end{pmatrix} \in M_d.$$

Remarks. The condition on λ_{l+1} in (b) is vacuous if $l \geqq k$. The semicontinuity conditions in (a) and (b) already imply $\lambda_j \geqq -\epsilon_n$ with $\epsilon_n \to 0$ for all j (in particular $j = l + 1$). Thus the condition on λ_{l+1} is one-sided and automatically carries over to λ_j, $j > l + 1$.

Proof. (a). Assume $h = T^*T$, $T \in RM(A)$. By 4.1 $h \in A_+^m$. If

$$T_n = \begin{pmatrix} r_n & s_n \\ u_n & v_n \end{pmatrix},$$

then $T \in RM(A)$ is equivalent to $r_n \to T_\infty$, $s_n \to 0$. Therefore

$$h_n - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} = T_n^*T_n - \begin{pmatrix} T_\infty^* T_\infty & 0 \\ 0 & 0 \end{pmatrix}$$

implies

$$\left\| \left[h_n - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix}\right] - \begin{pmatrix} 0 & 0 \\ u_n & v_n \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ u_n & v_n \end{pmatrix} \right\| \to 0 \text{ as } n \to \infty.$$

Since

$$\text{rank}\left[\begin{pmatrix} 0 & 0 \\ u_n & v_n \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ u_n & v_n \end{pmatrix}\right] \leqq l,$$

$$\lambda_{l+1}\left[h_n - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix}\right] \to 0.$$

Note:

$$\lambda_{l+1}(x) = \underset{\dim V = l}{\text{Min}} \ \underset{\substack{\theta \in V^\perp \\ \|\theta\| = 1}}{\text{Max}} \ (x\theta, \theta).$$

Now assume h satisfies the criterion in (a). Choose $\epsilon_n > 0$ such that

$$\lim \epsilon_n = 0 \quad \text{and} \quad \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} \leqq h_n + \epsilon_n.$$

Write

$$(h_\infty^{1/2} \ 0) = w_n(\epsilon_n + h_n)^{1/2},$$

where $w_n \in M_{k,d}$ and $\|w_n\| \leqq 1$ and

$$(r_n\ s_n) = w_n h_n^{1/2}.$$

Note that $(r_n\ s_n) \to (h_\infty^{1/2}\ 0)$ as $n \to \infty$, since

$$\| (\epsilon_n + h_n)^{1/2} - h_n^{1/2} \| \leqq \epsilon_n^{1/2}.$$

Now let $h'_n = h_n - (r_n\ s_n)^*(r_n\ s_n)$. Then $h'_n \geqq 0$ and, since

$$\left\| h'_n - \left[h_n - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| \to 0,$$

$\lambda_{l+1}(h'_n) \to 0$ as $n \to \infty$. Thus we can find $(u_n v_n)$ such that

$$(u_n\ v_n)^*(u_n\ v_n) \leqq h'_n \quad \text{and}$$

$$\|h'_n(u_n\ v_n)^*(u_n\ v_n)\| \to 0 \text{ as } n \to \infty.$$

Then if

$$t_n = \begin{pmatrix} r_n & s_n \\ u_n & v_n \end{pmatrix},$$

we can apply 5.17 to (t_n, h_n) to obtain T_n such that $\|T_n - t_n\| \to 0$ as $n \to \infty$ and $T_n^*T_n = h_n$. If $T_\infty = h_\infty^{1/2}$, then $T \in RM(A)$ and $T^*T = h$.

(b) can be deduced from (a) by using 2.4, 4.3, and the strictly positive element e introduced above.

$$(ehe)_n - \begin{pmatrix} (ehe)_\infty & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_n & \frac{1}{n} b_n \\ \frac{1}{n} b_n^* & \frac{1}{n^2} c_n \end{pmatrix} - \begin{pmatrix} h_\infty & 0 \\ 0 & 0 \end{pmatrix} \approx \begin{pmatrix} a_n - h_\infty & 0 \\ 0 & 0 \end{pmatrix}$$

for n large. Therefore

$$\left| \lambda_{l+1} \left[(ehe)_n - \begin{pmatrix} (ehe)_\infty & 0 \\ 0 & 0 \end{pmatrix} \right] - \lambda_{l+1}(a_n - h_\infty) \right| \to 0 \text{ as } n \to \infty.$$

(Here $\lambda_{l+1}(a_n - h_\infty) = 0$ if $l \geqq k$.)

Remark. A positive answer to 4.6 for this example follows from 5.18: $\{T^*T : T \in RM(A)\}$ and $\{T^*T : T \in QM(A)\}$ are norm closed.

5.G. $C_0(X) \otimes \mathcal{K}$. In 5.G A denotes $C_0(X) \otimes \mathcal{K}$ where X is a second countable, locally compact Hausdorff space. Of course $A = E_1$ is a special case. Let $z \in A^{**}$ be the central projection corresponding to the atomic representation of A ([**29**, 4.3.7]). Then zA^{**} can be identified with the space of bounded functions from X to $B(H)$. Since every semicontinuous element of A^{**} is universally measureable, each of the classes of semicontinuous elements can be identified with its projection onto zA^{**} ([**29**], Theorem 4.3.15). It is desirable to be a little more careful about this identification. If $h \in A_{sa}^{**}$ is universally measureable, then h, regarded as a functional on $\Delta(A)$, satisfies the barycenter formula. This means that

$$\varphi(h) = \int \theta(h) d\mu(\theta)$$

whenever μ is a probability measure on $\Delta(A)$ with resultant

$$\varphi \left(\varphi(a) = \int \theta(a) d\mu(\theta), \forall a \in A \right).$$

From direct integral theory we can easily conclude that if $h \in zA_{sa}^{**}$ is given by a Borel function from X to $B(H)$, then there is a unique $\tilde{h} \in A_{sa}^{**}$ such that $z\tilde{h} = h$ and $\tilde{h}$ satisfies the barycenter formula. (So far we have used the fact that A is separable and GCR.) We will say that $h \in zA_{sa}^{**}$ is lsc (in some sense) if $h = z\tilde{h}$ for some (unique) lsc $\tilde{h} \in A_{sa}^{**}$. It will turn out that h has to be given by a Borel function, so that $\tilde{h}$ is as above.

5.19. CRITERION FOR STRONG SEMICONTINUITY. *$h \in zA_{sa}^{**}$ is strongly* lsc *if and only if*

(i) *$h(x)$ is* lsc *in $B(H)$, $\forall x \in X$,*

(ii) *$\forall \epsilon > 0$, $\exists$ compact $F \subset X$ such that $h(x) \geqq -\epsilon$, $\forall x \notin F$, and*

(iii) *If $x_0 \in X$, $\mathcal{K} \ni K \leqq h(x_0)$, and $\epsilon > 0$, then there is a neighborhood U of x_0 such that $K \leqq h(x) + \epsilon$, $\forall x \in U$.*

Proof. First assume h is strongly lsc. By 3.22, $\exists a \in A_{sa}$ such that $a \leqq h$ (we should write $za \leqq h$). (i) and (ii) follow from this. (iii) follows from the same proof as for 5.13 (ii). (Actually (iii) follows from 5.13 by Remark 5.22 (ii) below.)

Now assume (i), (ii), and (iii). Since A is continuous trace,

$$\varphi \in P(A)^{-w^*} \Rightarrow \varphi = t\theta$$

for some $\theta \in P(A)$ and $0 \leqq t \leqq 1$. In particular supp $\varphi \leqq z$ and it makes sense to write $\varphi(h)$. We claim that h is an lsc function on $P(A)^{-w^*}$. In proving this, we will represent non-zero elements of $P(A)^{-w^*}$ by pairs (x, v), $x \in X$, $v \in H$, $0 < \|v\| \leqq 1$.

$$\langle a, (x, v) \rangle = (a(x)v, v), \ a \in A, \text{ and}$$

$$(x, v) = (y, w) \Leftrightarrow x = y \text{ and}$$

$$v = \lambda w, \ |\lambda| = 1.$$

Given any net (x_α, v_α), by passing to a subnet, we may assume $x_\alpha \to x \in X$ or $x_\alpha \to \infty$ and $v_\alpha \to v \in H$ weakly. If $x_\alpha \to \infty$, then

$$(x_\alpha, v_\alpha) \to 0 \in P(A)^{-w^*},$$

and $0 \leqq \underline{\lim}\langle h, (x_\alpha, v_\alpha)\rangle$ follows from (ii). If $x_\alpha \to x$, then $(x_\alpha, v_\alpha) \to (x, v)$ in $P(A)^{-w^*}$ (or $(x_\alpha, v_\alpha) \to 0$ if $v = 0$). If $\epsilon > 0$, by (i) there is $K \in \mathcal{K}$ such that

$$K \leqq h(x) \quad \text{and} \quad (h(x)v, v) \leqq (Kv, v) + \epsilon.$$

By (iii) $K \leqq h(x_\alpha) + \epsilon$ for α sufficiently large. Also

$$(Kv, v) \leqq (Kv_\alpha, v_\alpha) + \epsilon$$

for α sufficiently large. Therefore

$$\begin{aligned}(h(x)v, v) &\leqq (Kv, v) + \epsilon \\ &\leqq (Kv_\alpha, v_\alpha) + 2\epsilon \leqq (h(x_\alpha)v_\alpha, v_\alpha) + 3\epsilon.\end{aligned}$$

Hence

$$\langle h, (x, v)\rangle \leqq \underline{\lim}\langle h, (x_\alpha, v_\alpha)\rangle$$

and the claim is proved. If we fix $v \in H$, it follows that the function

$$x \mapsto \langle h, (x, v)\rangle = (h(x)v, v)$$

is lsc, and in particular Borel, on X. Thus h is a Borel function from X to $B(H)$ and there is $\tilde{h} \in A_{sa}^{**}$ such that $\tilde{h}$ satisfies the barycenter formula and $z\tilde{h} = h$.

Now let Δ_1 be the space of probability measures on $P(A)^{-w^*}$ with the usual (weak*) topology. Then the map $\mu \mapsto$ resultant μ is continuous from Δ_1 onto $\Delta(A)$, and $\Delta(A)$ may be regarded as a topological quotient space of Δ_1. To show that $\tilde{h}$ is an lsc function on $\Delta(A)$, it is sufficient to show that the pull-back of $\tilde{h}$ to Δ_1 is lsc. Since $\tilde{h}$ satisfies the barycenter formula, this pull-back is the function

$$\mu \mapsto \int \langle \tilde{h}, \varphi\rangle d\mu(\varphi) = \int \langle h, \varphi\rangle d\mu(\varphi).$$

Finally, it is a fact of functional analysis that if h is a bounded lsc function on a compact space, then the map

$$\mu \mapsto \int h(t)d\mu(t)$$

is lsc.

Remarks. It follows from the above proof that $h \in zA_{sa}^{**}$ is strongly lsc if and only if h is an lsc function on $P(A)^{-w^*}$. Also $\tilde{h} \in A_{sa}^{**}$ is strongly lsc if and only if $\tilde{h}$ satisfies the barycenter formula and $\tilde{h}$ is an lsc function on $P(A)^{-w^*}$. The second sentence is true for arbitrary C^*-algebras. The first is true at least for separable type I C^*-algebras perfect in the sense of **[8]**.

5.19 is actually true even if X is not second countable and even if $\mathcal{K}$ is replaced by $\mathcal{K}(H)$ for H non-separable. This can be proved by Michael's selection theorem. A key point is to prove that if $a_1, \dots, a_n \in A$, $a_1, \dots, a_n \leqq h$, and $\epsilon > 0$, then $\exists a \in A$ such that $a \leqq h$ and $a \geqq a_i - \epsilon$, $i = 1, \dots, n$. Although the proof via Michael's selection theorem is in some sense more elementary than the one given, it would use more space.

5.20. CRITERION FOR MIDDLE SEMICONTINUITY. *If* $h \in zA_{sa}^{**}$, *then* h *is middle* lsc *if and only if for every* $x_0 \in X$ *there is* $P_{x_0}: X \to B(H)_+$ *such that* $P_{x_0}(x) \to 0$ *strongly as* $x \to x_0$, $h(x_0) \leqq h(x) + P_{x_0}(x)$, $\forall x \in X$, *and* $\exists \lambda \in \mathbf{R}$ *such that* $\|P_{x_0}(x)\| \leqq \lambda$, $\forall x, x_0 \in X$.

Proof. If h is middle lsc choose $\lambda_1 > 0$ such that $h + \lambda_1$ is positive and strongly lsc. Let (P_k) be a sequence of finite rank projections in $B(H)$ such that $P_k \nearrow 1$. Fix x_0 and choose $U_1 \supset U_2 \supset \dots$ such that $\{U_k\}$ is a neighborhood basis at x_0 and

$$(h(x_0) + \lambda_1)^{1/2}P_k(h(x_0) + \lambda_1)^{1/2} \leqq h(x) + \lambda_1 + \frac{1}{k}, \forall x \in U_k$$

(5.19 (iii)). Define P_{x_0} by

$$P_{x_0}(x) = (h(x_0) + \lambda_1)^{1/2}(1 - P_k)(h(x_0) + \lambda_1)^{1/2} + \frac{1}{k},$$

$$x \in U_k \backslash U_{k+1}, P_{x_0}(x_0) = 0, \text{ and}$$

$$P_{x_0}(x) = 2\|h\|, x \notin U_1.$$

If λ and $\{P_{x_0}:x_0 \in X\}$ are given, choose λ_0 such that $h' = h + \lambda_0 \geqq 0$. We claim that $h' + \lambda$ is strongly lsc. 5.19 (i) and (ii) are automatic, since $h' + \lambda \geqq 0$. If x_0 is given, then

$$L_k = h'(x_0)^{1/2}P_k h'(x_0)^{1/2} + \lambda P_k \nearrow h'(x_0) + \lambda.$$

By 5.3, it is sufficient to verify 5.19 (iii) with K replaced by one of the L_k's. Fix k and $\epsilon > 0$ and note that since $\lambda - P_{x_0}(x) \to \lambda$ strongly as $x \to x_0$, there is a neighborhood U of x_0 such that

$$\lambda P_k \leqq (\lambda - P_{x_0}(x))^{1/2}P_k(\lambda - P_{x_0}(x))^{1/2} + \epsilon, \forall x \in U.$$

Then for $x \in U$,

$$\begin{aligned} L_k &\leqq h'(x_0) + (\lambda - P_{x_0}(x))^{1/2}P_k(\lambda - P_{x_0}(x))^{1/2} + \epsilon \\ &\leqq h'(x) + P_{x_0}(x) + \lambda - P_{x_0}(x) + \epsilon \\ &= h'(x) + \lambda + \epsilon. \end{aligned}$$

Thus 5.19 implies the claim.

Remark. P_{x_0} seems to be analogous to a modulus of continuity. The criterion of 5.20 is fairly easy to state, at least for $A = E_1$, but we think its simplicity is an illusion. Middle semicontinuity is the most difficult to work with.

5.21. CRITERION FOR WEAK SEMICONTINUITY. *If* $h \in zA^{**}_{sa}$, *then* h *is weakly* lsc *if and only if* $\forall x_0 \in X$, $\forall \epsilon > 0$, $\forall$ *finite rank projection* P, *there is a neighborhood* U *of* x_0 *such that*

$$Ph(x_0)P \leqq Ph(x)P + \epsilon, \forall x \in U.$$

Proof. Assume h is weakly lsc and x_0, P, ϵ are given. Choose $f \in C_0(X)$ such that $f = 1$ in a neighborhood of x_0, and let $a = f \otimes P \in A$. By 2.4 a^*ha is strongly lsc, and the existence of the required U follows from 5.19 (iii) for a^*ha.

Assume h satisfies the criterion and let $a \in A$. By 2.4 it is enough to show a^*ha is strongly lsc. 5.19 (i) and (ii) are obvious for a^*ha. To verify 5.19 (iii), let $x_0 \in X$ and $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$8\delta\|a\| \cdot \|h\| \leqq \epsilon,$$

a finite rank projection P such that $\|(1 - P)a\| < \delta$, and a neighborhood U of x_0 such that

$$\|a(x) - a(x_0)\| < \delta \quad \text{and}$$

$$Ph(x_0)P \leqq Ph(x)P + \frac{\epsilon}{4\|a\|^2}, \forall x \in U.$$

Then for $x \in U$,

$$\begin{aligned} a(x_0)^*h(x_0)a(x_0) &\leqq a(x_0)^*Ph(x_0)Pa(x_0) + \frac{\epsilon}{4} \\ &\leqq a(x_0)^*Ph(x)Pa(x_0) + \frac{\epsilon}{2} \\ &\leqq a(x_0)^*h(x)a(x_0) + \frac{3\epsilon}{4} \\ &\leqq a(x)^*h(x)a(x) + \epsilon. \end{aligned}$$

5.22. *Remarks.* (i) As in 5.15 (i), h is weakly lsc if and only if $h(x) \leqq k$, for every weak cluster point k of h at x.

(ii) Suppose $x_n \to x$ in X where x and the x_n's are all distinct. Then we have a surjective $\theta:A \to E_1$. h lsc (in any of the three senses) implies $\theta^{**}(h)$ lsc. If $\theta^{**}(h)$ is weakly lsc for all choices of x and (x_n), then h is weakly lsc. The same holds in the strong case for $h \geqq 0$.

5.H. *General separable continuous trace algebras and their subalgebras.* Let X be as in 5.G, $\mathscr{H}$ a separable continuous field of Hilbert spaces over X such that $\mathscr{H}(x) \neq 0$ $\forall x \in X$, and A the associated C^*-algebra. Then $A \otimes \mathscr{K} \cong C_0(X) \otimes \mathscr{K}$ and A is a corner of $A \otimes \mathscr{K}$. By 2.13, $h \in zA^{**}_{sa}$ is lsc (in any sense) if and only if its image in $z(A \otimes \mathscr{K})^{**}_{sa}$ is lsc. In order to derive criteria from those in 5.G, it is only necessary to express the results of 5.G in language that is independent of the choice of the isomorphism of $A \otimes \mathcal{K}$ with $C_0(X) \otimes \mathcal{K}$. The following is easily verified:

5.23. *$h \in zA_{sa}^{**}$ is strongly* lsc *if and only if*

(i) *Each $h(x_0)$ is an* lsc *element of $B(\mathcal{H}(x_0))$ (vacuous if* $\dim \mathcal{H}(x_0) < \infty$),

(ii) $\forall \epsilon > 0$, $\exists$ *compact $F \subset X$ such that $h(x) \geqq -\epsilon$, $\forall x \notin F$, and*

(iii) *If f is a continuous section of $\mathcal{K}(\mathcal{H})$, $f(x_0) \leqq h(x_0)$, and $\epsilon > 0$, then there is a neighborhood U of x_0 such that $f(x) \leqq h(x) + \epsilon$, $\forall x \in U$.*

5.24. *$h \in zA_{sa}^{**}$ is middle* lsc *if and only if $\exists \lambda > 0$ and functions P_{x_0}, $x_0 \in X$, such that*

(i) $P_{x_0}(x) \in B(\mathcal{H}(x))_+$,

(ii) *If v is a continuous section of $\mathcal{H}$, then*

$$\lim_{x \to x_0} \|P_{x_0}(x)v(x)\| = 0,$$

(iii) $h(x_0) \leqq h(x) + P_{x_0}(x)$, $\forall x_0, x \in X$,

(iv) $\|P_{x_0}(x)\| \leqq \lambda$, $\forall x_0, x \in X$.

5.25. *$h \in zA_{sa}^{**}$ is weakly* lsc *if and only if one of the following equivalent conditions is satisfied*:

(i) *For any continuous sections f and g of $\mathcal{K}(\mathcal{H})$ such that $f(x_0) = g(x_0)^*h(x_0)g(x_0)$ and any $\epsilon > 0$, there is a neighborhood U of x_0 such that*

$$f(x) \leqq g(x)^*h(x)g(x) + \epsilon, \forall x \in U.$$

(ii) *Suppose $x_n \to x$ and $h(x_n) \to k \in B(\mathcal{H}(x))$ in the sense that*

$$(h(x_n)v(x_n), w(x_n)) \to (kv(x), w(x))$$

for all continuous sections v, w of $\mathcal{H}$. Then $h(x) \leqq k$.

If A is a general separable continuous trace algebra, then locally A comes from continuous fields of Hilbert spaces on open subsets of $X = \hat{A}$. 2.24, 2.25, and 2.27 (iii) show that 5.23-5.25 are still correct when interpreted properly. 5.23 (iii), 5.24 (ii), and 5.25 are local properties, and if we just replace "section" with "local section" and realized that $\mathcal{H}$ is only locally defined, we can make sense of them. For 5.23 the following remark is needed: If $-\epsilon \leqq h \in \overline{A_{sa}^m}$, then by 3.16, $\exists a \in A$ such that $-\epsilon \leqq a \leqq h$. Using this and a partition of unity, one can show that 5.23 (i)-(iii) imply the existence of $a \in A$ such that $a \leqq h$. The proof of 5.24 showed that the λ of 5.24 (iv) is closely related to the λ' such that $h + \lambda'$ is strongly lsc. This eliminates the "hitch" discussed in 2.27 (iii).

Finally, if A is a C^*-subalgebra of a separable continuous trace algebra B, then her(A) is still continuous trace. By 2.14, $h \in A_{sa}^{**}$ is lsc (in any sense) if and only if its image in her(A)** is lsc.

5.I. *C^*-algebra extensions.* In this example we assume

$$0 \to I \to B \xrightarrow{\theta} A \to 0$$

and that semicontinuity in I and A is understood. For example, I might be $\mathcal{K}$ and A commutative (cf. 5.B, 5.E). We will derive a description of $\overline{B_{sa}^m}$ in terms of $\overline{I_{sa}^m}$ and $\overline{A_{sa}^m}$. In principle there is no need to consider weak and middle semicontinuity, since by [5] and 2.4, h is middle lsc if and only if $h + \lambda$ is strongly lsc for some $\lambda > 0$ and h is weakly lsc if and only if b^*hb is strongly lsc $\forall b \in B$.

Note that $B^{**} \cong I^{**} \oplus A^{**}$. Let $\rho:B^{**} \to I^{**}$ be the projection.

5.26. LEMMA. *If $h \in \overline{B_{sa}^m}$ and $\theta^{**}(h) \geqq 0$ in A^{**}, then $\rho(h) \in \overline{I_{sa}^m}$.*

Proof. Let $\epsilon > 0$. By [5] there is a net (b_α) in B_{sa} such that $b_\alpha \nearrow h + \epsilon$. Then $\theta(b_\alpha) \nearrow \theta^{**}(h) + \epsilon$. Dini's theorem implies $\theta(b_\alpha) \geqq -\epsilon$ for α sufficiently large. For such α, $b_\alpha = b'_\alpha + i_\alpha$, $b'_\alpha \geqq -\epsilon$, $i_\alpha \in I_{sa}$. Since

$$M(I)_+ \subset I_+^M \subset \overline{I_{sa}^m},$$

it follows that

$$\epsilon + \rho(b_\alpha) = \rho(\epsilon + b'_\alpha) + i_\alpha \in \overline{I_{sa}^m},$$

α large. Since $\rho(b_\alpha) \nearrow \rho(h) + \epsilon$, this shows

$$\rho(h) + 2\epsilon \in \overline{I_{sa}^m}, \forall \epsilon > 0,$$

which is sufficient.

5.27. THEOREM. *If $h \in B_{sa}^{**}$, then $h \in \overline{B_{sa}^m}$ if and only if $\theta^{**}(h) \in \overline{A_{sa}^m}$ and $\rho(h - b) \in \overline{I_{sa}^m}$ for all $b \in B_{sa}$ such that $\theta(b) \leqq \theta^{**}(h)$.*

Proof. The necessity follows from 5.26.

Assume the conditions. Then by 3.22, $\exists b \in B_{sa}$ such that $\theta(b) \leqq \theta^{**}(h)$. Changing notation (replace h by $h - b$), we may assume $\theta^{**}(h) \geqq 0$ so that $\rho(h) \in \overline{I_{sa}^m}$. Now by 3.24 and 3.25 there are bounded nets (a_α) in A_+ and (i_β) in I_{sa} such that $a_\alpha \leqq \theta^{**}(h)$, $i_\beta \leqq \rho(h)$, $a_\alpha \to \theta^{**}(h)$, and $i_\beta \to \rho(h)$. We claim there is $b_{\alpha,\beta} \in B_{sa}$ such that $b_{\alpha,\beta} \leqq h$, $\theta(b_{\alpha,\beta}) = a_\alpha$, and $\rho(b_{\alpha,\beta}) \geqq i_\beta$. To see this, choose $b' \in B_+$ such that $\theta(b') = a_\alpha$ and solve (by 3.16)

$$i_\beta - \rho(b') \leqq x \leqq \rho(h - b'), x \in I_{sa}.$$

($\rho(h - b') \in \overline{I_{sa}^m}$ by hypothesis, and $i_\beta - \rho(b') \in (I_{sa})_m^-$ since $\rho(b') \in M(I)_+$.) Let $b_{\alpha,\beta} = x + b'$. $b_{\alpha,\beta} \leqq h$ follows from $\rho(b_{\alpha,\beta}) \leqq \rho(h)$ and $\theta(b_{\alpha,\beta}) = a_\alpha \leqq \theta^{**}(h)$. Since $i_\beta \leqq \rho(b_{\alpha,\beta}) \leqq \rho(h)$ and $i_\beta \to \rho(h)$ σ-weakly, it is clear that $\rho(b_{\alpha,\beta}) \to \rho(h)$ σ-weakly (and hence σ-strongly). Since also $\theta(b_{\alpha,\beta}) = a_\alpha \to \theta^{**}(h)$, we conclude that $b_{\alpha,\beta} \to h$ on $\Delta(B)$; and hence h is lsc on $\Delta(B)$.

Remarks. (i) It is not necessary to verify $\rho(h - b)$ lsc for all b such that $\theta(b) \leqq \theta^{**}(h)$. Suppose for example that $a_\alpha \nearrow \theta^{**}(h)$ and that for each α, $\rho(h - b_\alpha) \in \overline{I_{sa}^m}$ for one (and hence all) $b_\alpha \in B_{sa}$ such that $\theta(b_\alpha) = a_\alpha$. If $\theta(b) \leqq \theta^{**}(h)$ for some $b \in B_{sa}$, then $\forall \epsilon > 0$, $\theta(b) \leqq a_\alpha + \epsilon$ for α sufficiently large (Dini). Then $\exists b_\alpha \in B_{sa}$ such that $\theta(b_\alpha) = a_\alpha$ and $b \leqq b_\alpha + \epsilon$. Therefore

$$\begin{aligned} &\epsilon + \rho(h - b) \\ &= \rho(h - b_\alpha) + \rho(\epsilon + b_\alpha - b) \in \overline{I_{sa}^m} + M(I)_+ \subset \overline{I_{sa}^m}, \forall \epsilon > 0, \end{aligned}$$

which is sufficient.

(ii) If $I = \mathcal{K}$, then, as is well known ([13]), $\pi \circ \rho_{|B}$ gives rise to $\tau : A \to B(H)/\mathcal{K}$, and τ determines the extension completely. Then the condition

$$\rho(h - b) \in \overline{I_{sa}^m}$$

becomes $\pi\rho(h) \geqq \tau(a)$ (for all $a \in A_{sa}$ such that $a \leqq \theta^{**}(h)$).

The simplest non-trivial example is the case where $A = c_0$ and $I = \mathcal{K}$. Thus let (P_n) be a sequence of mutually orthogonal, infinite rank projections in $B(H)$ such that $\sum P_n = 1$. Let B be the C^*-algebra generated by $\mathcal{K}$ and the P_n's. An element of B^{**} is represented by a pair $h = (h_1, h_2)$, $h_1 \in B(H)$, $h_2 \in l^\infty$. $\theta^{**}(h) \in \overline{A_{sa}^m}$ if and only if $(h_2)_-$ vanishes at ∞. If this is so define a_n by

$$(a_n)_k = \begin{cases} h_2(k), & k \leqq n \\ h_2(k), & h_2(k) < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $a_n \nearrow \theta^{**}(h) = h_2$, and $h \in \overline{B_{sa}^m}$ if and only if

$$\pi(h_1) \geqq \sum_1^n h_2(k)\pi(P_k) + \sum_{\substack{k>n \\ h_2(k)<0}} h_2(k)\pi(P_k), \forall n.$$

The infinite sum is norm convergent, since $(h_2)_- \in c_0$. To check whether h is weakly lsc, it is not necessary to consider b^*hb for all $b \in B$. It is enough to take

$$b = \sum_1^n P_k, \quad n = 1, 2, \dots .$$

In this example $\theta^{**}(b^*hb) \in A$. Then $h = (h_1, h_2) \in (\widetilde{B}_{sa}^m)^-$ if and only if

$$\pi\left(\left(\sum_1^n P_k\right) h_1 \left(\sum_1^n P_k\right)\right) \geqq \sum_1^n h_2(k)\pi(P_k), \forall n.$$

References

1. C. A. Akemann, *The general Stone-Weierstrass problem*, J. Funct. Anal. *4* (1969), 277-294.
2. ——— *Left ideal structure of C^*-algebras*, J. Funct. Anal. *6* (1970), 305-317.
3. ——— *Approximate units and maximal abelian C^*-subalgebras*, Pac. J. Math. *33* (1970), 543-550.
4. ——— *A Gelfand representation theory for C^*-algebras*, Pac. J. Math. *39* (1971), 1-11.
5. C. A. Akemann and G. K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J. *40* (1973), 785-795.
6. ——— *Ideal perturbations of elements in C^*-algebras*, Math. Scand. *41* (1977), 117-139.
7. C. A. Akemann, G. K. Pedersen and J. Tomiyama, *Multipliers of C^*-algebras*, J. Funct. Anal. *13* (1973), 277-301.
8. C. A. Akemann and F. Shultz, *Perfect C^*-algebras*, Memoirs A.M.S. *326* (1985).
9. L. G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pac. J. Math. *71* (1977), 335-348.
10. ——— *Close hereditary C^*-subalgebras and the structure of quasi-multipliers*, preprpint.
11. ——— *Generalized crossed products of C^*-algebras*, in preparation.
12. L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pac. J. Math. *71* (1977), 349-363.
13. R. C. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. *132* (1968), 79-99.
14. F. Combes, *Sur les faces d'une C^*-algèbre*, Bull. Sci. Math. *93* (1969), 37-62.
15. ——— *Quelques propriétés des C^*-algèbres*, Bull. Sci. Math. *94* (1970), 165-192.
16. Ch. Davis, *Notions generalizing convexity for functions defined on spaces of matrices*, Proc. Symp. Pure Math. *7* (Amer. Math. Soc., Providence, R.I., 1962), 187-201.
17. Ch. Davis, W. M. Kahan and H. F. Weinberger, *Norm-preserving dilations and their applications to optimal error bounds*, SIAM J. Numer. Anal. *19* (1982), 445-469.
18. J. Dixmier, *Les C^*-algèbres et leurs représentations* (Gauthier-Villars, Paris, 1964).
19. J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France *91* (1963), 227-284.
20. E. G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. *30* (1963), 391-412.
21. F. Hansen, *An operator inequality*, Math. Ann. *246* (1980), 249-250.
22. F. Hansen and G. K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. *258* (1982), 229-241.
23. G. G. Kasparov, *Hilbert C^*-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory *4* (1980), 133-150.
24. E. A. Michael, *Continuous selections I*, Ann. Math. *63* (1956), 361-382.
25. J. A. Mingo and W. J. Phillips, *Equivariant triviality theorems for Hilbert C^*-modules*, Proc. Amer. Math. Soc. *91* (1984), 225-230.
26. D. Olesen and G. K. Pedersen, *Derivations of C^*-algebras have semi-continuous generators*, Pac. J. Math. *53* (1974), 563-572.
27. S. K. Parrott, *On a quotient norm and the Sz.-Nagy-Foias lifting theorem*, J. Funct. Anal. *30* (1978), 311-328.
28. G. K. Pedersen, *Applications of weak* semicontinuity in C^*-algebra theory*, Duke Math. J. *39* (1972), 431-450.
29. ——— *C^*-algebras and their automorphism groups* (Academic Press, London, 1979).
30. ——— *SAW*-algebras and corona C^*-algebras, contributions to non-commutative topology*, J. Operator Theory (to appear).
31. N. T. Shen, *Embeddings of Hilbert bimodules*, preprint.

32. E. Størmer, *Two-sided ideals in C*-algebras*, Bull. Amer. Math. Soc. *73* (1967), 254-257.

33. M. Tomita, *Spectral theory of operator algebras, I*, Math. J. Okayama Univ. *9* (1959), 63-98.

Purdue University,
W. Lafayette, Indiana