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DIMENSION OF A TOPOLOGICAL TRANSFORMATION GROUP

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Throughout this paper, the Alexander-Spanier cohomology with compact supports will be used. Suppose X is a compact connected topological m-manifold which admits an effective action of a compact connected Lie group G ($m \geqq 19$). It is known [**3**] that X is either homeomorphic to the complex projective k-space CP^k ($m = 2k$), or

$$\dim G \leqq \langle \alpha \rangle + \langle m - \alpha \rangle,$$

for all α such that $H^\alpha(X; Q) \neq 0$, where $\langle k \rangle$ denotes $k(k+1)/2$ for a non-negative integer k. In this paper, we prove the corresponding result for the actions of compact connected Lie groups on the locally compact topological spaces. In [**5**], it is proved that if a compact connected (Lie) group G acts effectively on a connected locally compact m-dimensional space X with w conjugacy classes of isotropy subgroups, $w \geqq 2$, then $\dim G \leqq (w-1)\langle m - 1 \rangle$. We improve the bound on the dimension of G by proving the following result.

THEOREM. *Let G be a compact connected Lie group acting effectively on a connected locally compact m-dimensional space X with w distinct conjugacy classes of isotropy subgroups, $w \geqq 2$, $m \geqq 20$. Suppose the fixed point set F of G is not empty*, $\dim F < \alpha \leqq m - 1$ *for some α and $H^\alpha(X; Q) \neq 0$. Then precisely one of the following holds:*

(1) *There is exactly one type of orbits of the form $CP^k (m - 1 = 2k)$ and* $\dim G \leqq (w-2)\langle m-1 \rangle + \dim SU(k+1)$.

(2) $\dim G \leqq (w-2)\langle m-1 \rangle + \langle \beta \rangle + \langle m - \beta - 1 \rangle$, *where* $\beta = \max(\alpha, m - \alpha)$.

Proof. Suppose

$$\text{(3)} \qquad \dim G > (w-2)\langle m-1 \rangle + \langle \beta \rangle + \langle m - \beta - 1 \rangle.$$

We proceed to show that we only have statement (1). Now

$$\langle \beta \rangle + \langle m - \beta - 1 \rangle \geqq (m-1)^2/4 + (m-1)/2.$$

Hence

$$\text{(4)} \qquad \dim G > (w-2)\langle m-1 \rangle + (m-1)^2/4 + (m-1)/2.$$

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Let X_i, $i = 1, \ldots, w$, be the point set union of the orbits corresponding to w conjugacy classes of isotropy subgroups, and let K_i be the normal subgroups of G acting trivially on X_i such that G/K_i acts effectively on X_i with all orbits of the same type. We may assume that $K_w = G$ and $X_w = F$. Obviously, $X = X_1 \cup \ldots \cup X_{w-1} \cup F$, and G/K_i acts effectively on every orbit in X_i, $1 \leqq i \leqq w - 1$, which is at most $(m - 1)$-dimensional [**5**]. Hence

(5) $\dim G/K_i \leqq \langle m - 1 \rangle, \quad i = 1, \ldots, w - 1.$

The map

$$\phi : G \rightarrow G/K_1 \times \ldots \times G/K_{w-1}$$

defined by $\phi(g) = (gK_1, \ldots, gK_{w-1})$ for $g \in G$ is a monomorphism because the action of G on X is effective and $\bigcap_{i=1}^{w-1} K_i = \bigcap_{i=1}^{w} K_i$ is the identity of G. It follows from (4) that

(6) $$\sum_{i=1}^{w-1} \dim G/K_i \geqslant \dim G > (w - 2)\langle m - 1 \rangle + (m - 1)^2/4 + (m - 1)/2.$$

Express the groups G and G/K_i, $1 \leqq i \leqq w - 1$, in the following forms:

(7) $G = \bar{G}/N = (S_1 \times \ldots \times S_v \times T^q)/N,$

(8) $G/K_i = \bar{G}_i/N_i = (S_1{}^i \times \ldots \times S_{v_i}{}^i \times T^{q_i})/N_i,$

where T^q (respectively T^{q_i}) is a q-torus (q_i-torus), each S_j (respectively $S_j{}^i$) is a compact, connected, simply connected simple Lie group, or isomorphic to Spin (4) $\cong$ Spin (3) $\times$ Spin (3), and there is at most one Spin (3), and N (respectively N_i) is a finite normal subgroup of $\bar{G}$ (respectively $\bar{G}_i$).

It is easily seen from (6) that

(9) $\dim G/K_i > (m - 1)^2/4 + (m - 1)/2, \quad 1 \leqq i \leqq w - 1.$

Now for any fixed $x_i \in X_i$, let $M_i = (G/K_i)(x_i)$, the G/K_i orbit at x_i, $1 \leqq i \leqq w - 1$. Then $\dim M_i \leqq m - 1$. Since $m - 1 \geqq 19$, and G/K_i satisfy (9), we may modify the proof of the Main Lemma in [**3**] to the actions of G/K_i on M_i to obtain the following. For each i, $1 \leqq i \leqq w - 1$, exactly one of the following holds:

(α_i) M_i is homeomorphic to $CP^k (m - 1 = 2k)$, and G/K_i is locally isomorphic to $SU(k + 1)$.

(β_i) M_i is homeomorphic to $CP^k \times S^1 (m - 2 = 2k)$, and G/K_i is locally isomorphic to $U(k + 1)$.

(γ_i) M_i is a simple lens space finitely covered by $S^{2k+1} (m - 1 = 2k + 1)$, and G/K_i is locally isomorphic to $U(k + 1)$.

(δ_i) G/K_i contains a normal factor $S_1{}^i \cong$ Spin (n_i) (see (8)), where

(a_i) $n_i > (m + 1)/2$,

(b_i) $S_1{}^i$ acts almost effectively on the homogeneous space M_i with all orbits homeomorphic to either $S^{n_i - 1}$ or $RP^{n_i - 1}$.

Suppose there are i_1, i_2, $(i_1 \neq i_2)$ satisfying (α_i), $i = i_1, i_2$. Then

$$\dim G \leqq (w-3)\langle m-1\rangle + 2\dim SU(k+1) < (w-2)\langle m-1\rangle + (m-1)^2/4 + (m-1)/2.$$

This contradicts (4). If there is exactly one M_i satisfying (α_i), we have the statement (1). We will show that the remaining possibilities (β_i), (γ_i) and (δ_i) cannot occur.

In the case that there is an M_i satisfying either (β_i) or (γ_i), we have

$$\dim G \leqslant \sum_{j\neq i} \dim G/K_j + \dim U(k+1) \leqslant (w-2)\langle m-1\rangle + \langle\beta\rangle + \langle m-\beta-1\rangle,$$

which contradicts (3). Hence the possibilities (δ_i) hold for all i, $i = 1, \ldots, w-1$.

We may lift each $S_1^{\,i}$ in $\bar{G}_i$ to $\bar{G}$, and identify $S_1^{\,i}$ as a subgroup of $\bar{G}$, $1 \leqq i \leqq w-1$. The subgroups $S_1^{\,i}$ of $\bar{G}$ are all distinct, $1 \leqq i \leqq w-1$. Otherwise, there exist i, j, $i \neq j$, and $S_1^{\,i} = S_1^{\,j}$. Let

$$\bar{\phi} : \bar{G} \rightarrow \bar{G}_1 \times \ldots \times \bar{G}_{w-1}$$

be the homomorphism that covers ϕ. Define the homomorphism

$$\psi : \bar{G}_1 \times \ldots \times \bar{G}_{w-1} \rightarrow \bar{G}_1 \times \ldots \times \bar{G}_i/S_1^{\,i} \times \ldots \times \bar{G}_{w-1}$$

by $\psi(g_1, \ldots, g_i, \ldots, g_{w-1}) = (g_1, \ldots, g_{i-1}, g_iS_1^{\,i}, g_{i+1}, \ldots, g_{w-1})$. Then Ker $(\psi\bar{\phi})$ is a finite group. Hence

$$(10)\quad \dim \bar{G} = \dim G \leqslant \sum_{\substack{k=1\\k\neq i}}^{w-1} \dim G/K_k + \dim G/K_i - \dim S_1^{\,i}.$$

Let $t_c^{\,i}$ be the smallest integer such that $\dim S_c^{\,i} \leqq \langle t_c^{\,i}\rangle$. It follows from [**2**; **4**] (applied to the action of G/K_i on M_i) that

$$(11)\quad \sum_{c=1}^{v_i} t_c^{\,i} + q_i \leqslant \dim M_i \leqslant m-1.$$

But $t_1^{\,i} = n_i - 1 > (m-1)/2$ by (a_i), hence $\sum_{c=2}^{v_i} t_c^{\,i} + q_i < (m-1)/2$. From (8) we have

$$\dim G/K_i - \dim S_1^{\,i} \leqslant \sum_{c\geqq 2} \langle t_c^{\,i}\rangle + q_i \leqslant \left\langle \sum_{c\geqq 2} t_c^{\,i} + q_i \right\rangle \leqslant \langle[(m-1)/2]\rangle \langle (m-1)^2/4 + (m-1)/2.$$

Hence

$$\dim G < (w-2)\langle m-1\rangle + (m-1)^2/4 + (m-1)/2,$$

by (10). This is a contradiction to (4).

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Denote the subgroup $S_1^1 \times \ldots \times S_1^{w-1}$ of $\bar{G}$ by H. The group $S_1^1 \times \ldots \times S_1^{i-1} \times S_1^{i+1} \times \ldots \times S_1^{w-1}$ must act trivially on X_i, $1 \leqq i \leqq w-1$. Otherwise the orbits of H will have dimension at least $(n_i - 1) + (n_j - 1)$ for some j $(1 \leqq j \leqq w-1, j \neq i)$, and $(n_i - 1) + (n_j - 1) > m - 1$ by (a_i) which contradicts the fact that $\dim M_i \leqq m - 1$. It follows that $X_i/H = X_i/S_1^i$ and

$$X/H = X_1/S_1^1 \cup \ldots \cup X_{w-1}/S_1^{w-1} \cup F.$$

Now the action of S_1^i on X_i has all orbits either S^{n_i-1} or RP^{n_i-1}. This follows from (δ_i) and the fact that the action of S_1^i on any two G/K_i orbits in X_i are equivariant homeomorphic. Hence we have fibrations $X_i \to X_i/S_1^i$ with fibre S^{n_i-1} or RP^{n_i-1}, $1 \leqq i \leqq w-1$. Let $n_k = \min\{n_i : i = 1, \ldots, w-1\}$. Then

$$\dim X_i/S_1^i \leqq m - (n_i - 1) \leqq m - n_k + 1,$$

and

$$\dim X/H \leqq \max\{m - n_k + 1, \dim F\}. \tag{12}$$

We claim that $n_k \leqq \beta + 1$. Suppose, on the contrary, that $n_k \geqq \beta + 2$. Consider the projection $\pi : X \to X/H$. For each $\tilde{x} \in X/H$, $\pi^{-1}(\tilde{x})$ is S^{n_i-1}, RP^{n_i-1}, or a point, which is acyclic over Q up to $n_k - 2$. It follows from the Vietoris-Begle mapping theorem that

$$\pi^* : H^j(X/H; Q) \cong H^j(X; Q), \quad j \leqq n_k - 2.$$

However, $H^\alpha(X/H; Q) \neq 0$ since $\alpha \leqq \beta \leqq n_k - 2$. But

$$\dim X/H \leqq \max\{m - \beta - 1, \dim F\} < \alpha$$

from (12). This is, of course, impossible. Hence $n_k \leqq \beta + 1$.

Now we consider the action of G/K_k on M_k. From (δ_k) and (11) we have

$$S_1^k \cong \text{Spin}(n_k),\ n_k > (m+1)/2, \text{ and} \tag{13}$$
$$\beta \geqq t_1^k = n_k - 1 \geqq t_j^k, \quad 2 \leqq j \leqq v_k.$$

Let $t_1^k = \beta - u$, $u \geqq 0$. Then

$$\dim G/K_k = \dim \bar{G}_k \leqslant \langle \beta - u \rangle + \sum_{=2}^{vk} \langle t_j^k \rangle + q_k,$$

where

$$\sum_{j=2}^{vk} t_j^k + q_k - u \leqslant m - \beta - 1, \tag{14}$$

by (11). We consider two cases.

(i) $\sum_{j=2}^{v_k} t_j^k + q_k \leqq u$. Then

$$\dim G/K_k \leqslant \langle \beta - u \rangle + \langle \sum_{j=2}^{vk} t_j^k + q_k \rangle$$
$$\leqslant \langle \beta - u \rangle + \langle u \rangle \leqslant \langle \beta \rangle \leqslant \langle \beta \rangle + \langle m - \beta - 1 \rangle.$$

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(ii) $\sum_{j=2}^{v_k} t_j^k + q_k > u$. By repeated use of Lemma 2(b) in [3],

$$\langle \beta - u \rangle + \sum_{j=2}^{vk} \langle t_j^k \rangle + q_k \leqslant \langle \beta \rangle + \sum_{j=2}^{vk} \langle \tilde{t}_j^k \rangle + \tilde{q}_k,$$

where $0 \leqq \tilde{q}_k \leqq q_k$, $0 \leqq \tilde{t}_j^k \leqq t_j^k$, $(2 \leqq j \leqq v_k)$, and

$$\sum_{j=2}^{vk} \tilde{t}_j^k + \tilde{q}_k = \sum_{j=2}^{vk} t_j^k + q_k - u.$$

It follows that

$$\dim G/K_k = \dim \bar{G}_k \leqslant \langle \beta \rangle + \sum_{j=2}^{vk} \langle \tilde{t}_j^k \rangle + \tilde{q}_k$$

$$\leqslant \langle \beta \rangle + \left\langle \sum_{j=2}^{vk} \tilde{t}_j^k + \tilde{q}_k \right\rangle$$

$$= \langle \beta \rangle + \left\langle \sum_{j=2}^{vk} t_j^k + q_k - u \right\rangle$$

$$\leqslant \langle \beta \rangle + \langle m - \beta - 1 \rangle \text{ (From (14))}.$$

Hence

$$\dim G \leqq (w-2)\langle m - 1 \rangle + \langle \beta \rangle + \langle m - \beta - 1 \rangle,$$

a contradiction. This completes the proof of the theorem.

Remarks 1. The theorem is best possible. Let Y be the disjoint union of $(w - 2)$ copies of the $(m - 1)$-sphere S^{m-1} and $S^{\alpha-1} \times S^{m-\alpha}$ $(m - \alpha \geqq \alpha)$. Take X to be the suspension of Y. Let

$$G = SO(m) \times \ldots \times SO(m) \times SO(\alpha) \times SO(m - \alpha + 1),$$

with $(w - 2)$ copies of $SO(m)$. Now let each copy of $SO(m)$ in G act non-trivially and orthogonally on exactly one copy of S^{m-1}, and $SO(\alpha) \times SO(m - \alpha + 1)$ acts transitively and non-trivially just on $S^{\alpha-1} \times S^{m-\alpha}$ in Y. Extend the action of G to X leaving the two vertices of X fixed. Then there are w conjugacy classes of isotropy subgroups, $H^\alpha(X; Q) \neq 0$ and

$$\dim G = (w-2)\langle m - 1 \rangle + \langle m - \alpha \rangle + \langle \alpha - 1 \rangle.$$

For an example that satisfies statement (1) and

$$\dim G = (w-2)\langle m - 1 \rangle + \dim SU(k+1),$$

we simply replace $S^{\alpha-1} \times S^{m-\alpha}$ and the factor $SO(\alpha) \times SO(m - \alpha + 1)$ in the above example by CP^k $(2k = m - 1)$ and $SU(k + 1)$ respectively with $SU(k + 1)$ acting transitively on CP^k.

2. From the proof of the theorem, it is not difficult to see that if $w = 1$, we have the following result: Let G be a compact connected Lie group acting effectively on a connected locally compact m-dimensional space X with

exactly one type of orbits, $m \geqq 19$. Then X is either homeomorphic to CP^k $(2k = m)$, or

$$\dim G \leqq \langle \alpha \rangle + \langle m - \alpha \rangle$$

for all α such that $H^\alpha(X; Q) \neq 0$.

3. The same proof also shows that the theorem is true when the fixed point set F is empty.

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